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Problem Sheet 4

B4.3: Distribution Theory

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Question 1

This question provides a condition ensuring that the usual partial derivatives coincide with the distributional partial derivatives.

Prove Lemma 5.21 from the lecture notes: If the dimension $n \geqslant 2$ and $f \in \mathrm{C}^1(\mathbb{R}^n \setminus \{0\}) \cap \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ has usual partial derivatives $\partial_j f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ for each direction $1 \leqslant j \leqslant n$, then also

$$\int_{\mathbb{R}^n} \partial_j f \varphi \mathrm{d}x = -\int_{\mathbb{R}^n} f \partial_j \varphi \mathrm{d}x$$

holds for all $\varphi \in \mathscr{D}(\mathbb{R}^n)$. Give an example to show that it can fail for dimension $n = 1$. Show that for dimension $n = 1$ we instead have the following: If $f \in \mathrm{C}^1(\mathbb{R} \setminus \{0\}) \cap \mathrm{C}(\mathbb{R})$ and the usual derivative $f' \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$, then

$$\int_{\mathbb{R}} f' \varphi \mathrm{d}x = -\int_{\mathbb{R}} f \varphi' \mathrm{d}x$$

holds for all $\varphi \in \mathscr{D}(\mathbb{R})$.

Proof. Let $\varphi \in \mathscr{D}(\mathbb{R})$. Without loss of generality we fix $j = n$. For fixed $x_1, \dots, x_{n-1}$, we have the one-dimensional integration by parts as follows:

$$\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right) \frac{\partial f}{\partial x_n} \varphi \,\mathrm{d}x_n = -\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right) f \frac{\partial \varphi}{\partial x_n} \,\mathrm{d}x_n - f\varphi\Big|_{x_n=-\varepsilon}^{x_n=\varepsilon}$$

For $(x_1, \dots, x_{n-1}) \neq (0, \dots, 0)$, the last term on the RHS vanishes as $\varepsilon \to 0$ by the continuity of f. Therefore

$$\int_{\mathbb{R}^{n-1}} \lim_{\varepsilon \searrow 0} f\varphi\Big|_{x_n=-\varepsilon}^{x_n=\varepsilon} \mathrm{d}x^{n-1} = 0$$

since the integrand is zero almost everywhere.

Since φ is compactly supported and $\partial_n f$ is locally integrable, we have $\partial_n f \varphi \in \mathrm{L}^1(\mathbb{R}^n)$. By Fubini's theorem,

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{\partial f}{\partial x_n} \varphi \,\mathrm{d}x^n &= \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial f}{\partial x_n} \varphi \,\mathrm{d}x_n\right) \mathrm{d}x^{n-1} \\
&= \int_{\mathbb{R}^{n-1}} \lim_{\varepsilon \searrow 0} \left(\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right) \frac{\partial f}{\partial x_n} \varphi \,\mathrm{d}x_n\right) \mathrm{d}x^{n-1} \\
&= -\int_{\mathbb{R}^{n-1}} \lim_{\varepsilon \searrow 0} \left(\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right) \frac{\partial \varphi}{\partial x_n} f \,\mathrm{d}x_n\right) \mathrm{d}x^{n-1} - \int_{\mathbb{R}^{n-1}} \lim_{\varepsilon \searrow 0} f\varphi\Big|_{x_n=-\varepsilon}^{x_n=\varepsilon} \mathrm{d}x^{n-1} \\
&= -\int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} \frac{\partial \varphi}{\partial x_n} f \,\mathrm{d}x_n\right) \mathrm{d}x^{n-1} \\
&= -\int_{\mathbb{R}^n} \frac{\partial \varphi}{\partial x_n} f \,\mathrm{d}x^n
\end{aligned}$$

Nice.

For a counter-example in $\mathbb{R}$, consider the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1, & x < 0 \\ x, & x > 0 \end{cases}$$

Then the usual derivative is given by

$$f'(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

Consider a sequence of test functions $(\varphi_m) \subseteq \mathscr{D}(\mathbb{R})$ such that

$$\operatorname{supp} \varphi_m \subseteq [-1, 1], \qquad 0 \leqslant \varphi_m(x) \leqslant 1, \qquad \varphi_m(0) = 1, \qquad \lim_{m \to \infty} \varphi_m(x) = 0 \text{ for } x \neq 0$$

Then we have

$$\int_{\mathbb{R}} f \varphi_m' \,\mathrm{d}x + \int_{\mathbb{R}} f' \varphi_m \,\mathrm{d}x = \int_{-1}^{0} \varphi_m' \,\mathrm{d}x + \int_{0}^{1} x \varphi_m' \,\mathrm{d}x + \int_{0}^{1} \varphi_m \,\mathrm{d}x$$

$$= \int_0^1 \varphi_m \,\mathrm{d}x + (\varphi_m(0) - \varphi_m(-1)) + x\varphi_m(x)\big|_0^1 - (\varphi_m(1) - \varphi_m(0))$$
$$= \int_0^1 \varphi_m \,\mathrm{d}x - 2$$

By dominated convergence theorem,

$$\lim_{m\to\infty}\left(\int_{\mathbb{R}} f\varphi_m' \,\mathrm{d}x + \int_{\mathbb{R}} f'\varphi_m \,\mathrm{d}x\right) = \int_0^1 \lim_{m\to\infty} \varphi_m \,\mathrm{d}x - 2 = -2$$

contradicting that $\int_{\mathbb{R}} f\varphi_m' \,\mathrm{d}x + \int_{\mathbb{R}} f'\varphi_m \,\mathrm{d}x = 0$ for all $m \in \mathbb{N}$.

