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Problem Sheet 2

ASO: Multivariable Calculus

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Question 1

(a) Show that there exists a real-valued C^1 function g defined on a neighbourhood of the origin of $\mathbb{R}$ such that

$$g(x) = g(x)^3 + 2\,\mathrm{e}^{g(x)}\sin x$$

(b) Show that the equations

$$\begin{aligned} \mathrm{e}^x + \mathrm{e}^{2y} + \mathrm{e}^{3u} + \mathrm{e}^{4v} &= 4 \\ \mathrm{e}^x + \mathrm{e}^y + \mathrm{e}^u + \mathrm{e}^v &= 4 \end{aligned}$$

can be solved for u, v in terms of x, y near the origin.

Proof. (a) Let $f(x, y) = y^3 + 2\,\mathrm{e}^y \sin x - y$. We have $f(0, 0) = 0$. We shall show that f is C^1 in a neighbourhood of $(0, 0) \in \mathbb{R}^2$. We compute the partial derivatives:

$$\frac{\partial f}{\partial x} = 2\,\mathrm{e}^y \cos x \qquad \frac{\partial f}{\partial y} = 3y^2 + 2\,\mathrm{e}^y \sin x - 1$$

$\partial_x f$ and $\partial_y f$ are continuous everywhere. Then $f \in C^1(\mathbb{R}^2)$. Since $\partial_y f(0, 0) = -1 \neq 0$, by the implicit function theorem, there exists a neighbourhood I of 0 and a C^1 function $g : I \to \mathbb{R}$ such that $f(x, y) = 0$ implies that $y = g(x)$ for $x \in I$. The function g satisfies that $g(x) = g(x)^3 + 2\,\mathrm{e}^{g(x)} \sin x$ for $x \in I$.

(b) Let

$$f_1(x, y, u, v) = \mathrm{e}^x + \mathrm{e}^{2y} + \mathrm{e}^{3u} + \mathrm{e}^{4v} - 4 \qquad f_2(x, y, u, v) = \mathrm{e}^x + \mathrm{e}^y + \mathrm{e}^u + \mathrm{e}^v - 4$$

and $\boldsymbol{f}(u, v, x, y) = \begin{pmatrix} f_1(x, y, u, v) \\ f_2(x, y, u, v) \end{pmatrix}$. First we check that $\boldsymbol{f}$ is C_1. We compute the partial derivatives:

$$\begin{aligned} &\frac{\partial f_1}{\partial x} = \mathrm{e}^x \qquad \frac{\partial f_1}{\partial y} = 2\,\mathrm{e}^y \qquad \frac{\partial f_1}{\partial u} = 3\,\mathrm{e}^u \qquad \frac{\partial f_1}{\partial v} = 4\,\mathrm{e}^v \\ &\frac{\partial f_2}{\partial x} = \mathrm{e}^x \qquad \frac{\partial f_2}{\partial y} = \mathrm{e}^y \qquad \frac{\partial f_2}{\partial u} = \mathrm{e}^u \qquad \frac{\partial f_2}{\partial v} = \mathrm{e}^v \end{aligned}$$

All partial derivatives exist and are continuous. Then $\boldsymbol{f} \in C^1(\mathbb{R}^4)$. At $(0, 0, 0, 0) \in \mathbb{R}^4$: $\boldsymbol{f}(0, 0, 0, 0) = (0, 0)$. The differential map of $\boldsymbol{f}$ with respect to $\boldsymbol{y} := (u, v)$ at $(0, 0, 0, 0)$:

$$D_{\boldsymbol{y}} \boldsymbol{f}_{\mathbf{0}} = \begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix}$$

The map has non-zero determinant so it is invertible. By the implicit function theorem, there exists a open neighbourhood U of $(0, 0)$ and a C^1 function $\boldsymbol{g} : U \to \mathbb{R}^2$ such that $\boldsymbol{f}(x, y, u, v) = \mathbf{0}$ implies that $(u, v) = \boldsymbol{g}(x, y)$ for $(x, y) \in U$. The function $\boldsymbol{g}$ solves the equations for u, v in terms of x, y near the origin. □

Question 2

By considering the function defined by

$$f(x) = \frac{x}{2} + x^2 \sin\left(\frac{1}{x}\right) \text{ for } x \neq 0 \text{ and } f(0) = 0$$

show that the C^1 hypothesis cannot be removed from the statement of the inverse function theorem.

