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Problem Sheet 1

Coordinates, 4-Vectors and 4-Tensors

B5: General Relativity

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Question 1

Consider the following thought:

"Special relativity holds for frames moving at constant relative velocity, but of course acceleration requires general relativity because the frames are noninertial."

Ineffable twaddle. Special relativity certainly doesn't cower before simple kinematical accleration. On the other hand, acceleration, even just uniform accleration in one dimension, is not without its connections with general relativity. We shall explore some of them here. *For ease of notation, we set* $c = 1$. In part (d) we'll put c back.

(a) Let us first ask what we mean by "uniform acceleration." After all, a rocket approaching the speed of light c can't change its velocity at a uniform rate forever without exceeding c at some point. Go into the frame moving instantaneously at velocity v with the rocket relative to the "lab." In this frame, by definition the instantaneous rocket velocity v' vanishes. Wait a time dt' later, as measured in this frame. The rocket now has velocity dv' in this same frame. What we mean by constant acceleration is $dv'/dt' \equiv a'$ is constant. The acceleration measured in the fixed lab is certainly not constant! The question is, how is the lab acceleration $a = dv/dt$ related to the truly constant a' ?

To answer this, let $V = v/\sqrt{1-v^2}$, the spatial part of the 4-vector V^α associated with the ordinary velocity $\boldsymbol{v}$. The same relation holds for V' and v'. Assume for the moment that the primed and unprimed frames differ by some arbitrary velocity w. The 4 -velocity differentials are then given by:

$$dV' = \left(dV - w\,dV^0\right)/\sqrt{1-w^2}$$

where $V^0 = 1/\sqrt{1-v^2}$. Explain.

(b) Now, set $w = v$. We thereby go into the frame in which $v' = 0$; the rocket is instantaneously at rest. Prove that $dv = dv'\left(1-v^2\right)$. (Remember, v and v' are ordinary velocities.) From here, prove that

$$\frac{dv}{dt} = a'\left(1-v^2\right)^{3/2}$$

(c) Show that, starting from rest at $t = t' = 0$,

$$v = \frac{a't}{\sqrt{1+a'^2t^2}}, \quad a't = \sinh\left(a't'\right)$$

and hence show that (for $x = 0$ at $t = t' = 0$):

$$v = \tanh\left(a't'\right), \quad x = \frac{1}{a'}\left[\cosh\left(a't'\right) - 1\right]$$

The integrals are not difficult; do them yourselves.

(d) Let's use these results to construct a full coordinate transformation from the lab frame x, t to the accelerating x', t' frame. A good start is to guess a transform of the form

$$t = A\left(x'\right)\sinh\left(a't'\right) + B\left(x'\right), \quad x = A\left(x'\right)\cosh\left(a't'\right) + C\left(x'\right)$$

where A, B, and C depend only upon x'. Then on $x' =$ constant surfaces, $dx/dt = \tanh\left(a't'\right) = v$, which is indeed what we need.

By definition, constant t' surfaces are constant time surfaces in the $\left(x', t'\right)$ frame that moves instantaneously with velocity $v = \tanh\left(a't'\right)$ with respect to the (x, t) frame. On such a surface, $dt'/dx' = 0$. We fix the origin by demanding that as $t' \to 0, x \to x'$. We fix our clock by demanding that as $t' \to 0, t \to t'$ at the rocket location $x' = 0$. (This must be done locally: since A depends on x', this time agreement can be exact at only one value of x'.) Show that these constraints force B and C to be constant, and that B in particular must vanish.

Finally, put the speed of light c back into the equations, demand that x goes to x' at $t' = 0$, and show that

$$ct = \left(\frac{c^2}{a'} + x'\right)\sinh\left(a't'/c\right), \quad x = \left(\frac{c^2}{a'} + x'\right)\cosh\left(a't'/c\right) - \frac{c^2}{a'}$$

(e) Show that the invariant Minkowski line element may be written in x', t' coordinates as:

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 = \left(1 + \frac{a'x'}{c^2}\right)^2 c^2 dt'^2 - dx'^2$$

Provide a physical interpretation of your result in terms of a gravitational redshift. How do you interpret the region $x' \leqslant -c^2/a''$? (Review the results of 1 d.)