Nice example ... an easier choice would just be the Heaviside fn. Then you can just show $\int f'\varphi + f\varphi' \neq 0$ $\forall \varphi \in \mathscr{D}(\mathbb{R})$.

If $f \in \mathrm{C}^1(\mathbb{R} \cap \{0\}) \cap \mathrm{C}^0(\mathbb{R})$, then $f\varphi \in \mathrm{C}^0(\mathbb{R})$ for any $\varphi \in \mathscr{D}(\mathbb{R})$. By fundamental theorem of calculus,

$$\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right) f'\varphi \,\mathrm{d}x = -\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right) f\varphi' \,\mathrm{d}x + f(-\varepsilon)\varphi(-\varepsilon) - f(\varepsilon)\varphi(\varepsilon)$$

By continuity of $f\varphi$ at $x = 0$, as $\varepsilon \searrow 0$, we have

$$\int_{\mathbb{R}} f'\varphi \,\mathrm{d}x = \lim_{\varepsilon\searrow 0}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right) f'\varphi \,\mathrm{d}x = -\lim_{\varepsilon\searrow 0}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right) f\varphi' \,\mathrm{d}x = \int_{\mathbb{R}} f\varphi' \,\mathrm{d}x$$

□

Question 2. Boundary values in the sense of distributions for holomorphic functions.

(a) Prove that for each $n \in \mathbb{N}$,

$$(x + \mathrm{i}\varepsilon)^{-n} \to (x + \mathrm{i}0)^{-n} \quad \text{in } \mathscr{D}'(\mathbb{R}) \text{ as } \varepsilon \searrow 0$$

where the distribution $(x + \mathrm{i}0)^{-n}$ was defined in Problem 2 on Sheet 3.

A holomorphic function $f : H \to \mathbb{C}$ on the upper half-plane $H = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ is said to be of slow growth if for each $R > 0$ there exist $m = m_R \in \mathbb{N}_0$ and $c = c_R \geqslant 0$ so

$$|f(z)| \leqslant \frac{c}{\mathrm{Im}(z)^m}$$

holds for all $z \in H$ with $|\mathrm{Re}(z)| \leqslant R$ and $\mathrm{Im}(z) < 2$.

This was a really difficult question... will go through in class!

(b) Prove that if $f : H \to \mathbb{C}$ is holomorphic of slow growth, then it has a boundary value in the sense of distributions:

$$\langle f(x + \mathrm{i}0), \varphi \rangle := \lim_{\varepsilon\searrow 0} \int_{\mathbb{R}} f(x + \mathrm{i}\varepsilon)\varphi(x)\mathrm{d}x$$

exists for all $\varphi \in \mathscr{D}(\mathbb{R})$ and defines a distribution.

[*Hint: Assume first that $m = 0$ above and let $F : H \to \mathbb{C}$ be the holomorphic primitive with $F(\mathrm{i}) = 0$. Explain why F has a continuous extension to the closed upper half-plane $\overline{H}$ and use this to conclude the proof in this special case. Then use induction on m.*]

Proof. (a) We have shown in Problem 2.(b) on Sheet 3 that for any $\varphi \in \mathscr{D}(\mathbb{R})$,

$$\langle (x + \mathrm{i}0)^{-n}, \varphi \rangle = \lim_{\varepsilon\searrow 0} \int_{\mathbb{R}} \frac{\varphi(x)}{(x + \mathrm{i}\varepsilon)^n} \,\mathrm{d}x = \lim_{\varepsilon\searrow 0} \langle (x + \mathrm{i}\varepsilon)^{-n}, \varphi \rangle$$

Hence $(x + \mathrm{i}\varepsilon)^{-n} \to (x + \mathrm{i}0)^{-n}$ in $\mathscr{D}'(\mathbb{R})$ as $\varepsilon \searrow 0$.

will go through!

(b) (*I cannot see why F has a continuous extension on the real line. Perhaps it has some relation with the boundary correspondence theorem...*)

□

Unfortunately there were lots of restrictions on φ for this... Consider $(x+i0)^{-n}$ as a derivative of $\mathrm{Log}(x+i0)$ which is limit of $\mathrm{Log}(x+i\varepsilon)$ which we differentiate n-times to get something like $(x+i\varepsilon)^{-n}$ etc...

Question 3. Distributions defined by finite parts.

