Peize Liu

St. Peter's College
University of Oxford

Notes on
Complex Analysis

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Notes on Complex Analysis*

Peize Liu
St Peter's College, University of Oxford

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*These notes mainly follow Oxford second year complex analysis. Some off-syllabus topics are starred. These notes do not aim to be self-contained. The readers should be familiar with first-year **introductory analysis** and the language of **metric spaces**. Some **linear algebra**, **group theory**, and the basic arithmetic of **complex numbers** are also assumed. In addition, we may sometimes use intuitions in geometry and topology without giving rigorous proofs.

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Reading List

Lecture Notes

- K. McGerty, *A2: Metric Spaces and Complex Analysis* (2018-2019). **[McGerty]**

  The lecture notes for 2018-2019 Oxford second year math course A2: Metric Spaces and Complex Analysis, on which my notes is mainly based.

Basic Textbooks

- H. A. Priestley, *Introduction to Complex Analysis* (2nd edition). **[Priestley]**

  The primary textbook for **Oxford second year Complex Analysis course**. Very thorough overall and especially for residue calculus. Yet it does not contain advanced topics such as Riemann Mapping Theorem.

- J. Shi, *Functions of Complex Variables.* **[SJH]**

  An undergraduate complex analysis textbook adapted from Gong's book. Written in Chinese and is already out of print. Contains more detail and examples compared to Gong's book. Very typical Chinese–style textbook and is one of my favorite.

Advanced Textbooks

- D. Belyaev, *C4.8 Complex Analysis: Conformal Maps and Geometry* (2018-2019). **[Belyaev]**

  The lecture notes for 2018-2019 Oxford fourth year math course C4.8 Complex Analysis: Conformal Maps and Geometry. This is the second course on undergraduate complex analysis and it focuses mainly on conformal mappings, as suggested in the course title.

- E. M. Stein, *Princeton Lectures in Analysis II: Complex Analysis.* **[Stein]**

  One of the most popular book for undergraduate complex analysis. Contains a lot of advanced materials beyond this notes.

- S. Gong, *Concise Complex Analysis* (Revised edition). **[GS]**

  A relatively advanced book. Very concise and insightful. The way of introducing differential geometry into undergraduate complex analysis is very novel.

- W. Rudin, *Real and Complex Analysis* (3rd edition). **[Rudin]**
- S. Lang, *Complex Analysis* (GTM103) (4th edition). **[Lang]**

Papers

- Scott, Anne Elizabeth. *Cauchy Integral Theorem: a Historical Development of Its Proof.* Diss. Oklahoma State University, 1978.

  A paper that summarises the historical development of the proofs of Cauchy's Theorem.

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Chapter 0

Preliminaries

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0.1 Complex Numbers

Definition 0.1. Field of Complex Numbers $\mathbb{C}$

The field of complex numbers $\mathbb{C} := \mathbb{R}[x]/\langle x^2+1\rangle = \mathbb{R}(\sqrt{-1})$ is a quadratic extension of the field of real numbers. For convenience we write $\sqrt{-1} = \mathrm{i}$. Any complex number can be represented as $z = a + b\mathrm{i}$ where $a, b \in \mathbb{R}$, or as $z = r\,\mathrm{e}^{\mathrm{i}\theta}$ where $r \in \mathbb{R}$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. The addition and multiplication are as follows:

$$(a+b\mathrm{i}) + (c+d\mathrm{i}) = (a+c) + (b+d)\mathrm{i}$$

$$(a+b\mathrm{i})(c+d\mathrm{i}) = (ac-bd) + (ad+bc)\mathrm{i}$$

Remark. It is natural to identify $\mathbb{C}$ as $\mathbb{R}^2$ with extra multiplication structure. We can define the modulus, distance, and all associated concepts for metric spaces on the complex plane.

Definition 0.2. Norm

For $z = a + b\mathrm{i} \in \mathbb{C}$, we define its norm (modulus) to be $|z| := \sqrt{a^2+b^2}$.

