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Problem Sheet 1

C3.11: Riemannian Geometry

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Section A: Introductory

Question 1

Let

$$\mathcal{H}^n = \left\{ (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{j=1}^{n} x_j^2 - x_{n+1}^2 = -1, x_{n+1} > 0 \right\}$$

and let g be the restriction of

$$h = \sum_{j=1}^{n} \mathrm{d}x_j^2 - \mathrm{d}x_{n+1}^2$$

on $\mathbb{R}^{n+1}$ to $\mathcal{H}^n$.

(a) Show that g is a Riemannian metric on $\mathcal{H}^n$.

(b) Let $f(x) = Ax$ be a linear map on $\mathbb{R}^{n+1}$ given by $A = (a_{ij}) \in M_{n+1}(\mathbb{R})$ and let

$$G = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix. Show that f defines an isometry on $(\mathcal{H}^n, g)$ if and only if

$$A^{\mathrm{T}} G A = G \quad \text{and} \quad a_{n+1,n+1} > 0.$$

(c) Now let $n = 2, L > 0$ and $\alpha : [0, L] \to \mathcal{H}^2$ be given by $\alpha(t) = (\sinh t, 0, \cosh t)$. If $\tau_\alpha : T_{\alpha(0)}\mathcal{H}^2 \to T_{\alpha(L)}\mathcal{H}^2$ is the parallel transport map, compute $\tau_\alpha(\partial_1)$ and $\tau_\alpha(\partial_2)$.

Proof. (a) It is clear that h defines a symmetric type-$(0,2)$ tensor field on $\mathbb{R}^{n+1}$, and therefore the restriction g is a symmetric type-$(0,2)$ tensor field on $\mathcal{H}^n$. It remains to show that g is positive definite.

We can define a diffeomorphism $\varphi : \mathbb{R}^n \to \mathcal{H}^n$ is given by

$$\varphi(y_1, ..., y_n) = \left(y_1, ..., y_n, \sqrt{\sum_{j=1}^{n} y_j^2 + 1} \right)$$

Then

$$\begin{aligned}
\varphi^* g &= \sum_{j=1}^{n} \mathrm{d}y_j^2 - (\varphi^* \mathrm{d}x_{n+1})^2 \\
&= \sum_{j=1}^{n} \mathrm{d}y_j^2 - \left(\sum_{j=1}^{n} \frac{y_j}{\sqrt{\sum_{i=1}^{n} y_i^2 + 1}} \mathrm{d}y_j \right)^2 \\
&= \frac{1}{\sum_{i=1}^{n} y_i^2 + 1} \left(\sum_{j=1}^{n} \left(1 + \sum_{i=1}^{n} y_i^2 - y_j^2 \right) \mathrm{d}y_j^2 - 2 \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} y_j y_k \mathrm{d}y_j \mathrm{d}y_k \right)
\end{aligned}$$

We need some linear algebra to show that this is positive definite...

(b) *From on now we adopt Einstein's convention on summation of repeated indices from 1 to $n+1$. But we do not distinguish covariant/contravariant indices by lower/upper indices.*

Note that $f^*\mathrm{d}x_j = a_{jk}\mathrm{d}x_k$. Since $h = G_{ij}\mathrm{d}x^i\mathrm{d}x^j$, we have

$$f^*h = G_{ij}a_{ik}a_{j\ell}\mathrm{d}x_k\mathrm{d}x_\ell = (A^\top GA)_{k\ell}\mathrm{d}x_k\mathrm{d}x_\ell$$

If f is an isometry on $\mathbb{R}^{n+1}$, then $f^*h = h$, which is equivalent to $A^\top GA = G$.

Moreover, $x_{n+1} \circ f(x_{n+1}) = a_{n+1,n+1}$. f maps the half plane $\{x_{n+1} > 0\}$ into itself if and only if $a_{n+1,n+1} > 0$.