Recall from Sheet 2 that the distributional derivative of $\log|x|$ is the distribution $\operatorname{pv}\left(\frac{1}{x}\right)$ defined by the principal value integral

$$\left\langle \operatorname{pv}\left(\frac{1}{x}\right), \varphi \right\rangle := \lim_{\varepsilon \searrow 0} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) \frac{\varphi(x)}{x} \, \mathrm{d}x, \quad \varphi \in \mathscr{D}(\mathbb{R})$$

In order to represent the higher order derivatives one can use finite parts: Let $n \in \mathbb{N}$ with $n > 1$. We then define $\operatorname{fp}\left(\frac{1}{x^n}\right)$ for each $\varphi \in \mathscr{D}(\mathbb{R})$ by the finite part integral

$$\left\langle \operatorname{fp}\left(\frac{1}{x^n}\right), \varphi \right\rangle := \int_{-\infty}^{\infty} \frac{\varphi(x) - \sum_{j=0}^{n-2} \frac{\varphi^{(j)}(0)}{j!} x^j - \frac{\varphi^{(n-1)}(0)}{(n-1)!} x^{n-1} \mathbf{1}_{(-1,1)}(x)}{x^n} \, \mathrm{d}x$$

(a) Check that hereby $\operatorname{fp}\left(\frac{1}{x^n}\right)$ is a well-defined distribution on $\mathbb{R}$. Show that

$$\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{pv}\left(\frac{1}{x}\right) = -\operatorname{fp}\left(\frac{1}{x^2}\right) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x} \operatorname{fp}\left(\frac{1}{x^n}\right) = -n \operatorname{fp}\left(\frac{1}{x^{n+1}}\right)$$

for all $n > 1$. Is $\operatorname{fp}\left(\frac{1}{x^n}\right)$ homogeneous? (See Problem 4 on Sheet 2 for the definition of homogeneity.)

(b) Show that for $n > 1$ we have $x^n \operatorname{fp}\left(\frac{1}{x^n}\right) = 1$ and find the general solution to the equation $x^n u = 1$ in $\mathscr{D}'(\mathbb{R})$. What is the general solution to the equation $(x-a)^n v = 1$ in $\mathscr{D}'(\mathbb{R})$ when $a \in \mathbb{R} \setminus \{0\}$?

(c) Let $p(x) \in \mathbb{C}[x] \setminus \{0\}$ be a nontrivial polynomial. Describe the general solution $w \in \mathscr{D}'(\mathbb{R})$ to the equation

$$p(x) w = 1 \text{ in } \mathscr{D}'(\mathbb{R})$$

Proof. (a) First we check that $\operatorname{fp}\left(\frac{1}{x^n}\right)$ is a well-defined distribution.

$\psi(0) = \varphi(0) \neq 0$?

- $\left\langle \operatorname{fp}\left(\frac{1}{x^n}\right), \varphi \right\rangle$ is finite for any $\varphi \in \mathscr{D}(\mathbb{R})$.

  Define

  $$\psi(x) := \varphi(x) - \sum_{j=0}^{n-2} \frac{\varphi^{(j)}(0)}{j!} x^j - \frac{\varphi^{(n-1)}(0)}{(n-1)!} x^{n-1} \mathbf{1}_{(-1,1)}(x)$$

  Observe that $\psi^{(k)}(0) = 0$ for $0 \leqslant k \leqslant n-1$ and $\psi^{(n)}(0) = \varphi^{(n)}(0)$. Therefore by l'Hôptial's rule,

  $$\lim_{x \to 0} \frac{\psi(x)}{x^n} = \lim_{x \to 0} \frac{\psi^{(n)}(x)}{n!} = \frac{\varphi^{(n)}(0)}{n!}$$

  ✓ This is true but you can just use Taylor's theorem & things are much easier!

  Hence $\psi(x)/x^n$ is continuous on $\mathbb{R}$. Suppose that $\operatorname{supp} \varphi \subseteq [-a, a]$. Then for $|x| > a$, we have

  $$\frac{\psi(x)}{x^n} = \sum_{j=0}^{n-2} \frac{\varphi^{(j)}(0)}{j!} \frac{1}{x^{n-j}}$$

  and we know that

  $$\int_a^{\infty} \frac{1}{x^{n-j}} \, \mathrm{d}x = \frac{1}{n-j-1} \frac{1}{a^{n-j-1}}$$

  is finite for $j \leqslant n-2$. Therefore

  $$\left\langle \operatorname{fp}\left(\frac{1}{x^n}\right), \varphi \right\rangle = \int_{-a}^{a} \frac{\psi(x)}{x^n} \, \mathrm{d}x + \sum_{j=0}^{n-2} \frac{\varphi^{(j)}(0)}{j!} \left(\int_{-\infty}^{-a} + \int_{a}^{+\infty} \right) \frac{1}{x^{n-j}} \, \mathrm{d}x$$

  is finite. ✓

- It is clear that $\operatorname{fp}\left(\frac{1}{x^n}\right)$ is a linear functional because $\psi(x)$ is linear in all derivatives of $\varphi(x)$.