Remark. It is easy to check that $(\mathbb{C}, |\cdot|)$ is a normed vector space.

Definition 0.3. Complex Conjugation

For $z = a + b\mathrm{i} \in \mathbb{C}$, we define its complex conjugation to be $\bar{z} = a - b\mathrm{i}$.

Remark. $|z|^2 = z\bar{z}$.

Remark. The complex conjugation is the only non-trivial $\mathbb{R}$-module automorphism in $\mathbb{C}$, as $\mathrm{Gal}(\mathbb{C}|\mathbb{R}) = \{\mathrm{id}_{\mathbb{C}}, \bar{z}\}$. See Galois Theory for detailed discussions.

I am not going to repeat the definition of exponential / trigonometric functions or the closed / compact / connected sets on the complex plane, because they have been thoroughly studied in the preceding courses.

Definition 0.4. Domain.

We call $U \subseteq \mathbb{C}$ a domain, if U is open and connected.

Remark. A path-connected subset $U \subseteq \mathbb{R}^n$ is said to be simply-connected, if its fundamental group is trivial. That is, any closed path in U is homotopic to a constant path (a singleton) in U. See Section 1.2 for discussion of homotopy.

Theorem 0.5. $\mathbb{C}$ is algebraically closed.

Any non-constant polynomial has a root in $\mathbb{C}$.

Remark. This is the famous Fundamental Theorem of Algebra. We can use Extreme Value Theorem (continuous mapping between metric spaces preserves compactness) and some elementary estimation to proof this. An more elegant proof of this theorem makes use of Liouville's Theorem in Chapter 1. See Corollary 1.18.

Definition 0.6. Extended Complex Plane.

We introduce ∞ into the complex plane: $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. The arithmetic of ∞ is as follows:

$$\forall z \in \mathbb{C} : z + \infty = \infty + z = \infty$$
$$\forall z \in \mathbb{C} \setminus \{0\} : z \cdot \infty = \infty \cdot z = \infty;\; z/0 = \infty;\; z/\infty = 0.$$

This is an example of one-point compactification of the complex plane. See Section 4.1 for detailed discussion of the extended plane.

0.2 Complex Differentiation

Definition 0.7. Complex Differentiability, Holomorphicity

Suppose $U \subseteq \mathbb{C}$ is a domain. $f : U \to \mathbb{C}$ is (complex) differentiable at $z_0 \in U$, if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists. We denote the derivative $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$.

If f is differentiable at every $z \in U$, then we say that f is holomorphic in U.

Remark. We will present a necessary condition for complex differentiability, namely the Cauchy-Riemann Equations.

Theorem 0.8. Cauchy-Riemann Equations.

Suppose $U \in \mathbb{C}$ is a domain. Let $f : U \to \mathbb{C}$ be a complex-valued function. We can treat it as a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$, $f : (x, y) \mapsto (u, v)$. If f is complex differentiable, then u and v are real differentiable, and the partial derivatives satisfy:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Proof. The standard basis of $\mathbb{C}$ as a vector space over $\mathbb{R}$ is $\{1, \mathrm{i}\}$. The Jacobian matrix of f with respect to this basis is:

$$\begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix}$$

But with respect to the standard basis, multiplying a complex number $w = r + s\mathrm{i}$ is equivalent to left multiplying a matrix:

$$\begin{pmatrix} r & -s \\ s & r \end{pmatrix}$$

Hence we must have $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ by the complex differentiability. □

Corollary 0.9

The Cauchy-Riemann Equations is also equivalent to $\dfrac{\partial f}{\partial x} = \dfrac{1}{\mathrm{i}}\dfrac{\partial f}{\partial y}$.