Proof. (a) The four velocities $\mathrm{V} = (V^0, V)$ and $\mathrm{V}' = (V'^0, V')$ in the two inertial frames are related by a Lorentz boost:

$$V' = \frac{V - wV^0}{\sqrt{1-w^2}}$$

Taking differential:

$$\mathrm{d}V' = \frac{\mathrm{d}V - w\mathrm{d}V^0}{\sqrt{1-w^2}}$$

✓

Here w is fixed because we are working in the inertial frames.

(b) We have:

$$\mathrm{d}V'|_{v'=0} = \left(\frac{1}{(1-v'^2)^{3/2}}\right)_{v'=0} \mathrm{d}v' = \mathrm{d}v', \qquad \mathrm{d}V = \frac{1}{(1-v^2)^{3/2}}\mathrm{d}v, \qquad \mathrm{d}V^0 = \frac{v}{(1-v^2)^{3/2}}\mathrm{d}v$$

Substituting in the equation above, we have

$$\mathrm{d}v' = \frac{1}{1-v^2}\mathrm{d}v$$

By chain rule,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}v'}\frac{\mathrm{d}v'}{\mathrm{d}t'}\frac{\mathrm{d}t'}{\mathrm{d}t} = (1-v)^2 \cdot a' \cdot \gamma_v^{-1} = a'(1-v^2)^{3/2}$$

✓

(c) Integrating the equation above:

$$\int_0^v \frac{\mathrm{d}v}{(1-v^2)^{3/2}} = \int_0^t a'\,\mathrm{d}t \implies \frac{v}{\sqrt{1-v^2}} = a't \implies v(t) = \frac{a't}{\sqrt{1+a'^2t^2}}$$

✓

On the other hand, we also have

$$\frac{\mathrm{d}t}{\mathrm{d}t'} = \gamma_v = \frac{1}{\sqrt{1-v^2}}$$

Hence

$$\frac{\mathrm{d}t}{\mathrm{d}t'} = \sqrt{1+a'^2t^2} \implies \int_0^t \frac{\mathrm{d}t}{\sqrt{1+a'^2t^2}} = \int_0^{t'} \mathrm{d}t' \implies t' = \frac{1}{a'}\operatorname{arcsinh}(a't) \implies a't = \sinh(a't')$$

Now we subsitute the expression into $v(t)$:

$$v(t') = \frac{\sinh(a't')}{\sqrt{1+\sinh^2(a't')}} = \tanh(a't')$$

Integrating the expression to get $x(t')$:

$$x(t') = \int_0^t v(t)\,\mathrm{d}t = \int_0^{t'} \frac{v}{\sqrt{1-v^2}}\mathrm{d}t' = \int_0^{t'} a't\,\mathrm{d}t' = \int_0^{t'} \sinh(a't')\,\mathrm{d}t = \frac{1}{a'}(\cosh(a't') - 1)$$

✓

(d)
(e) } See tutorial

□

Question 2. Recognising tensors.

One way to prove that something is a vector or tensor is to show explicitly that it satisfies the coordinate transformation laws. This can be a long and arduous procedure if the tensor is complicated, like $R^{\lambda}_{\mu\nu\kappa}$. There is another way, usually much better!

Show that if V_ν is an arbitrary covariant vector and the combination $T^{\mu\nu}V_\nu$ is known to be a contravariant vector (note the free index μ), then

$$\left(T'^{\mu\nu} - T^{\lambda\sigma}\frac{\partial x'^{\mu}}{\partial x^{\lambda}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}}\right)V'_{\nu} = 0$$

Why does this prove that $T^{\mu\nu}$ is a tensor? Does your proof actually depend on the rank of the tensors involved?

Proof. Since $(T^{\mu\nu}V_{\nu})$ is a contravariant vector, the coordinate transformation is given by

$$\widetilde{T}^{\mu\nu}\widetilde{V}_{\nu} = \frac{\partial \widetilde{x}^{\mu}}{\partial x^{\lambda}}T^{\lambda\eta}V_{\eta}$$

Since (V_{ν}) is a covariant vector, the coordinate transformation is given by

$$\widetilde{V}_{\nu} = \frac{\partial x^{\eta}}{\partial \widetilde{x}^{\nu}}V_{\eta} \implies V_{\eta} = \frac{\partial \widetilde{x}^{\nu}}{\partial x^{\eta}}\widetilde{V}_{\nu}, \quad \text{(inverse function theorem)}$$

