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Problem Sheet 1

ASO: Projective Geometry

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In these questions, F denotes the base field.

Question 1

(i) If we identify $(x, y) \in F^2$ with the point $[1 : x : y] \in F\mathbb{P}^2$, what is the point at infinity shared by all lines of the form $y = mx + c$, where m is fixed?

(ii) Show that those projective transformations in $\mathrm{PGL}(3, F)$ which map the line at infinity to itself form a subgroup of $\mathrm{PGL}(3, F)$ which is isomorphic to

$$\mathrm{AGL}(2, F) = \{\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b} : \ A \in \mathrm{GL}(2, F), \ \boldsymbol{b} \in F^2\}$$

Which of these mappings fix the line at infinity pointwise?

Proof. (i) The projectivization of the line $\{(x, y) : y = mx + c\} \subseteq F^2$ is given by $\{[x_0 : x_1 : x_2] : cx_0 + mx_1 - x_2 = 0\} \subseteq F\mathbb{P}^2$. The point of infinity corresponds to the case when $x_0 = 0$. That is, $x_2 = mx_1$. So the point of infinity of the line $y = mx + c$ in $F\mathbb{P}^2$ is $[0 : 1 : m]$.

(ii) Suppose that $\tau \in \mathrm{PGL}(3, F)$ is induce by

$$T = \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ c_{2,1} & c_{2,2} & c_{2,3} \\ c_{3,1} & c_{3,2} & c_{3,3} \end{pmatrix} \in \mathrm{GL}(3, F)$$

If τ fixes the line of infinity, then for $[0 : x_1 : x_2] \in F\mathbb{P}^2$, we have:

$$\begin{pmatrix} c_{0,0} & c_{0,1} & c_{0,2} \\ c_{1,0} & c_{1,1} & c_{1,2} \\ c_{2,0} & c_{2,1} & c_{2,2} \end{pmatrix} \begin{pmatrix} 0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ y_1 \\ y_2 \end{pmatrix}$$

In particular, $c_{0,1}x_1 + c_{0,2}x_2 = 0$ for all $x_1, x_2 \in F$. Hence $c_{0,1} = c_{0,2} = 0$. Since T is invertible, $c_{0,0} \neq 0$. By rescaling we may assume that $c_{0,0} = 1$. We can write T as

$$T = \begin{pmatrix} 1 & 0 & 0 \\ b_1 & a_{1,1} & a_{1,2} \\ b_2 & a_{2,1} & a_{2,2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \boldsymbol{b} & A \end{pmatrix}$$

For $\boldsymbol{x} = (x_1, x_2)^{\mathrm{T}} \in F^2$, we embed in into $F\mathbb{P}^2$ by identification with $[1 : x_1 : x_2]$. We have:

$$T \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \boldsymbol{b} & A \end{pmatrix} \begin{pmatrix} 1 \\ \boldsymbol{x} \end{pmatrix} = A\boldsymbol{x} + \boldsymbol{b}$$

Hence we identify $\tau \in \mathrm{PGL}(3, F)$ with the affine transformation $\boldsymbol{x} \mapsto A\boldsymbol{x} + \boldsymbol{b}$. The subgroup of all such projective transformations is isomorphic to $\mathrm{AGL}(2, F)$.

If τ fixes the line at infinity pointwise, then $A\boldsymbol{x} = \boldsymbol{x}$ for all $\boldsymbol{x} \in F^2$. Hence $A = I_2$. The corresponding $T \in \mathrm{GL}(3, F)$ is given by

$$T = \begin{pmatrix} 1 & 0 & 0 \\ b_1 & 1 & 0 \\ b_2 & 0 & 1 \end{pmatrix}$$

□

Question 2

(i) Let $\mathbb{P}(U_1)$ and $\mathbb{P}(U_2)$ be two non-intersecting lines in the 3-dimensional projective space $F\mathbb{P}^3 := \mathbb{P}(F^4)$. Show that

$$F^4 = U_1 \oplus U_2$$

(ii) Deduce that three pairwise non-intersecting lines in $F\mathbb{P}^3$ have infinitely many transversals, i.e. projective lines meeting all three.

Proof. (i) If $\mathbb{P}(U_1) \cap \mathbb{P}(U_2) = \varnothing$, then $U_1 \cap U_2 = \{0\}$. We have $\dim(U_1 \oplus U_2) = \dim U_1 + \dim U_2 = 2 + 2 = 4 = \dim F^4$. But $U_1 \oplus U_2 \leqslant F^4$. Hence $F^4 = U_1 \oplus U_2$.