Corollary 0.10

If $f : U \to \mathbb{C}$ is complex differentiable, then the derivative is given by:

$$f'(z) = \frac{\partial u}{\partial x}(z) + \mathrm{i}\frac{\partial v}{\partial x}(z) = -\mathrm{i}\frac{\partial u}{\partial y}(z) + \frac{\partial v}{\partial y}(z)$$

Remark. We know from introductory analysis that a mapping $f : U \to \mathbb{R}^2$ is (real) differentiable given that the partial derivatives are continuously differentiable. Next proposition provides a sufficient condition for complex differentiability similar to this.

Proposition 0.11

Suppose $U \in \mathbb{C}$ is a domain. Let $f : U \to \mathbb{C}$ be a complex-valued function. Write $f : (x, y) \mapsto (u, v)$. If u, v are continuously differentiable on U and satisfy the Cauchy-Riemann Equations, then f is holomorphic in U.

Proof. Trivial. □

Remark. Actually we only require the continuity of either of the partial derivatives.

Definition 0.12. Laplacian, Harmonic Functions.

The differential operator $\nabla^2 := \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ acting on twice differentiable functions in $\mathbb{R}^2$ is called the Laplacian. $f : U \to \mathbb{R}$ is called a harmonic function if $f \in \ker \nabla^2$.

Definition 0.13. Wirtinger Derivatives.

The Wirtinger derivatives are defined to be:

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right), \ \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right)$$

Remark. If we factorise the Laplacian, we will obtain:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right) = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}$$

Proposition 0.14

Suppose $U \in \mathbb{C}$ is a domain. Let $f = u + v\mathrm{i} : U \to \mathbb{C}$ be holomorphic on U. Then:

$$\frac{\partial f}{\partial \bar{z}} = 0, \ f' = \frac{\partial f}{\partial z} = 2\frac{\partial u}{\partial z}$$

Sketch of Proof. It directly follows from the Cauchy-Riemann Equations and the definition of Wirtinger derivatives. □

Proposition 0.15

Suppose $U \in \mathbb{C}$ is a domain. Let $f = u + v\mathrm{i} : U \to \mathbb{C}$ be holomorphic on U. If u, v are twice continuously differentiable on U, then they are harmonic on U.

Proof.

$$\nabla^2 u = \nabla^2(\mathrm{Re} f) = \mathrm{Re}(\nabla^2 f) = \mathrm{Re}\left(4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} f\right) = 0$$
$$\nabla^2 v = \nabla^2(\mathrm{Im} f) = \mathrm{Im}(\nabla^2 f) = \mathrm{Im}\left(4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} f\right) = 0$$

Hence u, v are harmonic. □

Proposition 0.16

Suppose $U \in \mathbb{C}$ is a domain. Suppose that $g : U \to \mathbb{R}$ is a twice continuously differerntiable function and $\partial g/\partial z$ is holomorphic. Then g is harmonic on U.

Sketch of Proof. $\dfrac{\partial g}{\partial z}$ is holomorphic $\Longrightarrow \dfrac{\partial}{\partial \bar{z}}\dfrac{\partial g}{\partial z} = 0 \Longrightarrow \nabla^2 g = 0 \Longrightarrow g$ is harmonic. □

Definition 0.17. Harmonic Conjugate.

Suppose $U \in \mathbb{C}$ is a domain. Let $u : U \to \mathbb{R}$ be harmonic. Then $v : U \to \mathbb{R}$ is said to be a harmonic conjugate of u, if $f = u + v\mathrm{i}$ is holomorphic on U.

0.3 Branch Cuts

Definition 0.18. Multifunctions, Branches.

A multi-valued function or a multifunction on $U \subseteq \mathbb{C}$ is a mapping $f : U \to \mathscr{P}(\mathbb{C})$. That is, every point of U is assigned to multiple points in $\mathbb{C}$.

A branch of f on $V \subseteq U$ is a function $g : V \to \mathbb{C}$ such that $g(z) \in f(z)$ for $z \in V$. We will be interested in the holomorphic branches of a multifunction.

Remark. In order to distinguish multifunctions and their branches, we use $[f(x)]$ to emphasize that the image of x is a set instead of a number. This notation is not generally accepted.