- Continuity of $\operatorname{fp}\left(\frac{1}{x^n}\right)$.

  Suppose that $\{\varphi_m\} \subseteq \mathscr{D}(\mathbb{R})$ and $\varphi \in \mathscr{D}(\mathbb{R})$ such that $\varphi_m \to \varphi$ in $\mathscr{D}(\mathbb{R})$. Let $a > 0$ such that $\operatorname{supp} \varphi_m, \operatorname{supp} \varphi \subseteq [-a, a]$.

Similarly we can define

$$\psi_m(x) := \varphi_m(x) - \sum_{j=0}^{n-2} \frac{\varphi_m^{(j)}(0)}{j!} x^j - \frac{\varphi_m^{(n-1)}(0)}{(n-1)!} x^{n-1} \mathbf{1}_{(-1,1)}(x)$$

Then $\psi_m(x) \to \psi(x)$ as $m \to \infty$ pointwise (in fact it is also locally uniform). Since $\psi(x)/x^n$ is bounded on $\mathbb{R}$, by dominated convergence theorem,

$$\lim_{m\to\infty} \left\langle \mathrm{fp}\left(\frac{1}{x^n}\right), \varphi_m \right\rangle = \lim_{m\to\infty} \int_{\mathbb{R}} \frac{\psi_m(x)}{x^n}\,\mathrm{d}x = \int_{\mathbb{R}} \frac{\psi(x)}{x^n}\,\mathrm{d}x = \left\langle \mathrm{fp}\left(\frac{1}{x^n}\right), \varphi \right\rangle$$

Hence $\mathrm{fp}\left(\frac{1}{x^n}\right)$ is continuous in $\mathscr{D}'(\mathbb{R})$.

Next, we check that $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{pv}\left(\frac{1}{x}\right) = -\mathrm{fp}\left(\frac{1}{x^2}\right)$ by brute force.

$$\begin{aligned}
-\left\langle \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{pv}\left(\frac{1}{x}\right), \varphi \right\rangle = \left\langle \mathrm{pv}\left(\frac{1}{x}\right), \varphi' \right\rangle &= \lim_{\varepsilon\searrow 0}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right)\frac{\varphi'(x)}{x}\,\mathrm{d}x \\
&= \lim_{\varepsilon\searrow 0}\left(\frac{\varphi(x)}{x}\bigg|_{-\infty}^{-\varepsilon} + \int_{-\infty}^{-\varepsilon}\frac{\varphi(x)}{x^2}\,\mathrm{d}x + \frac{\varphi(x)}{x}\bigg|_{\varepsilon}^{\infty} + \int_{\varepsilon}^{\infty}\frac{\varphi(x)}{x^2}\,\mathrm{d}x\right) \\
&= \lim_{\varepsilon\searrow 0}\left(-\frac{\varphi(\varepsilon)+\varphi(-\varepsilon)}{\varepsilon} + \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right)\frac{\varphi(x)}{x^2}\,\mathrm{d}x\right)
\end{aligned}$$

Note that by l'Hôptial's rule,

$$\lim_{\varepsilon\searrow 0}\frac{\varphi(\varepsilon)+\varphi(-\varepsilon)-2\varphi(0)}{\varepsilon} = \lim_{\varepsilon\searrow 0}\big(\varphi'(0)-\varphi'(0)\big) = 0$$

Therefore

$$-\left\langle \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{pv}\left(\frac{1}{x}\right), \varphi \right\rangle = \lim_{\varepsilon\searrow 0}\left(-\frac{2\varphi(0)}{\varepsilon} + \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right)\frac{\varphi(x)}{x^2}\,\mathrm{d}x\right)$$

On the other hand,

$$\begin{aligned}
\left\langle \mathrm{fp}\left(\frac{1}{x^2}\right), \varphi \right\rangle &= \int_{\mathbb{R}} \frac{\varphi(x)-\varphi(0)-\varphi'(0)x\mathbf{1}_{(-1,1)}(x)}{x^2}\,\mathrm{d}x \\
&= \lim_{\varepsilon\searrow 0}\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right)\frac{\varphi(x)-\varphi(0)-\varphi'(0)x\mathbf{1}_{(-1,1)}(x)}{x^2}\,\mathrm{d}x \\
&= \lim_{\varepsilon\searrow 0}\left(\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right)\frac{\varphi(x)}{x^2}\,\mathrm{d}x - \varphi(0)\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right)\frac{1}{x^2}\,\mathrm{d}x - \varphi'(0)\left(\int_{-1}^{-\varepsilon} + \int_{\varepsilon}^{1}\right)\frac{1}{x}\,\mathrm{d}x\right) \\
&= \lim_{\varepsilon\searrow 0}\left(\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right)\frac{\varphi(x)}{x^2}\,\mathrm{d}x - \frac{2\varphi(0)}{\varepsilon}\right)
\end{aligned}$$

which proves the claim.