(ii) **The statement is true only if F is an infinite field.**

Suppose that $\mathbb{P}(U_1)$, $\mathbb{P}(U_2)$ and $\mathbb{P}(U_3)$ are three non-intersecting projective lines. By (i) we have $U_3 \subseteq F^4 = U_1 \oplus U_2$. For $\langle u_1 \rangle \in \mathbb{P}(U_3)$, there exists $\langle u_1 \rangle \in \mathbb{P}(U_1)$ and $\langle u_2 \rangle \in \mathbb{P}(U_2)$ such that $u_3 = u_1 + u_2$. Take $U_4 = \langle u_1, u_2 \rangle$. Then $\langle u_1 \rangle, \langle u_2 \rangle, \langle u_3 \rangle \in \mathbb{P}(U_4)$ and therefore $\mathbb{P}(U_4)$ transverses all $\mathbb{P}(U_1)$, $\mathbb{P}(U_2)$ and $\mathbb{P}(U_3)$. Moreover, for distinct $\langle u_3 \rangle \in \mathbb{P}(U_3)$, the projective line $\mathbb{P}(U_4)$ is distinct. Suppose that $\langle u_3 \rangle \neq \langle u_3' \rangle$ and $\langle u_3 \rangle, \langle u_3' \rangle \in \mathbb{P}(U_4)$, then $U_4 = \langle u_3, u_3' \rangle \implies U_4 = U_3$. This is impossible because $\mathbb{P}(U_3)$ does not intersect with the other two projective lines. Since F is infinite, so is $\mathbb{P}(U_3)$. We hence constructed infinitely many transversals of the three projective lines. □

Question 3

Let L_1, L_2 be two non-empty projective linear subspaces of a projective space $\mathbb{P}(V)$, corresponding to linear subspaces $U_1, U_2 \subseteq V$. Show that the span

$$\langle L_1, L_2 \rangle = \mathbb{P}(U_1 + U_2)$$

is the union of projective lines P_1P_2 with $P_i \in L_i$.

Proof. For $P_1 = \langle v_1 \rangle \in L_1$ and $P_2 = \langle v_2 \rangle \in L_2$, the projective line P_1P_2 is the projectivization of the linear subspace $\langle v_1, v_2 \rangle$. Since $v_1 \in U_1$ and $v_2 \in U_2$, $\langle v_1, v_2 \rangle \subseteq U_1 + U_2$. Therefore $\mathbb{P}(\langle v_1, v_2 \rangle) \subseteq \mathbb{P}(U_1 + U_2) = \langle L_1, L_2 \rangle$. In other words, the projective line P_1P_2 lies in the span $\langle L_1, L_2 \rangle$.

Conversely, for $\langle u \rangle \in \langle L_1, L_2 \rangle$, $u \in U_1 + U_2$. There exists $u_1 \in U_1$ and $u_2 \in U_2$ such that $u = u_1 + u_2$. Let $P_1 = \langle u_1 \rangle$ and $P_2 = \langle u_2 \rangle$. $\langle u \rangle \subseteq \langle u_1, u_2 \rangle$ implies that $\langle u \rangle$ is a projective point on the projective line P_1P_2.

We conclude that $\langle L_1, L_2 \rangle$ is the union of all projective lines P_1P_2 with $P_i \in L_i$. □

Question 4

(i) List the elements of $\mathrm{PGL}(2, \mathbb{F}_2)$. What is the order of $\mathrm{PGL}(2, F)$ if $|F| = q$?

(ii) By considering the action of $\mathrm{PGL}(2, \mathbb{F}_2)$ on $\mathbb{F}_2\mathbb{P}^1$, show that $\mathrm{PGL}(2, \mathbb{F}_2) \cong S_3$. Is $\mathrm{PGL}(2, \mathbb{F}_3) \cong S_4$? Is $\mathrm{PGL}(2, \mathbb{F}_5) \cong S_6$?