Remark. We will see that the complex analogues of some real functions, such as power and logarithm, are in nature multifunctions. The many-valuedness often arises from the argument function.

Example 0.19. Argument.

The argument is a multifunction $\arg : \mathbb{C} \setminus \{0\} \to \mathscr{P}(\mathbb{R})$ such that for $z \in \mathbb{C}$,

$$[\arg(z)] := \{\theta \in \mathbb{R} : z = |z|\,\mathrm{e}^{\mathrm{i}\theta}\}$$

We know that $[\arg(z)]$ is in the form of $\{\theta + 2\pi k : k \in \mathbb{Z}\}$.

We can restrict the image to a 2π segment on $\mathbb{R}$ to eliminate the many-valuedness. For example, the choice $(-\pi, \pi]$ gives a single-valued function, namely the **principal argument**:

$$\mathrm{Arg}(z) \in [\arg(z)] \cap (-\pi, \pi]$$

However, $\mathrm{Arg} : \mathbb{C} \setminus \{0\} \to (-\pi, \pi]$ is not continuous on $\mathbb{C} \setminus \{0\}$. It has a jump discontinuity across the negative real axis. In fact, we can never impose a restriction on the image which make the argument into a continuous function.

Proof. Suppose that $\theta : \mathbb{C} \setminus \{0\} \to \mathbb{R}$ is such a continuous function. Define $k : \mathbb{R} \to \mathbb{R}$ by:

$$k(t) = \frac{1}{2\pi}\left(\theta(e^{it}) + \theta(e^{-it})\right)$$

Then k is continuous on $\mathbb{R}$. Moreover, there exists $m, n : \mathbb{R} \to \mathbb{Z}$ such that

$$k(t) = \frac{1}{2\pi}((t + 2m(t)\pi) + (-t + 2n(t)\pi)) = m(t) + n(t)$$

We can see that k is continuous and takes integer values. Hence k is constant on $\mathbb{R}$. But for $t = \pi$, $\theta(e^{i\pi}) = \theta(e^{-i\pi})$. Hence

$$k(\pi) = \frac{2(\pi + 2m(\pi)\pi)}{2\pi} = 2m(\pi) + 1$$

is odd. On the other hand, for $t = 0$,

$$k(0) = \frac{2(0 + 2m(0)\pi)}{2\pi} = 2m(0)$$

is even. Contradiction. □

Example 0.20. Logarithm.

We know that the exponential function $\exp : \mathbb{R} \to (0, +\infty)$ is differentiable and bijective. So we can define its inverse function, namely the logarithm, on $(0, +\infty)$, which is also bijective and differentiable. However, the exponential function on the complex plane is not injective, so its "inverse" would be a multifunction.

We define the complex logarithm for $z \neq 0$:

$$[\log z] := \exp^{-1}(\{z\}) = \{w \in \mathbb{C} : z = e^w\}$$

If we write $z = |z|e^{i\theta}$ and $w = a + bi$, then

$$e^w = z \implies e^a e^{ib} = |z|e^{i\theta} \implies a = \ln|z|,\ b \in \theta + 2\pi\mathbb{Z}$$

That is,

$$[\log z] = \{\ln|z| + i\theta : \theta \in \arg(z)\} = \ln|z| + i \cdot \arg(z)$$

The imaginary part of the complex logarithm is exactly the argument. So by Example 0.19, there is no continuous branch of the complex logarithm.

When we pick the principal argument, the corresponding single-valued logarithm is called **principal logarithm**:

$$\operatorname{Log} z := \ln|z| + i\operatorname{Arg}(z)$$

The principal logarithm is holomorphic on $\mathbb{C} \setminus \mathbb{R}^-$, with $(\operatorname{Log} z)' = 1/z$. The question that on which domain does the complex logarithm have a holomorphic branch is fully addressed in Corollary 1.39.

Example 0.21. Power Functions.