Next, we check that $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{fp}\left(\frac{1}{x^n}\right) = -n\,\mathrm{fp}\left(\frac{1}{x^{n+1}}\right)$. Simply note that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\varphi(x) - \sum_{j=0}^{n-1}\frac{\varphi^{(j)}(0)}{j!}x^j - \frac{\varphi^{(n)}(0)}{(n)!}x^n\mathbf{1}_{(-1,1)}(x)\right) = \varphi'(x) - \sum_{j=0}^{n-2}\frac{\varphi^{(j+1)}(0)}{j!}x^j - \frac{\varphi^{(n)}(0)}{(n-1)!}x^{n-1}\mathbf{1}_{(-1,1)}(x)$$

Therefore by integration by parts,

$$\begin{aligned}
\left\langle \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{fp}\left(\frac{1}{x^n}\right), \varphi \right\rangle = -\left\langle \mathrm{fp}\left(\frac{1}{x^n}\right), \varphi' \right\rangle &= -\int_{\mathbb{R}}\frac{1}{x^n}\left(\varphi'(x) - \sum_{j=0}^{n-2}\frac{\varphi^{(j+1)}(0)}{j!}x^j - \frac{\varphi^{(n)}(0)}{(n-1)!}x^{n-1}\mathbf{1}_{(-1,1)}(x)\right)\mathrm{d}x \\
&= -\frac{1}{x^n}\left(\varphi(x) - \sum_{j=0}^{n-1}\frac{\varphi^{(j)}(0)}{j!}x^j - \frac{\varphi^{(n)}(0)}{(n)!}x^n\mathbf{1}_{(-1,1)}(x)\right)\Bigg|_{-\infty}^{\infty} \\
&\quad - n\int_{\mathbb{R}}\frac{1}{x^{n+1}}\left(\varphi(x) - \sum_{j=0}^{n-1}\frac{\varphi^{(j)}(0)}{j!}x^j - \frac{\varphi^{(n)}(0)}{(n)!}x^n\mathbf{1}_{(-1,1)}(x)\right)\mathrm{d}x \\
&= -n\,\mathrm{fp}\left(\frac{1}{x^{n+1}}\right)
\end{aligned}$$

Let $\widetilde{\varphi}(x) = \varphi(x/r)$. Then $\widetilde{\varphi}^{(j)}(0) = r^{-j}\varphi^{(j)}(0)$. The dilated function

$$\widetilde{\psi}(x) := \widetilde{\varphi}(x) - \sum_{j=0}^{n-2} \frac{\widetilde{\varphi}^{(j)}(0)}{j!} x^j - \frac{\widetilde{\varphi}^{(n-1)}(0)}{(n-1)!} x^{n-1} \mathbf{1}_{(-1,1)}(x) = \varphi\left(\frac{x}{r}\right) - \sum_{j=0}^{n-2} \frac{\varphi^{(j)}(0)}{j!} \left(\frac{x}{r}\right)^j - \frac{\varphi^{(n-1)}(0)}{(n-1)!} \left(\frac{x}{r}\right)^{n-1} \mathbf{1}_{(-1,1)}(x) = \psi\left(\frac{x}{r}\right)$$

Hence

$$\left\langle d_r \operatorname{fp}\left(\frac{1}{x^n}\right), \varphi \right\rangle = r^{-1} \left\langle \operatorname{fp}\left(\frac{1}{x^n}\right), d_{1/r}\varphi \right\rangle = r^{-1} \int_{\mathbb{R}} \frac{\widetilde{\psi}(x)}{x^n} \, \mathrm{d}x = r^{-n-1} \int_{\mathbb{R}} \frac{\psi(x/r)}{(x/r)^n} \, \mathrm{d}x = r^{-n} \int_{\mathbb{R}} \frac{\psi(x)}{x^n} \, \mathrm{d}x = \left\langle r^{-n} \operatorname{fp}\left(\frac{1}{x^n}\right), \varphi \right\rangle$$

Hence $\operatorname{fp}\left(\frac{1}{x^n}\right)$ is homogeneous of degree $-n$. Much easier to just use the results of sheet 2 for this!

(b) Note that for $j < n$,

$$\frac{\mathrm{d}}{\mathrm{d}x^j}(x^n \varphi(x))\bigg|_{x=0} = \sum_{i=0}^{j} \binom{j}{i} \frac{n!}{i!} x^{n-i} \varphi^{(j-i)}(0)\bigg|_{x=0} = 0$$

Therefore

$$\left\langle x^n \operatorname{fp}\left(\frac{1}{x^n}\right), \varphi \right\rangle = \left\langle \operatorname{fp}\left(\frac{1}{x^n}\right), x^n \varphi \right\rangle = \int_{\mathbb{R}} \frac{x^n \varphi(x)}{x^n} \, \mathrm{d}x = \int_R \varphi(x) \, \mathrm{d}x = \langle 1, \varphi \rangle$$

which implies that $x^n \operatorname{fp}\left(\frac{1}{x^n}\right) = 1$.