Proof. (i) The elements of $\mathrm{PGL}(2, \mathbb{F}_2) = \mathrm{GL}(2, \mathbb{F}_2)$ are

$$\begin{array}{ccc} (0,0) & \mapsto & (0,0) \\ (1,0) & \mapsto & (1,0) \\ (0,1) & \mapsto & (0,1) \\ (1,1) & \mapsto & (1,1) \end{array} \qquad \begin{array}{ccc} (0,0) & \mapsto & (0,0) \\ (1,0) & \mapsto & (0,1) \\ (0,1) & \mapsto & (1,0) \\ (1,1) & \mapsto & (1,1) \end{array} \qquad \begin{array}{ccc} (0,0) & \mapsto & (0,0) \\ (1,0) & \mapsto & (1,0) \\ (0,1) & \mapsto & (1,1) \\ (1,1) & \mapsto & (0,1) \end{array}$$

$$\begin{array}{rclcrclcrcl} (0,0) & \mapsto & (0,0) & \quad & (0,0) & \mapsto & (0,0) & \quad & (0,0) & \mapsto & (0,0) \\ (1,0) & \mapsto & (1,1) & & (1,0) & \mapsto & (0,1) & & (1,0) & \mapsto & (1,1) \\ (0,1) & \mapsto & (0,1) & & (0,1) & \mapsto & (1,1) & & (0,1) & \mapsto & (1,0) \\ (1,1) & \mapsto & (1,0) & & (1,1) & \mapsto & (1,0) & & (1,1) & \mapsto & (0,1) \end{array}$$

We first count the order of $\mathrm{GL}(2,F)$. Note that $T : F^2 \to F^2$ is uniquely determined by its action on the standard basis $\{(1,0),(0,1)\}$. Suppose that $T:\ (1,0) \mapsto (x_1,x_2),\ (0,1) \mapsto (y_1,y_2)$. If $T \in \mathrm{GL}(2,F)$, then $\{(x_1,x_2),(y_1,y_2)\}$ is linearly independent. Equivalently, $x_1y_2 - x_2y_1 \neq 0$.

We shall count the cardinality of $\{(x_1,x_2,y_1,y_2) \in F^4 : x_1y_2 - x_2y_1 = 0\}$.

- If $x_1 = 0$:
  - If $x_2 = 0$:
    - $y_1, y_2 \in F$ are arbitrary. There are q^2 combinations.
  - If $x_2 \neq 0$:
    - $y_1 = 0$ and $y_2 \in F$ are arbitrary. There are $q(q-1)$ combinations.
- If $x_1 \neq 0$:
  - If $x_2 = 0$:
    - $y_2 = 0$ and $y_1 \in F$ are arbitrary. There are $q(q-1)$ combinations.
  - If $x_2 \neq 0$:
    - If $y_1 = 0$:
      - $y_2 = 0$. There are $(q-1)^2$ combinations.
    - If $y_1 \neq 0$:
      - $y_2 = x_1^{-1}x_2y_1$. There are $(q-1)^3$ combinations.

In total, the set has $q^2 + q(q-1) + q(q-1) + (q-1)^2 + (q-1)^3 = q^3 + q^2 - q$ elements. The cardinality of $\mathrm{GL}(2,F)$:

$$\operatorname{card}\mathrm{GL}(2,F) = \operatorname{card}\{(x_1,x_2,y_1,y_2) \in F^4 : x_1y_2 - x_2y_1 \neq 0\} = q^4 - (q^3 + q^2 - q) = q(q-1)^2(q+1)$$

Note that $\mathrm{PGL}(2,F) = \mathrm{GL}(2,F)/\sim$ where $S \sim T \iff \exists\, \lambda \in F\backslash\{0\} :\ S = \lambda T$. Each equivalent class of $\mathrm{GL}(2,F)$ has exactly $q-1$ elements. Hence the order of $\mathrm{PGL}(2,F)$ is $q(q-1)(q+1) = q^3 - q$.

(ii) The elements of $\mathbb{F}_2\mathbb{P}^1$ are

$$L_1 = \{(0,0),(1,0)\} \qquad L_2 = \{(0,0),(0,1)\} \qquad L_3 = \{(0,0),(1,1)\}$$

The projective transformations of $\mathbb{F}_2\mathbb{P}^1$ are bijections of $\mathbb{F}_2\mathbb{P}^1$. Hence $\mathrm{PGL}(2,\mathbb{F}_2) \leqslant S_3$. But we know that $|\mathrm{PGL}(2,\mathbb{F}_2)| = |S_3| = 6$. Hence $\mathrm{PGL}(2,\mathbb{F}_2) \cong S_3$.