We deduce that f defines an isometry on $\mathcal{H}^n$ if and only if $A^\top GA = G$ and $a_{n+1,n+1} > 0$.

(c) We can parameterise $\mathcal{H}^2$ by the "hyperbolic spherical coordinates". Let $\psi : (0, 2\pi) \times \mathbb{R} \to \mathcal{H}^2$ given by

$$\psi(\theta, \phi) := (\cos\theta \sinh\phi, \sin\theta \sinh\phi, \cosh\phi)$$

Then

$$\begin{aligned}\psi^*g &= (-\sin\theta\sinh\phi\mathrm{d}\theta + \cos\theta\cosh\phi\mathrm{d}\phi)^2 + (\cos\theta\sinh\phi\mathrm{d}\theta + \sin\theta\cosh\phi\mathrm{d}\phi)^2 - \sinh^2\phi\mathrm{d}\phi^2 \\ &= \mathrm{d}\phi^2 + \sinh^2\phi\,\mathrm{d}\theta^2\end{aligned}$$

Note that $\alpha(t) = \psi \circ \beta(t)$, where $\beta(t) = (0, t)$. Therefore $\dot\alpha(t) = \psi_*\dot\beta(t) = \partial_\phi$. The Christoffel symbols

$$\begin{aligned}\Gamma^\phi_{\phi\phi} &= \frac{1}{2}g^{\phi a}\left(2\partial_\phi g_{a\phi} - \partial_a g_{\phi\phi}\right) = \frac{1}{2}g^{\phi\phi}\partial_\phi g_{\phi\phi} = 0 \\ \Gamma^\theta_{\phi\phi} &= \frac{1}{2}g^{\theta a}\left(2\partial_\phi g_{a\phi} - \partial_a g_{\phi\phi}\right) = 0 \\ \Gamma^\phi_{\phi\theta} &= \frac{1}{2}g^{\phi a}\left(\partial_\phi g_{a\theta} + \partial_\theta g_{a\phi} - \partial_a g_{\theta\phi}\right) = 0 \\ \Gamma^\theta_{\phi\theta} &= \frac{1}{2}g^{\theta a}\left(\partial_\phi g_{a\theta} + \partial_\theta g_{a\phi} - \partial_a g_{\theta\phi}\right) = \frac{1}{2}g^{\theta\theta}\partial_\phi g_{\theta\theta} = \coth\phi\end{aligned}$$

Suppose that $X(t)$ is a parallel vector field along α. Then $\nabla_\phi X(t) = 0$. In local coordinates we have

$$\frac{\mathrm{d}X^\phi}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}X^\theta}{\mathrm{d}t} + \coth\phi(t)X^\theta = \frac{\mathrm{d}X^\theta}{\mathrm{d}t} + X^\theta\coth t = 0$$

The solution is given by $X(t) = a\partial_\phi + \dfrac{b}{\sinh t}\partial_\theta$. The push-forward of tangent vectors:

$$\begin{aligned}\partial_\phi &= \cosh\phi(\cos\theta\partial_1 + \sin\theta\partial_2) + \sinh\phi\partial_3 \\ \partial_\theta &= \sinh\phi(-\sin\theta\partial_1 + \cos\theta\partial_2)\end{aligned}$$

Hence $X(t) = a(\cosh t\partial_1 + \sinh t\partial_3) + b\partial_2$. Hence

$$\tau_\alpha(\partial_1) = \cosh L\,\partial_1 + \sinh L\,\partial_3, \qquad \tau_\alpha(\partial_2) = \partial_2$$

□

Question 2

Let (M, g) be a connected Riemannian manifold and let $\widetilde{M}$ be the universal cover of M.

(a) Show that there exists a unique Riemannian metric $\widetilde{g}$ on $\widetilde{M}$ such that the covering map $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ is a local isometry.

(b) Show that the fundamental group of M acts on $(\widetilde{M}, \widetilde{g})$ by isometries.