Similar to real functions, the complex power function is defined in terms of logarithm:

$$[z^\alpha] := [e^{\alpha \log z}] \ \text{ for } z \in \mathbb{C} \setminus \{0\}$$

Therefore power functions are also multifunctions. Especially, the rule $[z^\alpha z^\beta] = [z^{\alpha+\beta}]$ still holds, whereas $[(z_1 z_2)^\alpha] = [z_1^\alpha z_2^\alpha]$ does not hold generally.

For fractional powers, $[z^{1/n}]$ $(n \in \mathbb{Z}^+)$, the image generally contains n points and can be described by the n-th roots of unity. Let $\omega_n := e^{2\pi i/n}$. Then $[1^{1/n}] = \{1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}\}$.

$$\implies [z^{1/n}] = \sqrt[n]{|z|}\, e^{i\operatorname{Arg}(z)/n} \{1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}\}$$

Remark. Given a multifunction f, suppose the many-valuedness arises because the definition of f depends explicitly or implicitly on the argument of one or more points a. Such points are usually excluded from the domain of definition. But from the reasoning in Example 0.19, we can never choose a continuous branch of f in the neighbourhood of a. Such points are called **branch points**. The formal definition is presented below.

Definition 0.22. Branch Points, Branch Cuts.

Suppose $f : U \to \mathscr{P}(\mathbb{C})$ is a multifunction defined on open set $U \subseteq \mathbb{C}$. $z_0 \in \overline{U}$ is said to be a branch point of f, if for $r > 0$, there is no continuous branch of f defined on $B(z_0, r) \setminus \{z_0\}$.

It is also useful to work in the extended complex plane. If U is unbounded, we define $\tilde{f}(1/z) = f(z)$. We say ∞ is a branch point of f, if 0 is a branch point of $\tilde{f}$. Equivalently, ∞ is a branch point of f if only if for $r > 0$, there is no continuous branch of f defined on $U \cap (\mathbb{C} \setminus B(0, r))$.

A branch cut of f is a curve in $\mathbb{C}$ on whose complement we can choose a continuous branch of f. A branch cut must contain all branch points.

Remark. The argument function and complex logarithm both have branch points at 0 and ∞. When we pick the principal argument, we are in fact performing a branch cut at the negative real axis $\mathbb{R}^- = (-\infty, 0]$. Generally we may only consider cuts that are between branch points (or infinity).

Example 0.23

Consider the multifunction $[(z^2-1)^{1/2}]$. We observe that 1 and -1 are two branch points. If we shift to the polar coordinates and write:

$$z = 1 + r\,\mathrm{e}^{\mathrm{i}\theta} = -1 + s\,\mathrm{e}^{\mathrm{i}\varphi}$$

Then $[(z^2-1)^{1/2}] = [\sqrt{rs}\,\mathrm{e}^{\mathrm{i}(\theta+\varphi)/2}]$. If $f(z) = F(r, s, \theta, \varphi)$ is a holomorphic branch of $[(z^2-1)^{1/2}]$, then $F(r, s, \theta, \varphi)$ should be uniquely determined by z. More explicitly, we require that $\mathrm{e}^{\mathrm{i}(\theta+\varphi)/2}$ stays unchanged after $f(z)$ goes along a closed path γ in the cut plane. The branch cut is chosen to restrict the movement on the plane such that all inadmissible closed paths are outlawed.

The idea of winding number introduced in Section 1.4 can help determine the admissibility of closed paths. In this case, suppose $\gamma(t) = 1 + r(t)\,\mathrm{e}^{\mathrm{i}\theta(t)}$ continuously parametrises the closed path. Then the value of θ remains unchanged if 1 is in the exterior of γ, and the value increases by an integer multiple of 2π if 1 is in the interior of γ. This is similar for -1. If we want $\mathrm{e}^{\mathrm{i}(\theta+\varphi)/2}$ be unchanged, then the closed paths that go around either (but not both) of the branch points should be outlawed. We conclude that a holomorphic branch of $[(z^2-1)^{1/2}]$ exists if we perform the branch cut at $[-1, 1]$.