The case of $\mathrm{PGL}(2,\mathbb{F}_3)$ is similar. The elements of $\mathbb{F}_3\mathbb{P}^1$ are:

$$L_1 = \{(0,0),(1,0),(2,0)\} \quad L_2 = \{(0,0),(0,1),(0,2)\} \quad L_3 = \{(0,0),(1,1),(2,2)\} \quad L_4 = \{(0,0),(1,2),(2,1)\}$$

We have $\mathrm{PGL}(2,\mathbb{F}_3) \leqslant S_4$. By part (i) we know that $|\mathrm{PGL}(2,\mathbb{F}_3)| = 3^3 - 3 = 24 = |S_4|$. Therefore we have $\mathrm{PGL}(2,\mathbb{F}_3) \cong S_4$.

For $\mathrm{PGL}(2,\mathbb{F}_5)$, by part (i) we know that $|\mathrm{PGL}(2,\mathbb{F}_5)| = 5^3 - 5 = 120$. But $|S_6| = 6! = 720$. Therefore $\mathrm{PGL}(2,\mathbb{F}_5) \ncong S_6$. $\square$

Question 5

Let a, b, c, d be four distinct points in $\mathbb{C}$. Show that a, b, c, d lie on a cirline if and only if the cross-ratio $(ab : cd)$ is real.

Proof. By Proposition 7.2 in the notes, projective transformations preserve cross-ratio. From Part A Complex Analysis we know that $\mathrm{PGL}(2, \mathbb{C}) = \mathrm{Mob}$, the group of Möbius transformations, and that Möbius transformations preserves circles in $\mathbb{CP}^1$ (which are cirlines in $\mathbb{C}$).

Consider the Möbius transformation $z \mapsto \dfrac{(z-b)(c-d)}{(z-d)(c-b)}$. Under such map, we have

$$b \mapsto 0,\ c \mapsto 1,\ d \mapsto \infty,\ a \mapsto (ab : cd).$$

If $(ab : cd)$ is real, then $(ab : cd), 0, 1, \infty \in \mathbb{CP}^1$ lies on the same circle in $\mathbb{CP}^1$. It follows that $a, b, c, d \in \mathbb{C}$ lies on the same cirline in $\mathbb{C}$. Conversely, if a, b, c, d lies on a cirline in $\mathbb{C}$, then $(ab : cd), 0, 1, \infty \in \mathbb{CP}^1$ lies on the same circle in $\mathbb{CP}^1$. In particular, $(ab : cd), 0, 1$ lies on the same line in $\mathbb{C}$. It follows that $(ab : cd) \in \mathbb{R}$. □

Question 6

We say x_0, x_1 and x_2, x_3 are *harmonically separated* if $(x_0x_1 : x_2x_3) = -1$, where the x_i are distinct points in a projective line $F\mathbb{P}^1$. Let a, b, c, d be four points in general position in the projective plane $F\mathbb{P}^2$ and let e, f, g be the diagonal points, i.e. $e = ac \cap bd$, $f = ab \cap cd$, $g = ad \cap bc$. Let ge meet ab at h. Prove that a, b and h, f are harmonically separated.

Proof. Since $a, b, c, d \in F\mathbb{P}^2$ are in general position, we can apply a projective transformation which maps them to $[1:0:0], [0:1:0], [0:0:1], [1:1:1]$, in which the cross-ratio is preserved. Hence without loss of generality we may assume that $a = [1:0:0], b = [0:1:0], c = [0:0:1], d = [1:1:1]$. Then we have $e = ac \cap bd = [1:0:1]$, $f = ab \cap cd = [1:1:0]$, $g = ad \cap bc = [0:1:1]$, and $h = ge \cap ab = [1:-1:0]$. It follows that a, b, h, f lie on the same projective line, which is the projectivization of $\langle (1,0,0), (0,1,0) \rangle$. We can compute the cross-ratio:

$$(ab : hf) = \frac{(a_0h_1 - h_0a_1)(b_0f_1 - f_0b_1)}{(a_0f_1 - f_0a_1)(b_0h_1 - h_0b_1)} = \frac{(1 \cdot (-1) - 0)(0 - 1 \cdot 1)}{(1 \cdot 1 - 0)(0 - 1 \cdot 1)} = -1$$

Hence ab and hf are harmonically separated. □

Question 7

(i) Let $\tau \in \mathrm{PGL}(2, \mathbb{C})$, other than the identity. Show that τ fixes either one or two points. Show that this need not be true over other fields.