Proof. (a) The covering map $\pi : \widetilde{M} \to M$ is a local diffeomorphism and hence is an immersion. By Proposition

1.3, the pull-back $\widetilde{g} := \pi \circ g$ is a Riemannian metric on $\widetilde{M}$, and therefore $\pi : (\widetilde{M}, \widetilde{g}) \to (M, g)$ is a local isometry. Suppose that h is another Riemannian metric on $\widetilde{M}$ such that $\pi : (\widetilde{M}, h) \to (M, g)$ is a local isometry. Then locally $h = \pi^* g = \widetilde{g}$. This holds for any point on $\widetilde{M}$. Hence $\widetilde{g} = h$ globally, and $\widetilde{g}$ is unique.

(b) For $\alpha \in \pi_1(M)$, α acts on $\widetilde{M}$ by a Deck transformation $f_\alpha : \widetilde{M} \to \widetilde{M}$. That is, $\pi \circ f_\alpha = \pi$. Then

$$\widetilde{g} = \pi^* g = f_\alpha^* \pi^* g = f_\alpha^* \widetilde{g}$$

Hence f_α is an isometry. □

Section B: Core

Question 3

Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds with Levi-Civita connections ∇_1 and ∇_2 respectively. Recall that $T_{(p_1,p_2)}(M_1 \times M_2) \cong T_{p_1} M_1 \times T_{p_2} M_2$ for all $(p_1, p_2) \in M_1 \times M_2$. Define g on $M_1 \times M_2$ by

$$g_{(p_1,p_2)}\left((X_1, X_2), (Y_1, Y_2)\right) = (g_1)_{p_1}(X_1, Y_1) + (g_2)_{p_2}(X_2, Y_2).$$

(a) Show that g is a Riemannian metric on $M_1 \times M_2$.

(b) Show that the Levi-Civita connection ∇ of g on $M_1 \times M_2$ satisfies

$$\nabla_{(X_1,X_2)}(Y_1, Y_2) = \left((\nabla_1)_{X_1} Y_1, (\nabla_2)_{X_2} Y_2\right)$$

for all vector fields X_1, Y_1 on M_1 and X_2, Y_2 on M_2.

Proof. (a) Note that the fibre-wise isomorphism of vector spaces implies the global diffeomorphism of bundles $\mathrm{T}(M_1 \times M_2) \cong \mathrm{T}M_1 \times \mathrm{T}M_2$. So any $X \in \Gamma(\mathrm{T}(M_1 \times M_2))$ must take the form (X_1, X_2) for $X_1 \in \Gamma(\mathrm{T}M_1)$ and $X_2 \in \Gamma(\mathrm{T}M_2)$. Let $\pi_i : M_1 \times M_2 \to M_i$ be the projection map. Then $g : \mathrm{T}(M_1 \times M_2) \times \mathrm{T}(M_1 \times M_2) \to \mathrm{C}^\infty(M_1 \times M_2)$ is given by

$$g((X_1, X_2), (Y_1, Y_2)) = g_1(X_1, Y_1) \circ \pi_1 + g_2(X_2, Y_2) \circ \pi_2$$

you mean for $X_i \in \Gamma(\pi_i^* TM_i)$, right? Otherwise just consider $X := (f(x_1,x_2) X_1, 0)$ for some $X_1 \in \Gamma(TM_1)$.

We need to show that g is a symmetric type-$(0,2)$ tensor field. The symmetry is immediate from definition. To show that it is a tensor field, it suffices to show that g is $\mathrm{C}^\infty(M_1 \times M_2)$-linear in each slot (*by Question 1 of Sheet 1, C7.6 General Relativity II*). It is tautological. Let $(f_1, f_2) \in \mathrm{C}^\infty(M_1 \times M_2)$. Then

$$\begin{aligned} g((f_1, f_2) \cdot (X_1, X_2), (Y_1, Y_2)) &= g_1(f_1 X_1, Y_1) \circ \pi_1 + g_2(f_2 X_2, Y_2) \circ \pi_2 \\ &= f_1 g_1(X_1, Y_1) \circ \pi_1 + f_2 g_2(X_2, Y_2) \circ \pi_2 \\ &= (f_1 \cdot f_2) \cdot (g_1(X_1, Y_1) \circ \pi_1 + g_2(X_2, Y_2) \circ \pi_2) \\ &= (f_1 \cdot f_2) g((X_1, X_2), (Y_1, Y_2)) \end{aligned}$$