Proof. For this well-known function, we know that it is continuous and differentiable on $\mathbb{R}$, but its derivative:

$$f'(x) = \begin{cases} \dfrac{1}{2} + 2x \sin\dfrac{1}{x} - \cos\dfrac{1}{x}, & x \neq 0; \\ \dfrac{1}{2}, & x = 0. \end{cases}$$

is discontinuous at $x = 0$. So $f \notin C^1(I)$ for any neighbourhood I of 0. Suppose that the inverse function theorem applies. Then f is locally invertible at $x = 0$. In particular it is bijective and continuous in some interval I containing 0. By Prelim Analysis we know that f is strictly monotonic in the interval. So its derivative f' has fixed sign in I. But $f'(x) \sim \frac{1}{2} - \cos\frac{1}{x}$ changes sign infinitely many times near $x = 0$, which is a contradiction.

We conclude that the condition of f being C^1 in the inverse function theorem is necessary. □

Question 3

Deduce the inverse function theorem from the implicit function theorem.

Proof. Suppose that $\Omega \subset \mathbb{R}^n$ is open and $\boldsymbol{f} \in C^1(\Omega, \mathbb{R}^n)$. Let $\boldsymbol{x}_0 \in \Omega$ and $\boldsymbol{y}_0 = \boldsymbol{f}(\boldsymbol{x}_0)$. Suppose that $D\boldsymbol{f}(\boldsymbol{x}_0)$ is invertible.

Consider $\boldsymbol{g} : \Omega \times \boldsymbol{f}(\Omega) \to \mathbb{R}^n$ given by

$$\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{y}$$

We have $D_{\boldsymbol{x}}\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) = D\boldsymbol{f}(\boldsymbol{x})$ and $D_{\boldsymbol{y}}\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) = -I_n$. In particular, $\boldsymbol{g} \in C^1(\Omega \times \boldsymbol{f}(\Omega), \mathbb{R}^n)$.

Since $D\boldsymbol{f}(\boldsymbol{x}_0)$ is invertible, $D_{\boldsymbol{x}}\boldsymbol{g}(\boldsymbol{x}_0, \boldsymbol{y}_0)$ is invertible. By the **Implicit Function Theorem**, there exists open neighbourhoods U of y_0 and V of x_0, and a function $\boldsymbol{f}^{-1} \in C^1(U, V)$ such that

$$\forall (\boldsymbol{x}, \boldsymbol{y}) \in V \times U : \ \boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}) \iff \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}) = 0 \iff \boldsymbol{x} = \boldsymbol{f}^{-1}(\boldsymbol{y})$$

We deduce that $\boldsymbol{f}$ is a diffeomorphism on $\boldsymbol{f}^{-1}(U)$. In addition, the differential of $\boldsymbol{f}^{-1}$ at $\boldsymbol{y}_0$ is given by

$$D\boldsymbol{f}^{-1}(\boldsymbol{y}_0) = -\left(D_{\boldsymbol{x}}\boldsymbol{g}(\boldsymbol{x}_0, \boldsymbol{y}_0)\right)^{-1} D_{\boldsymbol{y}}\boldsymbol{g}(\boldsymbol{x}_0, \boldsymbol{y}_0) = \left(D\boldsymbol{f}(\boldsymbol{x}_0)\right)^{-1}$$

In this way we have derived the full inverse function theorem from the implicit function theorem. □

Question 4

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function.

(a) Show that the graph of f

$$\{(x, y, z) \in \mathbb{R}^3 : \ z = f(x, y)\}$$

is a 2-dimensional submanifold of $\mathbb{R}^3$.

(b) Identify the normal space to M at a point, $(x, y, f(x, y))$ and give a basis for the tangent space at that point.

Proof. (a) This is trivial by Proposition 6.3 in the notes.