(ii) If τ fixes two points, show that there is an inhomogeneous coordinate z on $\mathbb{CP}^1$ with respect to which

$$\tau(z) = \lambda z, \qquad \lambda \neq 0, 1$$

Is the same true over other fields?

(iii) Let A_1, A_2, A_3 be three distinct points in $\mathbb{CP}^1$ and let $n \geqslant 3$ be an integer. Show that there is $\tau \in \mathrm{PGL}(2, \mathbb{C})$ such that $\tau(A_1) = A_2$, $\tau(A_2) = A_3$ and τ has order n.

Proof. Throughout this question, we do not distinguish between $\mathbb{CP}^1$ and $\mathbb{C}_\infty$. We identify $\mathbb{C}$ as an open subset of $\mathbb{CP}^1$ via the embedding $z \mapsto [1 : z]$. Then $\infty = [0 : 1]$.

(i) The projective transformations in $\mathrm{PGL}(2, \mathbb{C})$ is uniquely determined by its action on three distinct points. Hence if $\tau \in \mathrm{PGL}(2, \mathbb{C})$ fixes three or more points, then it must be the identity map. So it suffices to show that τ fixes at least one point in $\mathbb{CP}^1$.

We know that τ is given by a Möbius transformation. Suppose that $\tau(z) = \dfrac{az+b}{cz+d}$ $\quad (ad - bc \neq 0)$. We consider the equation with respect to $z \in \mathbb{CP}^1$:

$$z = \frac{az+b}{cz+d} \iff cz^2 - (a-d)z - b = 0$$

If $c \neq 0$, by Fundamental Theorem of Algebra the equation has a finite solution, which corresponds to a fixed point of τ in $\mathbb{C}$. If $c = 0$, then $d \neq 0$ and $\tau(z) = \frac{a}{d}z + \frac{b}{d}$ always fixes $z = \infty$.

We conclude that τ fixed either one of two points.

The statement does not hold for general fields. For instance, consider $\mathrm{PGL}(2, \mathbb{F}_2)$. In Question 4.(i) we have shown that it is isomorphic to S_3. There is a projective transformation $L_1 \mapsto L_2$, $L_2 \mapsto L_3$, $L_3 \mapsto L_1$ that has no fixed points.

(ii) Suppose that τ fixes $z_1 = [a_1 : b_1]$ and $z_2 = [a_2 : b_2]$. Consider $\sigma \in \mathrm{PGL}(2, \mathbb{C})$ induced by the matrix

$$\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

We have $\sigma(0) = z_1$ and $\sigma(\infty) = z_2$. Then $\sigma^{-1} \circ \tau \circ \sigma \in \mathrm{PGL}(2, \mathbb{C})$ fixes 0 and ∞. Suppose that $\sigma^{-1} \circ \tau \circ \sigma(z) = \frac{az+b}{cz+d}$. Then $b = 0$, $c = 0$. So $\sigma^{-1} \circ \tau \circ \sigma(z) = \frac{a}{d}z$. Moreover, $\frac{a}{d} \neq 0$ because $\sigma^{-1} \circ \tau \circ \sigma$ is invertible; $\frac{a}{d} \neq 1$ because $\tau \neq \mathrm{id}_{\mathbb{C}_\infty} \implies \sigma^{-1} \circ \tau \circ \sigma(z) \neq \mathrm{id}_{\mathbb{C}_\infty}$. We can write the action of τ explicitly as follows: for $[1 : z] \in \mathbb{CP}^1$, $\tau([a_1 + a_2 z : b_1 + b_2 z]) = [a_1 + \lambda a_2 z : b_1 + \lambda b_2 z]$.

(iii) Let $\rho \in \mathrm{PGL}(2, \mathbb{C})$ such that $\rho(A_1) = 1$, $\rho(A_2) = \mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}$, and $\rho(A_3) = \mathrm{e}^{\frac{4\pi \mathrm{i}}{n}}$. Let $\sigma_n(z) = \mathrm{e}^{\frac{2\pi \mathrm{i}}{n}} z$. We claim that $\tau := \rho^{-1} \circ \sigma_n \circ \rho \in \mathrm{PGL}(2, \mathbb{C})$ satisfies the desired properties:

$\tau(A_1) = \rho^{-1} \circ \sigma_n \circ \rho(A_1) = \rho^{-1} \circ \sigma_n(1) = \rho^{-1}\left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}\right) = A_2.$