So g is $\mathrm{C}^\infty(M_1 \times M_2)$-linear in the first slot. By symmetry it is $\mathrm{C}^\infty(M_1 \times M_2)$-linear in the second slot. Hence $g \in \Gamma(\mathrm{S}^2 \mathrm{T}^*(M_1 \times M_2))$.

Finally we need to show that g is positive definite. This is also tautological:

$$g_{(p_1,p_2)}\left((X_1, X_2), (Y_1, Y_2)\right) = (g_1)_{p_1}(X_1, Y_1) + (g_2)_{p_2}(X_2, Y_2) \geqslant 0 + 0 = 0$$

for $(X_1, X_2), (Y_1, Y_2) \in \Gamma(\mathrm{T}(M_1 \times M_2))$. Hence g is positive. If the equation above is equal to zero, by positivity of g_1 and g_2, we must have

$$(g_1)_{p_1}(X_1, Y_1) = (g_2)_{p_2}(X_2, Y_2) = 0$$

Hence $X_1 = Y_1 = 0$ and $X_2 = Y_2 = 0$. Hence g is definite. We conclude that g is a Riemannian metric on $M_1 \times M_2$.

(b) It suffices to show that $\nabla : \Gamma(\mathrm{T}(M_1 \times M_2)) \times \Gamma(\mathrm{T}(M_1 \times M_2)) \to \Gamma(\mathrm{T}(M_1 \times M_2))$ defined by

$$\nabla_{(X_1,X_2)}(Y_1, Y_2) = ((\nabla_1)_{X_1} Y_1, (\nabla_2)_{X_2} Y_2)$$

satisfies the conditions of a Levi-Civita connection. Then we can invoke the fundamental theorem of Riemannian geometry to deduce that it is the unique Levi-Civita connection on $M_1 \times M_2$.

The verification that ∇ is an affine connection on $M_1 \times M_2$ is entirely tautological and I would just assume this. The torsion tensor T of ∇ is given by

you need to "define it" for vector fields like $X = (X_1, X_2)$ with $X_i \in \pi_i^* TM_i$, that is the v.f. X_i depends on both the coordinates $(x_1, x_2) \in M_1 \times M_2$. Then you show that Leibniz rule is retained. Or to 'extend the definition' via Leibniz rule if you want.

$$\begin{aligned} T((X_1, X_2), (Y_1, Y_2)) &= \nabla_{(X_1,X_2)}(Y_1, Y_2) - \nabla_{(Y_1,Y_2)}(X_1, X_2) - [(X_1, X_2), (Y_1, Y_2)] \\ &= ((\nabla_1)_{X_1} Y_1, (\nabla_2)_{X_2} Y_2) - ((\nabla_1)_{Y_1} X_1, (\nabla_2)_{Y_2} X_2) - ([X_1, Y_1], [X_2, Y_2]) \\ &= (T_1(X_1, Y_1), T_2(X_2, Y_2)) = 0 \end{aligned}$$