Combining the equations:

$$\widetilde{T}^{\mu\nu}\widetilde{V}_{\nu} = \frac{\partial \widetilde{x}^{\mu}}{\partial x^{\lambda}}T^{\lambda\eta}\frac{\partial \widetilde{x}^{\nu}}{\partial x^{\eta}}\widetilde{V}_{\nu} \implies \left(\widetilde{T}^{\mu\nu} - T^{\lambda\eta}\frac{\partial \widetilde{x}^{\mu}}{\partial x^{\lambda}}\frac{\partial \widetilde{x}^{\nu}}{\partial x^{\eta}}\right)\widetilde{V}_{\nu} = 0$$

Since (V_{ν}) is arbitrary, we have

$$\widetilde{T}^{\mu\nu} - T^{\lambda\eta}\frac{\partial \widetilde{x}^{\mu}}{\partial x^{\lambda}}\frac{\partial \widetilde{x}^{\nu}}{\partial x^{\eta}} = 0 \implies \widetilde{T}^{\mu\nu} = T^{\lambda\eta}\frac{\partial \widetilde{x}^{\mu}}{\partial x^{\lambda}}\frac{\partial \widetilde{x}^{\nu}}{\partial x^{\eta}}$$

Hence $(T^{\mu\nu})$ is a type (2,0) tensor.

The proof can be generalised as follows: Suppose that $(A^{i_1\cdots i_m}{}_{j_1\cdots j_n})$ represents N^{m+n} smooth functions. $(B^{k_1\cdots k_p}{}_{\ell_1\cdots\ell_q})$ is a type (p,q) tensor field such that any index contraction of

$$A^{i_1\cdots i_m}{}_{j_1\cdots j_n}B^{k_1\cdots k_p}{}_{\ell_1\cdots\ell_q}$$

is a tensor field of suitable type, then A is a type (m,n) tensor field. This is known (from the SR lectures) as the **quotient rule**. □

Question 3. What about $d^2x_{\mu}/d\tau^2$?

The geodesic equation in standard form gives us an expresssion for $d^2x^{\mu}/d\tau^2$ in terms of the affine connection, $\Gamma^{\mu}{}_{\nu\lambda}$. For the covariant coordinate x_{μ}, show that

$$\frac{d^2x_{\mu}}{d\tau^2} = \frac{1}{2}\frac{dx^{\nu}}{d\tau}\frac{dx^{\rho}}{d\tau}\frac{\partial g_{\nu\rho}}{\partial x^{\mu}}$$

Refer to section 4.7 in the notes if help is needed. Under what conditions is $dx_0/d\tau = V_0 \equiv V_t$ a constant of the motion?

Remark. I think it is better to call $\Gamma^{\mu}{}_{\nu\lambda}$ the **Christoffel symbols**, which are the local components of the affine connection ∇. The relation is given by $\nabla_{\partial_\nu}\partial_\lambda = \Gamma^{\mu}{}_{\nu\lambda}\partial_\mu$. The **Levi-Civita connection** on a (pseudo-)Riemannian manifold (M,g) is the unique map $\nabla : \Gamma(\mathrm{T}M)\times\Gamma(\mathrm{T}M) \to \Gamma(\mathrm{T}M)$, $(X,Y)\mapsto \nabla_X Y$, such that for any $X,Y,Z \in \Gamma(\mathrm{T}M)$ and smooth functions a,b on M,

1. $\nabla_{aX+bY}Z = a\nabla_X Z + b\nabla_Y Z$;
2. $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$;
3. $\nabla_X(aY) = a\nabla_X Y + X(a)Y$;
4. $X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y,\nabla_X Z)$ (compatible with metric g);
5. $\nabla_X Y - \nabla_Y X = [X,Y]$ (torsion-free).

$\nabla_X Y$ is called the covariant derivative of Y along X. The result is known as the **fundamental theorem of Riemannian Geometry**.