To find the solutions of $x^n u = 1$, we consider the equation $x^n u = 0$ instead. The solution u must satisfy $\operatorname{supp} u = \{0\}$. By Theorem 5.32, u is a linear combination of the derivatives of the delta function δ_0. Note that

$$\left\langle x^n \delta_0^{(k)}, \varphi \right\rangle = \left\langle \delta_0, (-1)^k \frac{\mathrm{d}}{\mathrm{d}x^k}(x^n \varphi(x)) \right\rangle = (-1)^k \frac{\mathrm{d}}{\mathrm{d}x^k}(x^n \varphi(x))\bigg|_{x=0} = 0$$

for $k < n$. For $k \geqslant n$, RHS contains a term proportional to $\varphi(0)$ and hence does not vanish identically for any $\varphi \in \mathscr{D}(\mathbb{R})$. In summary, the general solution to $x^n u = 0$ is given by

$$u = \sum_{k=0}^{n-1} c_k \delta_0^{(k)}$$

Hence the general solution to $x^n u = 1$ is given by

$$u = \operatorname{fp}\left(\frac{1}{x^n}\right) + \sum_{k=0}^{n-1} c_k \delta_0^{(k)}$$

for some $c_0, \ldots, c_{n-1} \in \mathbb{R}$.

For the equation $(x-a)^n v = 1$, we first note that

$$(x-a)^n \tau_{-a} \operatorname{fp}\left(\frac{1}{x^n}\right) = 1$$

where τ_{-a} is the translation operation by $-a$. It is because for $\varphi \in \mathscr{D}(\mathbb{R})$,

$$\left\langle (x-a)^n \tau_{-a} \operatorname{fp}\left(\frac{1}{x^n}\right), \varphi \right\rangle = \left\langle \operatorname{fp}\left(\frac{1}{x^n}\right), x^n \varphi(x+a) \right\rangle = \int_{\mathbb{R}} \varphi(x+a) \, \mathrm{d}x = \int_{\mathbb{R}} \varphi(x) \, \mathrm{d}x = \langle 1, \varphi \rangle$$

Following the similar argument, we deduce that the general solution to $(x-a)^n v = 1$ is given by

$$v = \tau_{-a} \operatorname{fp}\left(\frac{1}{x^n}\right) + \sum_{k=0}^{n-1} c_k \delta_a^{(k)}$$

for some $c_0, \ldots, c_{n-1} \in \mathbb{R}$. □

Question 4

A function $f:\mathbb{R}\to\mathbb{R}$ is *convex* if for all $x_0, x_1 \in \mathbb{R}$ and $\lambda \in (0,1)$ we have

$$f(\lambda x_1 + (1-\lambda)x_0) \leqslant \lambda f(x_1) + (1-\lambda) f(x_0)$$

A function $a:\mathbb{R}\to\mathbb{R}$ satisfying (1) with equality everywhere is called an *affine function.*

(a) Show that an affine function must have the form $a(x) = a_1 x + a_0$ for some constants a_0, $a_1 \in \mathbb{R}$. Show also that a function $f:\mathbb{R}\to\mathbb{R}$ is convex if and only if it for each compact interval $[\alpha,\beta]\subseteq\mathbb{R}$ has the property:

$$\text{when } a \text{ is affine and } f(x)\leqslant a(x) \text{ for } x\in\{\alpha,\beta\}, \text{ then } f\leqslant a \text{ on } [\alpha,\beta]$$

(b) Show that a convex function $f:\mathbb{R}\to\mathbb{R}$ satisfies the 3 slope inequality:

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leqslant \frac{f(x_3)-f(x_1)}{x_3-x_1} \leqslant \frac{f(x_3)-f(x_2)}{x_3-x_2}$$

holds for all triples $x_1 < x_2 < x_3$. Deduce that a convex function must be continuous and that it is differentiable except for in at most countably many points.

Show that a convex function must be locally Lipschitz continuous: for each $r > 0$ there exists $L = L_r \geqslant 0$ so $|f(x)-f(y)| \leqslant L|x-y|$ holds for all $x, y \in [-r, r]$.

(c) Assume that $f:\mathbb{R}\to\mathbb{R}$ is twice differentiable. Show that f is convex if and only if

$$f''(x) \geqslant 0$$

holds for all $x\in\mathbb{R}$.

(d) Let $u \in \mathscr{D}'(\mathbb{R})$ and assume that $u'' \geqslant 0$ in $\mathscr{D}'(\mathbb{R})$. Show that u is represented by a convex function.

Proof. (a) ~ (c) are standard Prelim Analysis questions.