Hence ∇ is torsion-free. Finally,

$$\begin{aligned} (Z_1, Z_2)(g((X_1, X_2), (Y_1, Y_2))) &= Z_1(g_1(X_1, Y_1)) \circ \pi_1 + Z_2(g_2(X_2, Y_2)) \circ \pi_2 \\ &= g_1((\nabla_1)_{Z_1} X_1, Y_1) \circ \pi_1 + g_1(X_1, (\nabla_1)_{Z_1} Y_1) \circ \pi_1 \\ &+ g_2((\nabla_2)_{Z_2} X_2, Y_2) \circ \pi_2 + g_2(X_1, (\nabla_2)_{Z_2} Y_2) \circ \pi_2 \\ &= g(((\nabla_1)_{Z_1} X_1, (\nabla_2)_{Z_2} X_2), (Y_1, Y_2)) + g((X_1, X_2), ((\nabla_1)_{Z_1} Y_1, (\nabla_2)_{Z_2} Y_2)) \\ &= g(\nabla_{(Z_1,Z_2)}(X_1, X_2), (Y_1, Y_2)) + g((X_1, X_2), \nabla_{(Z_1,Z_2)}(Y_1, Y_2)) \end{aligned}$$

Hence $\nabla : \Gamma(\mathrm{T}_s^{r+1}(M_1 \times M_2)) \to \Gamma(\mathrm{T}_s^r(M_1 \times M_2))$ satisfies $\nabla g = 0$. It is compatible with the metric g. We conclude that ∇ is the Levi-Civita connection on $M_1 \times M_2$. □

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Question 4

Let (H^2, h) be the upper half-space with the hyperbolic metric

$$h = \frac{\mathrm{d}x_1^2 + \mathrm{d}x_2^2}{x_2^2}.$$

(a) Calculate the Christoffel symbols of h in the coordinates (x_1, x_2) on H^2 using the definition or formula for the Christoffel symbols.

Let $\alpha : [0, L] \to (H^2, h)$ be the curve $\alpha(t) = (t, 1)$ and let τ_α be the parallel transport along α.

(b) Let $X_0 = \partial_2 \in T_{(0,1)} H^2$. Calculate $\tau_\alpha(X_0)$ and show that, viewed as a vector in Euclidean $\mathbb{R}^2$, it makes an angle L with the vertical axis.

Let

$$G = \{u : \mathbb{R} \to \mathbb{R} : u(x_1, x_2)(t) = x_1 + t x_2, x_1 \in \mathbb{R}, x_2 > 0\}$$

and define a manifold structure on G so that $f : G \to H^2$ given by $f(u(x_1, x_2)) = (x_1, x_2)$ is a diffeomorphism. Define a Riemannian metric g on G by $g = f^* h$.

(c) Show that, for all $v \in G$, the map $L_v : G \to G$ given by $L_v(u) = v \circ u$ is an isometry of g.

Proof. *Throughout this question, the Einstein's convention is abopted. And we carefully distinguish between covariant/cotangential and contravariant/tangential indices.*

(a) If $h = h_{ij}\mathrm{d}x^i\mathrm{d}x^j$, then we have $h_{ij}(x) = (x^2)^{-2}\delta_{ij}$. The inverse matrix is $h^{ij}(x) = (x^2)^2\delta^{ij}$. The formula for the Christoffel symbols in terms of the metric (Koszul formula) is given by

$$\begin{aligned}\Gamma^i_{jk} &= \frac{1}{2}h^{i\ell}\left(\partial_j h_{\ell k} + \partial_k h_{\ell j} - \partial_\ell h_{jk}\right)\\ &= \frac{1}{2}(x^2)^2\delta^{i\ell}(\partial_j (x^2)^{-2}\delta_{\ell k} + \partial_k (x^2)^{-2}\delta_{j\ell} - \partial_\ell (x^2)^{-2}\delta_{jk})\end{aligned}$$

Note that $\partial_j((x^2)^{-2}) = -2(x^2)^{-3}\delta_{2j}$. Then

$$\Gamma^i_{jk} = -\frac{1}{x^2}\left(\delta^i_k\delta_{2j} + \delta^i_j\delta_{2k} - \delta_{jk}\delta^i_2\right)$$

The six Christoffel symbols are given by

$$\Gamma^1_{11} = 0, \qquad \Gamma^1_{12} = \Gamma^1_{21} = -\frac{1}{x^2}, \qquad \Gamma^1_{22} = 0, \qquad \Gamma^2_{11} = \frac{1}{x^2}, \qquad \Gamma^2_{12} = \Gamma^2_{21} = 0, \qquad \Gamma^2_{22} = -\frac{1}{x^2}$$