Proof. Let $\gamma : \tau \mapsto (x^{\mu}(\tau))$ be a geodesic on the spacetime M (where a local coordinate chart (U,x^{μ}) is fixed). The local form of geodesic equation is given by

$$\nabla_{\gamma'}\gamma' = 0 \implies \frac{\mathrm{d}^2x^{\mu}}{\mathrm{d}\tau^2} + \Gamma^{\mu}{}_{\nu\lambda}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau}\frac{\mathrm{d}x^{\lambda}}{\mathrm{d}\tau} = 0$$

The expansion of the Christoffel symbols in terms of the metric components is given by

$$\Gamma^{\mu}{}_{\nu\lambda} = \frac{1}{2} g^{\mu\eta} \left(\frac{\partial g_{\lambda\eta}}{\partial x^{\nu}} + \frac{\partial g_{\nu\eta}}{\partial x^{\lambda}} - \frac{\partial g_{\nu\lambda}}{\partial x^{\eta}} \right)$$

The parallel transports of a vector field and a dual vector field along the geodesic are given by

$$\frac{\mathrm{D}V^{\mu}}{\mathrm{D}\tau} := \nabla_{\gamma'} V^{\mu} = \frac{\mathrm{d}V^{\mu}}{\mathrm{d}\tau} + \Gamma^{\mu}{}_{\nu\lambda} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau} V^{\lambda}, \qquad \frac{\mathrm{D}V_{\mu}}{\mathrm{D}\tau} = g_{\mu\nu} \frac{\mathrm{D}V^{\nu}}{\mathrm{D}\tau} = \frac{\mathrm{d}V_{\mu}}{\mathrm{d}\tau} - \Gamma^{\lambda}{}_{\mu\nu} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau} V_{\lambda}$$

Next, return to the question. Note that $\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau}$ is a tangent vector field along γ. Since γ is a geodesic, in particular we have $\frac{\mathrm{D}}{\mathrm{D}\tau}\left(\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau}\right) = 0$. Therefore we can directly compute:

Not sure I agree with this. Let $V^{\mu} = \frac{dx^{\mu}}{d\tau}$ a vector. Then $\frac{d^2 x_{\mu}}{d\tau^2} = \frac{d}{d\tau}(g_{\mu\nu} V^{\nu}) = g_{\mu\nu} \frac{d}{d\tau} V^{\nu} = g_{\mu\nu} D_{\rho} V^{\nu} \frac{dx^{\rho}}{d\tau}$ is a perfectly good covector to me

$$\begin{aligned}
\frac{\mathrm{d}^2 x_{\mu}}{\mathrm{d}\tau^2} &= \frac{\mathrm{D}}{\mathrm{D}\tau}\left(\frac{\mathrm{d}x^{\rho}}{\mathrm{d}\tau}\right) + \Gamma^{\lambda}{}_{\mu\nu} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau} \frac{\mathrm{d}x_{\lambda}}{\mathrm{d}\tau} \\
&= g_{\mu\rho} \frac{\mathrm{D}}{\mathrm{D}\tau}\left(\frac{\mathrm{d}x^{\rho}}{\mathrm{d}\tau}\right) + g_{\lambda\rho} \Gamma^{\lambda}{}_{\mu\nu} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau} \frac{\mathrm{d}x^{\rho}}{\mathrm{d}\tau} \\
&= g_{\lambda\rho} \Gamma^{\lambda}{}_{\mu\nu} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau} \frac{\mathrm{d}x^{\rho}}{\mathrm{d}\tau} \\
&= \frac{1}{2} g_{\lambda\rho} g^{\lambda\eta} \left(\frac{\partial g_{\mu\eta}}{\partial x^{\nu}} + \frac{\partial g_{\nu\eta}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\eta}} \right) \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau} \frac{\mathrm{d}x^{\rho}}{\mathrm{d}\tau} \\
&= \frac{1}{2} \delta^{\eta}{}_{\rho} \left(\frac{\partial g_{\mu\eta}}{\partial x^{\nu}} + \frac{\partial g_{\nu\eta}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\eta}} \right) \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau} \frac{\mathrm{d}x^{\rho}}{\mathrm{d}\tau} \\
&= \frac{1}{2} \left(\frac{\partial g_{\mu\rho}}{\partial x^{\nu}} + \frac{\partial g_{\nu\rho}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}} \right) \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau} \frac{\mathrm{d}x^{\rho}}{\mathrm{d}\tau} \\
&= \frac{1}{2} \frac{\partial g_{\nu\rho}}{\partial x^{\mu}} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau} \frac{\mathrm{d}x^{\rho}}{\mathrm{d}\tau}
\end{aligned}$$

It is important to notice that this is true in particular when $\partial_0 g_{\mu\nu} = 0$. This happens (in suitable coordinates) when the system enjoys time-translational invariance

When $\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} \frac{\mathrm{d}x^{\rho}}{\mathrm{d}\tau} \frac{\partial g_{\mu\rho}}{\partial x^0} =: V^{\mu} V^{\rho} \frac{\partial g_{\mu\rho}}{\partial x^0} = 0$, V_0 is a constant of motion along γ.