$\tau(A_2) = \rho^{-1} \circ \sigma_n \circ \rho(A_2) = \rho^{-1} \circ \sigma_n\left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{n}}\right) = \rho^{-1}\left(\mathrm{e}^{\frac{4\pi \mathrm{i}}{n}}\right) = A_3.$

$\tau^n = (\rho^{-1} \circ \sigma_n \circ \rho)^n = \rho^{-1} \circ \sigma_n^n \circ \rho = \rho^{-1} \circ \mathrm{id} \circ \rho = \mathrm{id}.$ □

Question 8

Use the strategy outlined in the lectures to prove Pappus' Theorem: *Let A, B, C and $A'.B', C'$ be similar collinear triples of distinct points in the projective plane $F\mathbb{P}^2$. Then the three intersection points $AB' \cap A'B$, $BC' \cap B'C$ and $CA' \cap C'A$ are collinear.* Proceed by the following steps.

(i) Prove the theorem in the degenerate case when A, B, C', B' are *not* in general position.

(ii) If these 4 points are in general position, explain why without loss of generality we may take them to be

$$A = [1, 0, 0], \quad B = [0, 1, 0], \quad C' = [0, 0, 1] \quad B' = [1, 1, 1].$$

Proof. (i) If A, B, C', B' are not in general position, we may consider the case that C' lies in the projective line ABC. The other cases are similar.

If $C' \in ABC$, then $BC' \cap B'C = CA' \cap C'A = C$. Then C and $AB' \cap A'B$ are of course on the same projective line.

(ii) It follows from general position theorem that there exists a unique projective transformation such that

$$A \mapsto [1, 0, 0], \quad B \mapsto [0, 1, 0], \quad C' \mapsto [0, 0, 1] \quad B' \mapsto [1, 1, 1].$$

Clearly projective transformations preserve projective lines. So without loss of generality we can take

$$A = [1, 0, 0], \quad B = [0, 1, 0], \quad C' = [0, 0, 1] \quad B' = [1, 1, 1].$$

Since $C \in AB$, $C = [a, b, 0]$ for some $a, b \in F$. Since $A' \in C'B'$, $A' = [c : c : d]$ for some $c, d \in F$. A direct calculation shows that:

$$\langle x \rangle = AB' \cap A'B = [c : d : d] \quad \langle y \rangle = BC' \cap B'C = [0 : b - a : -a] \quad \langle z \rangle = CA' \cap C'A = [(a-b)c : 0 : -bd]$$

Then we have $(b-a)x-dy+z=0$. Hence $AB' \cap A'B$, $BC' \cap B'C$ and $CA' \cap C'A$ are collinear. $\square$

Question 9

Every line in the real affine plane $\mathbb{R}^2$ can be written in the form $ax+by+c=0$ where a, b are not both zero. Of course, $\lambda ax + \lambda by + \lambda c = 0$ is an equation of the same line where $\lambda \neq 0$. Hence the space of lines can be identified with

$$M = \frac{\mathbb{R}^2 \backslash \{(0,0)\} \times \mathbb{R}}{\mathbb{R}^*}$$

Identify M as a subspace of $\mathbb{RP}^2$. What is the topology of M?

Proof. We know that $a_1x+b_1y+c_1$ and $a_2x+b_2y+c_2=0$ represents the same line if and only if there exists $\lambda \in \mathbb{R}\backslash\{0\}$ such that $(a_1, b_1, c_1) = \lambda(a_2, b_2, c_2)$. Therefore the identification $\{(x, y) \in \mathbb{R}^2 : ax + by + c = 0\} \mapsto [a : b : c]$ gives a well-defined embedding of M into $\mathbb{RP}^2$. More specfically, $M = \mathbb{RP}^2 \backslash \{[0:0:1]\}$.

To determine the topology of M, we consider $\mathbb{RP}^2$ as $S^2/\{x \sim -x\}$, the 2-sphere with antipodal points identified. Then $M = S^2 \backslash \{\pm(0,0,1)\}/\{x \sim -x\}$. We can use the Mercator projection that projects $S^2 \backslash \{\pm(0,0,1)\}$ onto an open cylinder $S^1 \times (-1,1)$. The equivalence relation $(\theta, t) \sim (-\theta, -t)$ induces the quotient topology on $S^1 \times (-1,1)$, which is homeomorphic to an open Möbius strip. In other words, M is homeomorphic to an open Möbius strip as a subspace of $\mathbb{RP}^2$. $\square$