(a) Suppose that $f:[-x_0,x_0]\to\mathbb{R}$ satisfies

$$\forall\, x_1, x_2 \in [-x_0,x_0]\ \forall\,\lambda\in[0,1] \quad f(\lambda x_1 + (1-\lambda)x_2) = \lambda f(x_1) + (1-\lambda) f(x_2)$$

Then for $x\in[-x_0,x_0]$, let $x_2 = -x_0$, $x_1 = x_0$, and $\lambda = \frac{1}{2} + \frac{x}{2x_0}$. We have

$$f(x) = \left(\frac{1}{2}+\frac{x}{2x_0}\right) f(x_0) + \left(\frac{1}{2}-\frac{x}{2x_0}\right) f(-x_0) = \frac{f(x_0)-f(-x_0)}{2x_0}x + \frac{1}{2}(f(x_0)+f(-x_0)) = a_1 x + a_0$$

If $a:\mathbb{R}\to\mathbb{R}$ is affine, then it is affine restricted to $[-x_0,x_0]$ for any $x_0 > 0$. We deduce that $a(x) = a_1 x + a_0$ for some $a_0, a_1 \in \mathbb{R}$. on $[x_0, x_0]$. Now explain why a_1, a_0 are unchanged when going from $[-x_0, x_0]$ to $\mathbb{R}$.

" $\Longrightarrow$ " Suppose that $f:\mathbb{R}\to\mathbb{R}$ is convex. Let $a:\mathbb{R}\to\mathbb{R}$ be an affine function such that $f(\alpha)\leqslant a(\alpha)$ and $f(\beta)\leqslant a(\beta)$. For $x\in[\alpha,\beta]$, let $x_1=\alpha$, $x_0=\beta$ and $\lambda = \dfrac{x-\alpha}{\beta-\alpha}$. Then

$$f(x) \leqslant \frac{x-\alpha}{\beta-\alpha} f(\alpha) + \frac{\beta-x}{\beta-\alpha} f(\beta) \leqslant \frac{x-\alpha}{\beta-\alpha} a(\alpha) + \frac{\beta-x}{\beta-\alpha} a(\beta) = a(x)$$

Hence $f\leqslant a$ on $[\alpha,\beta]$.

" $\Longleftarrow$ " Suppose that the converse holds. Then for any $[x_1,x_0]\subseteq\mathbb{R}$, let $a:\mathbb{R}\to\mathbb{R}$ be an affine function such that $a(x_1) = f(x_1)$ and $a(x_0) = f(x_0)$. Explicitly,

$$a(x) = \frac{f(x_0)-f(x_1)}{x_0-x_1}(x-x_1) + f(x_1)$$

Since $f\leqslant a$ on $[x_1,x_0]$, for any $\lambda\in[0,1]$ we have

$$f(\lambda x_1 + (1-\lambda)x_0) \leqslant a(\lambda x_1 + (1-\lambda)x_0) = \lambda a(x_1) + (1-\lambda) a(x_0) = \lambda f(x_1) + (1-\lambda) f(x_0)$$

Hence f is convex.

(b) Consider the affine function $a : \mathbb{R} \to \mathbb{R}$ with $a(x_1) = f(x_1)$ and $a(x_3) = f(x_3)$. Explicitly,

$$a(x) = \frac{f(x_3) - f(x_1)}{x_3 - x_1}(x - x_1) + f(x_1)$$

By part (a), $f \leqslant a$ on $[x_1, x_3]$. In particular,

$$f(x_2) \leqslant a(x_2) = \frac{f(x_3) - f(x_1)}{x_3 - x_1}(x_2 - x_1) + f(x_1) \implies \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leqslant \frac{f(x_3) - f(x_1)}{x_3 - x_1}$$

In addition, we note that

$$a(x_2) = \frac{f(x_3) - f(x_1)}{x_3 - x_1}(x_2 - x_1) + f(x_1) = \frac{f(x_3) - f(x_1)}{x_3 - x_1}(x_2 - x_3) + f(x_3)$$

So $f(x_2) \leqslant a(x_2)$ also implies that

$$\frac{f(x_3) - f(x_1)}{x_3 - x_1} \leqslant \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

In summary, for $x_1 < x_2 < x_3$,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leqslant \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leqslant \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

Let f be a convex function. We fix $x_0 \in \mathbb{R}$. Consider the function $g : \mathbb{R} \setminus \{x_0\} \to \mathbb{R}$ defined by

$$g(x) := \frac{g(x) - g(x_0)}{x - x_0}$$

Then by 3 slope inequality g is non-decreasing on $\mathbb{R} \setminus \{x_0\}$. In particular, the one-sided derivatives $\lim_{x \nearrow x_0} g(x)$ and $\lim_{x \searrow x_0} g(x)$ exists. Hence

$$\lim_{x \nearrow x_0} (f(x) - f(x_0)) = \lim_{x \searrow x_0} (f(x) - f(x_0)) = 0$$

which implies that $\lim_{x \to x_0} f(x) = f(x_0)$. Hence f is continuous at $x_0 \in \mathbb{R}$.

As the one-sided derivatives of f exists everywhere, f' can only have jump discontinuities. It follows that f is differentiable except for in at most countably many points.