Remark. The notation $\frac{\mathrm{d}^2 x_{\mu}}{\mathrm{d}\tau^2}$ and the concept of *covariant coordinates* x_{μ} are fundamental ambiguous. We can only talk about covariant and contravariant coordinates when we work in the tangent and cotangent space (which are identified by the metric g). For this question, the only concept that make sense is the dual of the tangent vector $\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau}$ along the geodesic, which may be denoted by $\frac{\mathrm{d}x_{\mu}}{\mathrm{d}\tau}$. $\frac{\mathrm{d}^2 x_{\mu}}{\mathrm{d}\tau^2}$, however, is merely the derivative of the smooth map $\tau \mapsto \frac{\mathrm{d}x_{\mu}}{\mathrm{d}\tau} \in \mathrm{T}^*_{\gamma(\tau)} M$, and is not a dual vector field. We have to transform it to a dual vector field before raising its index. That is why we need the parallel transport.

Question 4. Practise with the Ricci Tensor.

(a) Consider the 2D surface given by

$$z^2 = x^2 + y^2$$

where x, y, z are Cartesian coordinates in 3D Euclidian space. This represents a pair of cones centred on the origin, one cone opening upward, the other opening downward. The opening angle is 45° measured from the z axis. Justify this description.

(b) A point in the 2D conic surface can be determined by R, the cylindrical radius of the point measured from the z-axis, and ϕ, the usual azimuthal angle. Show that the metric for the 2D surface in these coordinates is

$$ds^2 = 2dR^2 + R^2 d\phi^2$$

(Hint: Start with a standard metric in good old 3D Euclidian space, then enforce the constraint that $z^2 = x^2 + y^2 = R^2$. This is known as "embedding." The Nash Embedding Theorem states that pretty much any Riemannian hypersurface can always be embedded in some higher dimensional Euclidian space.)

(c) Is this 2D surface curved, in the mathematical sense of having nonvanishing components of the curvature tensor $\mathscr{R}^{\lambda}_{\kappa\mu\nu}$? (We use $\mathscr{R}$ for the tensor, R for the radial coordinate.) Answer the question by showing that the metric of part 4 b) can be transformed to new coordinates R', ϕ', for which

$$ds^2 = dR'^2 + R'^2 d\phi'^2$$

(The transformation law is extremely simple!) Why does this result alone answer the posed question? Can you give a physical interpretation of your mathematical transformation?

(d) Next, consider a different 2D surface: $z = (\alpha/2)\left(x^2 + y^2\right)$ where α is an arbitrary constant parameter. Show that this is a paraboloid of revolution, i.e. a parabola spun around the z -axis. Prove that the metric within this surface is given by

$$ds^2 = \left(1 + \alpha^2 R^2\right) dR^2 + R^2 d\phi^2$$

(e) Prove that this surface is distorted by curvature. Calculate, for example, $\mathscr{R}_{\phi\phi}$ and show that it is not zero, but given by

$$\mathscr{R}_{\phi\phi} = -\frac{\alpha^2 R^2}{\left(1 + \alpha^2 R^2\right)^2}$$

You should show en route that the only nonvanishing affine connection coefficients are

$$\Gamma^{R}_{RR} = \frac{\alpha^2 R}{1 + \alpha^2 R^2}, \quad \Gamma^{\phi}_{\phi R} = \Gamma^{\phi}_{R\phi} = \frac{1}{R}, \quad \Gamma^{R}_{\phi\phi} = -\frac{R}{1 + \alpha^2 R^2}$$

Proof. (a) In cylindrical coordinates (R, φ, z), the equation $z^2 = x^2 + y^2$ becomes $z = \pm R$. The intersection of the surface with the half-plane $\varphi = \text{const}$ is two rays whose angle is $\pi/4$ measured from the z-axis. So $z^2 = x^2 + y^2$ is the surface of revolution of the two rays about the z-axis and hence is a pair of cones as described in the question. ✓