(b) Let $g(x, y, z) = f(x, y) - z$. The graph of f is the loci of g in $\mathbb{R}^3$. By Proposition 6.8, the tangent space of the manifold at $\boldsymbol{x} := (x, y, f(x, y))$ is given by

$$T_{\boldsymbol{x}}M = \ker Dg(\boldsymbol{x}) = \ker \begin{pmatrix} \frac{\partial f}{\partial x}(\boldsymbol{x}) & \frac{\partial f}{\partial y}(\boldsymbol{x}) & -1 \end{pmatrix} = \left\langle \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x}(\boldsymbol{x}) \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y}(\boldsymbol{x}) \end{pmatrix} \right\rangle$$

The basis of $T_{\boldsymbol{x}}M$ is $\left\{ \left(1, 0, \frac{\partial f}{\partial x}(\boldsymbol{x})\right)^{\mathrm{T}}, \left(0, 1, \frac{\partial f}{\partial y}(\boldsymbol{x})\right)^{\mathrm{T}} \right\}$.

The normal space at $\boldsymbol{x}$ is simply the orthogonal complement of the tangent space. In this case, the normal space of the manifold is 1-dimensional, generated by the vector

$$\left(0, 1, \frac{\partial f}{\partial y}(\boldsymbol{x})\right) \times \left(1, 0, \frac{\partial f}{\partial x}(\boldsymbol{x})\right) = \left(\frac{\partial f}{\partial x}(\boldsymbol{x}), \frac{\partial f}{\partial y}(\boldsymbol{x}), -1\right)$$

The result is unsurprising. □

Question 5

For which values of c does the equation $x^2 + y^2 - z^2 = c$ define a 2-dimensional submanifold of $\mathbb{R}^3$?

Describe the loci defined by the above equation, paying particular attention to any values for which the locus is not a manifold.

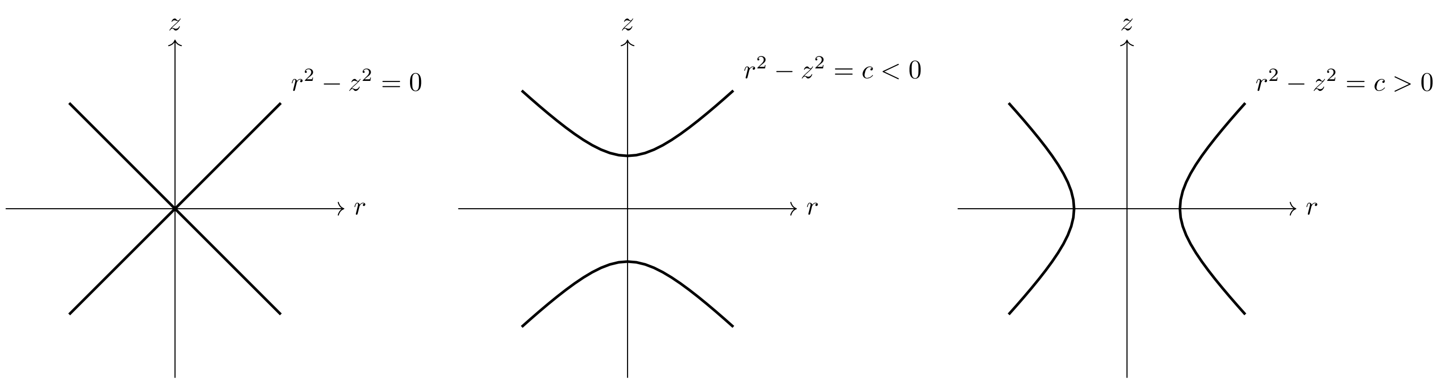
Proof. Let $f(x, y, z) = x^2 + y^2 - z^2 - c$. The differential of f at $\boldsymbol{x} = (x, y, z)$ is given by

$$Df(x, y, z) = (2x, 2y, 2z)$$

The set $f(x, y, z) = 0$ is a (C^1) manifold if $\operatorname{rank} Df(x, y, z) = 1$ for every (x, y, z) where $f(x, y, z) = 0$. We observe that $\operatorname{rank} Df(x, y, z) ne 1 \iff (x, y, z) = (0, 0, 0)$. Hence $f(x, y, z) = 0$ is a manifold if and only if $f(0, 0, 0) \neq 0$, which is equivalent to $c \neq 0$.