For $r > 0$, let

$$L_r := \sup_{x \in [-r, r]} \frac{|f(r) - f(x)|}{r - x}$$

We know that $L_r < \infty$ because by 3-slope inequality, for all $x < r$,

$$\left| \frac{f(r) - f(x)}{r - x} \right| \leqslant \left| \frac{f(2r) - f(r)}{2r - r} \right|$$

For x, y with $-r \leqslant y < x \leqslant r$, by 3 slope inequality,

$$\frac{|f(x) - f(y)|}{|x - y|} \leqslant \frac{|f(x) - f(r)|}{|x - r|} \leqslant L_r$$

Hence $|f(x) - f(y)| \leqslant L_r |x - y|$ for all $x, y \in [-r, r]$. f is locally Lipschitz continuous.

(c) " $\implies$ " Suppose that f is a twice differentiable convex function. Fix $a, b \in \mathbb{R}$ with $a < b$. For $x \in (a, b)$, by 3 slope inequality,

$$\frac{f(x) - f(a)}{x - a} \leqslant \frac{f(x) - f(b)}{x - b}$$

Therefore

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \inf_{x \in (a,b)} \frac{f(x) - f(a)}{x - a} \leqslant \sup_{x \in (a,b)} \frac{f(x) - f(b)}{x - b} = \lim_{x \to b} \frac{f(x) - f(b)}{x - b} = f'(b)$$

Hence f' is non-decreasing. We deduce that $f''(x) \geqslant 0$ for all $x \in \mathbb{R}$.

" $\impliedby$ " Suppose that $f'' \geqslant 0$. Then f' is non-decreasing on $\mathbb{R}$. We fix $x_0, x_2 \in \mathbb{R}$ with $x_0 < x_2$. Let $\lambda \in (0, 1)$, and $x_1 := \lambda x_0 + (1 - \lambda) x_2$.

Since $x_0 < x_1 < x_2$, by Lagrange's mean value theorem, there exists $\xi \in (x_0, x_1)$ and $\zeta \in (x_1, x_2)$ such that

$$f'(\xi) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \qquad f'(\zeta) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Then

$$\xi < \zeta \implies f'(\xi) \leqslant f'(\zeta) \implies \frac{f(x_1) - f(x_0)}{x_1 - x_0} \leqslant \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Rearranging the inequality:

$$(x_2 - x_0) f(x_1) \leqslant (x_2 - x_1) f(x_2) + (x_1 - x_0) f(x_0)$$

Hence

$$f(\lambda x_0 + (1-\lambda)x_2) = f(x_1) \leqslant \frac{x_1 - x_0}{x_2 - x_0} f(x_0) + \frac{x_2 - x_1}{x_2 - x_0} f(x_2) = \lambda f(x_0) + (1-\lambda) f(x_2)$$

Hence f is convex.

(d) By Theorem 5.10, $u'' \geqslant 0$ in $\mathscr{D}'(\mathbb{R})$ implies that there exists a increasing function $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ such that $u' = f$. Let

$$F(x) := \int_0^x f(t)\,\mathrm{d}t$$

why?

be a primitve of f in the usual sense. Then $F' = f$ in $\mathscr{D}'(\mathbb{R})$: for $\varphi \in \mathscr{D}(\mathbb{R})$,

$$\begin{aligned}
\langle F', \varphi \rangle = -\langle F, \varphi' \rangle &= -\int_{\mathbb{R}} F(x)\varphi'(x)\,\mathrm{d}t \\
&= -\int_{\mathbb{R}} \left(\int_0^x f(t)\,\mathrm{d}t \right) \varphi'(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}} \left(\int_t^{\infty} -\varphi'(x)\,\mathrm{d}x \right) f(t)\,\mathrm{d}t && \text{(Fubini's theorem)} \\
&= \int_{\mathbb{R}} f(t)\varphi(t)\,\mathrm{d}t = \langle f, \varphi \rangle
\end{aligned}$$

Hence $(u - F)' = 0$ in $\mathscr{D}'(\mathbb{R})$. By constancy theorem, $u - F = c$ for some $c \in \mathbb{R}$. Hence $u = F + c$ is a regular distribution.

For $x_0 < x_1 < x_2$, since f is increasing, we have

$$\frac{F(x_1) - F(x_0)}{x_1 - x_0} = \frac{1}{x_1 - x_0} \int_{x_0}^{x_1} f(t)\,\mathrm{d}t \leqslant f(x_1 -) \leqslant f(x_1 +) \leqslant \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f(t)\,\mathrm{d}t = \frac{F(x_2) - F(x_1)}{x_2 - x_1}$$

By the similar proof in part (c), we deduce that F is convex. Hence $u = F + c$ is represented by a convex function. □

I take issue with the first line which I have highlighted - I don't think that you can assume this.
Instead try replicating proof from LN where you consider $\rho_\varepsilon * u$ & show $\varepsilon \longmapsto \rho_\varepsilon * u$ is ↑ fn of ε & $\rho_\varepsilon * u$ is convex. Then let $\varepsilon \to 0^+$.