(b) The local parametrisation of the surface in $\mathbb{R}^3$ is given by $\mathbf{r}(R, \varphi) = (R\cos\varphi, R\sin\varphi, \pm R)$. The metric tensor is given by

$$g = g_{ij} \mathrm{d}x^i \mathrm{d}x^j, \quad g_{ij} = \left\langle \frac{\partial \mathbf{r}}{\partial R}, \frac{\partial \mathbf{r}}{\partial \varphi} \right\rangle$$

Here $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^3$. (*In classical differential geometry,* $g_{11} = E$, $g_{12} = g_{21} = F$, $g_{22} = G$, *and* $g = I$ *is called the first fundamental form of the surface.*)

$$\frac{\partial \mathbf{r}}{\partial R} = (\cos\varphi, \sin\varphi, \pm 1); \qquad \frac{\partial \mathbf{r}}{\partial \varphi} = (-R\sin\varphi, R\cos\varphi, 0)$$

Hence

$$g_{11} = 2; \qquad g_{12} = g_{21} = 0; \qquad g_{22} = R^2$$ ✓

We deduce that $g = 2\mathrm{d}R^2 + R^2 \mathrm{d}\varphi^2$.

(c) Let us try the rescaling $R = \lambda R'$, $\varphi = \eta\varphi'$. Then $\mathbf{r}(R', \varphi') = (\lambda R'\cos(\eta\varphi'), \lambda R'\sin(\eta\varphi'), \pm\lambda R')$. Similarly we can compute

$$g_{11} = 2\lambda^2; \qquad g_{12} = g_{21} = 0; \qquad g_{22} = \lambda^2 \eta^2 R'^2$$

We can take $\lambda = 1/\sqrt{2}$ and $\eta = \sqrt{2}$. Therefore the metric tensor is given by

$$g = \mathrm{d}R'^2 + R'^2 \mathrm{d}\varphi'^2$$

This is the metric of $\mathbb{R}^2$ parametrised by the polar coordinates (R', φ'). The surface is locally isometric to $\mathbb{R}^2$ and hence is flat. (On $\mathbb{R}^2$, all Christoffel symbols are zero, so are the components of the Riemann curvature tensor.) ✓

(d) The local parametrisation of the surface $z = \frac{\alpha}{2}(x^2 + y^2)$ is given by $\mathbf{r}(R, \varphi) = \left(R\cos\varphi, R\sin\varphi, \frac{\alpha}{2} R^2\right)$.

$$\frac{\partial \mathbf{r}}{\partial R} = (\cos\varphi, \sin\varphi, \alpha R); \qquad \frac{\partial \mathbf{r}}{\partial \varphi} = (-R\sin\varphi, R\cos\varphi, 0)$$

Hence

$$g = (1 + \alpha^2 R^2)\mathrm{d}R^2 + R^2 \mathrm{d}\varphi^2$$

(e) The Lagrangian of the surface is given by

$$\mathscr{L} = g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau} = (1+\alpha^2R^2)\left(\frac{\mathrm{d}R}{\mathrm{d}\tau}\right)^2 + R^2\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)^2$$

By Euler-Lagrange equation:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\mathscr{L}}{\partial\dot{x}^i} - \frac{\partial\mathscr{L}}{\partial x^i} = 0$$

We find that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(2\dot{R}(1+\alpha^2R^2)\right) - 2R(\dot{\varphi}^2+\alpha^2\dot{R}^2) = 0 \implies \ddot{R} + \frac{\alpha^2R^2}{1+\alpha^2R^2}\dot{R}^2 - \frac{R}{1+\alpha^2R^2}\dot{\varphi}^2 = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(2R^2\dot{\varphi}) = 0 \implies \ddot{\varphi} + \frac{2}{R}\dot{R}\dot{\varphi} = 0$$

Comparing with the geodesic equations $\ddot{x}^i + \Gamma^i{}_{jk}\dot{x}^j\dot{x}^k = 0$, we obtain the Christoffel symbols:

$$\Gamma^R{}_{RR} = \frac{\alpha^2R^2}{1+\alpha^2R^2},\quad \Gamma^R{}_{\varphi\varphi} = -\frac{R}{1+\alpha^2R^2},\quad \Gamma^\varphi{}_{R\varphi} = \Gamma^\varphi{}_{\varphi R} = \frac{1}{R},\quad \Gamma^R{}_{R\varphi} = \Gamma^R{}_{\varphi R} = \Gamma^\varphi{}_{RR} = \Gamma^\varphi{}_{\varphi\varphi} = 0$$