To describe the loci of the equation, we first note that the equation has azimuthal symmetry. So we change to the cylindrical coordinates, in which the manifold is described by $r^2 - z^2 = c$.

When $c = 0$, the loci $f(r, \theta, z) = r^2 - z^2 = 0$ is a double cone. $(0, 0, 0)$ is a singularity of the surface. When $c \neq 0$, the loci $f(r, \theta, z) = r^2 - z^2 - c = 0$ is a hyperboloid, and is a sub-manifold of $\mathbb{R}^3$.



□

Question 6

Find the maximum value of

$$g(x_1, ..., x_n) = \prod_{i=1}^{n} x_i$$

subject to the constraint $\sum_{i=1}^{n} x_i = 1$ and the condition that the x_i are non-negative.

Deduce the arithmetic mean / geometeric mean inequality

$$\left(\prod_{i=1}^{n} a_i \right)^{\frac{1}{n}} \leqslant \frac{1}{n} \sum_{i=1}^{n} a_i$$

for non-negative real numbers $a_1, ..., a_n$.

Proof. Let $f(x_1, ..., x_n) = \sum_{i=1}^{n} x_i - 1$ and $g(x_1, ..., x_n) = \prod_{i=1}^{n} x_i$. It is clearly that $f, g \in C^1(\mathbb{R})$.

Consider $M := \{\boldsymbol{x} \in \mathbb{R}^n : f(\boldsymbol{x}) = 0,\ x_i > 0\}$. M is an open set in $\mathbb{R}^n$, and is clearly a submanifold of $\mathbb{R}^n$. Since $\overline{M}$ is compact and g is continuous, g attains its supremum at some $\boldsymbol{z} \in \overline{M}$. But $g = 0$ on ∂M and $g > 0$ in M. We deduce that $\boldsymbol{z} \in M$.

By the theorem of Lagrange multipliers, there exists $\lambda \in \mathbb{R}$ such that $\boldsymbol{\nabla} g(\boldsymbol{z}) = \lambda \boldsymbol{\nabla} f(\boldsymbol{z})$. We compute the gradients:

$$\boldsymbol{\nabla} f(\boldsymbol{x}) = (1, \ldots, 1)^{\mathrm{T}} \qquad \boldsymbol{\nabla} g(\boldsymbol{x}) = x_1 \cdots x_n \left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right)^{\mathrm{T}} = g(\boldsymbol{x}) \left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right)^{\mathrm{T}}$$

Then we obtain that $\dfrac{g(\boldsymbol{z})}{z_i} = \lambda$ for each $i \in \{1, \ldots, n\}$. Therefore $z_1 = \cdots = z_n$. $f(\boldsymbol{z}) = 0$ implies that $nz_i - 1 = 0$. Hence $z_1 = \cdots = z_n = \dfrac{1}{n}$. $g(\boldsymbol{z}) = \dfrac{1}{n^n}$. $\lambda = \dfrac{1}{n^{n-1}}$.

For any $a_1, \ldots, a_n > 0$, we define $b_i := \dfrac{a_i}{\sum_{i=1}^n a_i}$ for each i. So $\displaystyle\sum_{i=1}^n b_i = 1$. The point $(b_1, \ldots, b_n) \in M$. We have:

$$g(b_1, \ldots, b_n) \leqslant g(z_1, \ldots, z_n) = \frac{1}{n^n} \quad \Longrightarrow \quad \frac{\prod_{i=1}^n a_i}{\left(\sum_{i=1}^n a_i\right)^n} \leqslant \frac{1}{n^n} \quad \Longrightarrow \quad \left(\prod_{i=1}^n a_i\right)^{\frac{1}{n}} \leqslant \frac{1}{n} \sum_{i=1}^n a_i$$

Hence we obtain the AM-GM inequality. $\square$