(b) Let $X(t)$ be the parallel vector field along $\alpha(t)$, that is, $\nabla_{\dot\alpha(t)}X(t) = 0$. In local coordinates we have

$$\frac{\mathrm{d}X^i}{\mathrm{d}t} + \Gamma^i_{jk}\dot\alpha^j X^k = 0, \qquad X(0) = \partial_2$$

Note that $\dot\alpha(t) = (1, 0)$. We have the system of first-order ODEs

$$\frac{\mathrm{d}X^1}{\mathrm{d}t} - X^2(t) = 0, \qquad \frac{\mathrm{d}X^2}{\mathrm{d}t} + X^1(t) = 0, \qquad (X^1(0), X^2(0)) = (0, 1)$$

By inspection we can write down the unique solution $(X^1(t), X^2(t)) = (\sin t, \cos t)$. Therefore the parallel transport of X_0 is $\tau_\alpha(X_0) = X(L) = \sin L\partial_1 + \cos L\partial_2$. It makes an angle L with the vertical axis x^2.

(c) Suppose that $v \in G$ is given by $v\colon t \mapsto y^1 + ty^2$. $L_v : G \to G$ induces the map $F_v : H^2 \to H^2$ via the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{L_v} & G \\ f\downarrow & & \downarrow f \\ H^2 & \xrightarrow{F_v} & H^2 \end{array}$$

By construction L_v is an isometry of (G, g) if and only if F_v is an isometry of (H^2, h). For $u\colon t \mapsto x^1+tx^2$, $v\circ u\colon t \mapsto y^1+(x^1+tx^2)y^2 = y^1+x^1y^2+tx^2y^2$. Hence $F_v(x^1, x^2) = (u^1, u^2) := (y^1+x^1y^2, x^2y^2)$. The pull-back of the metric is given by

$$(F_v)^*h = h_{ij}(F_v^{-1}(x))(F_v)^*(\mathrm{d}x^i)(F_v)^*(\mathrm{d}x^j) = h_{ij}(F_v^{-1}(x))(y^2)^{-1}\cdot(y^2)^{-1}\mathrm{d}x^i\mathrm{d}x^j = \frac{(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2}{(x^2)^2} = h$$

Hence F_v is an isometry. ∎

Section C: Optional

Question 5

Let $\mathcal{S}^2$ be the unit sphere in $\mathbb{R}^3$ endowed with the round metric g, let $U = \mathcal{S}^2 \setminus \{(0,0,1)\}$ and let $\varphi : U \to \mathbb{R}^2$ be

$$\varphi(x_1, x_2, x_3) = \frac{(x_1, x_2)}{1 - x_3}$$

so that

$$\varphi^{-1}(y_1, y_2) = \frac{\left(2y_1, 2y_2, y_1^2 + y_2^2 - 1\right)}{y_1^2 + y_2^2 + 1}$$

(a) Show that

$$\left(\varphi^{-1}\right)^* g = \frac{4\left(\mathrm{d}y_1^2 + \mathrm{d}y_2^2\right)}{\left(1 + y_1^2 + y_2^2\right)^2}.$$

Let $\beta : [0, 2\pi] \to \mathbb{R}^2$ be given by $\beta(t) = (\cos t, \sin t)$.

(b) Using the fact that $\varphi^{-1} : \left(\mathcal{S}^2 \setminus \{0,0,1\}, g\right) \to \left(\mathbb{R}^2, \left(\varphi^{-1}\right)^* g\right)$ is an isometry or otherwise, show that the restrictions of the vector fields

$$y_1\partial_1 + y_2\partial_2 \quad \text{and} \quad -y_2\partial_1 + y_1\partial_2$$

to β are parallel along β with respect to the metric $\left(\varphi^{-1}\right)^* g$.

Proof. □