The components of the Ricci curvature tensor and the Christoffel symbols are related by

$$R_{ij} = \frac{\partial\Gamma^k{}_{ik}}{\partial x^j} - \frac{\partial\Gamma^k{}_{ij}}{\partial x^k} + \Gamma^k{}_{i\ell}\Gamma^\ell{}_{jk} - \Gamma^k{}_{ij}\Gamma^\ell{}_{k\ell}$$

Hence

$$\begin{aligned} R_{\varphi\varphi} &= \frac{\partial\Gamma^k{}_{\varphi k}}{\partial\varphi} - \frac{\partial\Gamma^k{}_{\varphi\varphi}}{\partial x^k} + \Gamma^k{}_{\varphi\ell}\Gamma^\ell{}_{\varphi k} - \Gamma^k{}_{\varphi\varphi}\Gamma^\ell{}_{k\ell} \\ &= -\frac{\partial\Gamma^R{}_{\varphi\varphi}}{\partial R} + \Gamma^R{}_{\varphi\varphi}\Gamma^\varphi{}_{\varphi R} \\ &= \frac{\mathrm{d}}{\mathrm{d}R}\frac{R}{1+\alpha^2R^2} - \frac{R^2}{1+\alpha^2R^2} \\ &= -\frac{\alpha^2R^2}{(1+\alpha^2R^2)^2} \end{aligned}$$

□

Question 5. What is "the spatial part" of a metric?

It is easy, even trivial, to get the proper time from a metric. One simply sets all the spatial $dx^i = 0$ in the invariant interval $g_{\mu\nu}dx^\mu dx^\nu$, and reads off a proper time of

$$d\tau = \sqrt{-g_{00}}\,dx^0/c$$

This is what a local inertial observer reads off on their watch. So to get " the spatial part" of the metric, call it dl^2, do we just take whatever is left over from setting $dx^0 = 0$, i.e. $dl^2 = g_{ij}dx^i dx^j$? Not quite. How does an observer actually measure a distance? They take a light ray, bounce it off a mirror a distance dl away, measure the (proper) time on their watch $d\tau$ for the light to go and come back, and then set $dl = c\,d\tau/2$. Let's go with that.

(a) Show that for a diagonal metric tensor (all $g_{0i} = g_{i0} = 0$), this procedure gives

$$dl^2 = g_{ij}dx^i dx^j$$

just as we expect.

(b) Show that for a general metric tensor $g_{\mu\nu}$, with $g_{0i} = g_{0i}$ present, this procedure gives

$$dl^2 = \gamma_{ij}dx^i dx^j,\ \text{where } \gamma_{ij} = g_{ij} - (g_{0i}g_{0j}/g_{00})$$

The metric tensor of a rotating black hole (the Kerr metric) actually has $g_{0\phi} = g_{\phi 0}$ components, so this formula is very relevant here. We see that the spatial part of the metric may contain mixed time-indexed terms!

(c) Using the $g_{\mu\nu}g^{\nu\rho} = \delta_\mu^\rho$ relations, show that

$$g^{ij}\gamma_{jk} = \delta_k^i$$

and that, defining γ^{ij} by raising indices via $\gamma^{ij} \equiv g^{ik}g^{jm}\gamma_{km}$, leads to

$$\gamma^{ij} = g^{ij}$$

the "pure spatial part" of $g^{\mu\nu}.\gamma^{ij}$ defined this way is indeed the inverse of γ_{ij}. Hence, we are justified in regarding $\gamma_{ij}dx^i dx^j$ as the invariant interval in its own three-dimensional space, with inverse γ^{ij}, within the more encompassing four-dimensional $g_{\mu\nu}$ spacetime. (Note that this also shows that the indices on γ_{ij} may be raised with γ^{ij}.)

(d) Show that $\det g_{\mu\nu} = g_{00} \det \gamma_{ij}$, which is consistent with identifying γ_{ij} as the spatial metric. You may find it useful to recall that the determinant of a matrix is unchanged when a multiple of one row is added to another.

Proof. (a) Let us skip this and do (b).

(b) We will be slightly informal for this part. (I believe there is a rigorous way of doing this.)

Suppose that the light signal leaves point A in the space at $t = \mathrm{d}x^0_{(1)}/c$. It hits the mirror at point B in the space at $t = 0$ and then return to A at $t = \mathrm{d}x^0_{(2)}/c$. Locally the light travels along a null vector. We have

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = g_{00}\mathrm{d}(x^0)^2 + 2g_{0i}\mathrm{d}x^0\mathrm{d}x^i + g_{ij}\mathrm{d}x^i\mathrm{d}x^j = 0$$

This is a quadratic equation of $\mathrm{d}x^0$. The two solutions are given by

$$\mathrm{d}x^0_{(1)} = \frac{-g_{0i}\mathrm{d}x^i - \sqrt{(g_{0i}g_{0j} - g_{00}g_{ij})\mathrm{d}x^i\mathrm{d}x^j}}{g_{00}}, \qquad \mathrm{d}x^0_{(2)} = \frac{-g_{0i}\mathrm{d}x^i + \sqrt{(g_{0i}g_{0j} - g_{00}g_{ij})\mathrm{d}x^i\mathrm{d}x^j}}{g_{00}}$$

The proper time is given by

$$\mathrm{d}\tau = \sqrt{-g_{00}}\frac{\mathrm{d}x^0_{(2)} - \mathrm{d}x^0_{(1)}}{c} = \frac{2}{c}\sqrt{\frac{g_{00}g_{ij} - g_{0i}g_{0j}}{g_{00}}\mathrm{d}x^i\mathrm{d}x^j}$$

Hence the spatial distance is given by

$$\mathrm{d}\ell = \frac{c}{2}\mathrm{d}\tau = \sqrt{\frac{g_{00}g_{ij} - g_{0i}g_{0j}}{g_{00}}\mathrm{d}x^i\mathrm{d}x^j} \implies \mathrm{d}\ell^2 = \left(g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}}\right)\mathrm{d}x^i\mathrm{d}x^j =: \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

If $g_{0i} = 0$, then the procedure gives

$$\mathrm{d}\ell^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

which is consistent with the result in (a).

(c) Note that

$$g^{ij}g_{jk} = g^{i\mu}g_{\mu k} - g^{i0}g_{0k} = \delta^i_k - g^{i0}g_{0k}$$

and

$$g^{ij}g_{0j} = g^{i\mu}g_{0\mu} - g^{i0}g_{00} = \delta^i_0 - g^{i0}g_{00} = -g^{i0}g_{00}$$

Hence

$$g^{ij}\gamma_{jk} = g^{ij}g_{jk} - \frac{g^{ij}g_{0j}g_{0k}}{g_{00}} = \delta^i_k - g^{i0}g_{0k} - \frac{-g^{i0}g_{00}g_{0k}}{g_{00}} = \delta^i_k$$

Raising the indices of γ_{ij}:

$$\gamma^{ij} = g^{ik}g^{jm}\gamma_{km} = g^{ik}\delta^j_k = g^{ij}$$

(γ^{ij}) is the inverse matrix of (γ_{ij}):

$$\gamma^{ij}\gamma_{jk} = g^{ij}\gamma_{jk} = \delta^i_k$$

(d) The determinant can be computed with some elementary row operations:

$$\det(g_{\mu\nu}) = \det\begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} = \det\begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ 0 & g_{11} - g_{01}g_{10}/g_{00} & g_{12} - g_{02}g_{10}/g_{00} & g_{13} - g_{03}g_{10}/g_{00} \\ 0 & g_{21} - g_{01}g_{20}/g_{00} & g_{22} - g_{02}g_{20}/g_{00} & g_{23} - g_{03}g_{20}/g_{00} \\ 0 & g_{31} - g_{01}g_{30}/g_{00} & g_{32} - g_{02}g_{30}/g_{00} & g_{33} - g_{03}g_{30}/g_{00} \end{pmatrix}$$

$$= g_{00}\det\begin{pmatrix} g_{11} - g_{01}g_{10}/g_{00} & g_{12} - g_{02}g_{10}/g_{00} & g_{13} - g_{03}g_{10}/g_{00} \\ g_{21} - g_{01}g_{20}/g_{00} & g_{22} - g_{02}g_{20}/g_{00} & g_{23} - g_{03}g_{20}/g_{00} \\ g_{31} - g_{01}g_{30}/g_{00} & g_{32} - g_{02}g_{30}/g_{00} & g_{33} - g_{03}g_{30}/g_{00} \end{pmatrix} = g_{00}\det(g_{ij} - g_{i0}g_{0j}/g_{00})$$

$$= g_{00}\det(\gamma_{ij}) \qquad \square$$