In this cut plane, there are two holomorphic branches:

$$f_\pm(z) = \pm|z^2-1|^{1/2}\,\mathrm{e}^{\mathrm{i}(\theta+\varphi)/2},\ \ \theta \in [\arg(z-1)] \cap (-\pi, \pi],\ \ \varphi \in [\arg(z+1)] \cap [0, 2\pi)$$

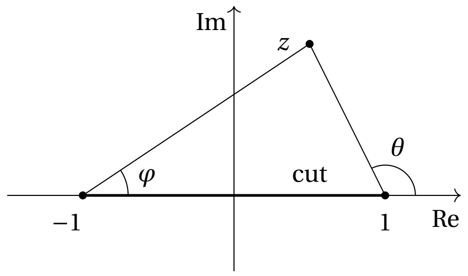


Figure 1: Branch cut along $[-1, 1]$.

Example 0.24

Consider the multifunction $[\log(z^2-1)]$. The branch points are ± 1 and ∞. The polar form:

$$[\log(z^2-1)] = [\ln(rs) + \mathrm{i}(\theta+\varphi)] \ \text{ for } z = 1 + r\,\mathrm{e}^{\mathrm{i}\theta} = -1 + s\,\mathrm{e}^{\mathrm{i}\varphi}$$

In this case, any closed path that encloses either or both of ± 1 is outlawed. We can achieve this by performing a branch cut at $(-\infty, -1] \cup [1, +\infty)$. Holomorphic branches are given, for $k \in \mathbb{Z}$, by:

$$f_k(z) = \ln(rs) + \mathrm{i}(\theta + \varphi + 2k\pi),\ \ \theta \in [\arg(z-1)] \cap [0, 2\pi),\ \ \varphi \in [\arg(z+1)] \cap (-\pi, \pi]$$

0.4 Paths and Integration

0.4.1 Paths.

Recall that a path in $\mathbb{R}^n$ is described by a continuous parametrisation $\gamma : [a,b] \to \mathbb{R}^n$. We adopt this concept in $\mathbb{C}$. We say that a path $\gamma : [a,b] \to \mathbb{C}$ is:

1. **closed**, if $\gamma(a) = \gamma(b)$;
2. **simple**, if $\forall x, y \in (a,b) : \gamma(x) \neq \gamma(y)$;
3. $\mathbf{C^1}$ or **smooth**, if γ is continuously differentiable on (a,b);
4. **piecewise smooth**, if there exists a partition of $[a,b]$: $a = x_0 < x_1 < ... < x_n = b$ such that $\gamma\big|_{[x_{i-1},x_i]}$ is C^1. A piecewise-smooth simple closed curve is also called a **contour**.

Sometimes we denote the image of the path $\gamma^* := \gamma([a,b])$.

Remark. A smooth path does not necessarily have a well-defined tangent at every point, especially where $\gamma'(t) = 0$. Some texts insist that a smooth path must have non-vanishing derivative everywhere. We shall not adopt this convention.

Definition 0.25. Length, Rectifiability.

Suppose $\gamma : [a,b] \to \mathbb{C}$ is a path. Given a partition of $[a,b]$, $\mathscr{P} : a = t_0 < t_1 < ... < t_n = b$, we define:

$$\Lambda(\gamma;\mathscr{P}) := \sum_{i=1}^{n} \left|\gamma(t_i) - \gamma(t_{i-1})\right|$$

We say that γ is rectifiable, or is of bounded variation, if $\sup_{\mathscr{P}} \Lambda(\gamma;\mathscr{P})$ exists, where the supremum is taken over all partitions $\mathscr{P}$ of $[a,b]$. We denote $L(\gamma) := \sup_{\mathscr{P}} \Lambda(\gamma;\mathscr{P})$ the length of the path γ.

Proposition 0.26

Suppose $\gamma : [a,b] \to \mathbb{C}$ is a smooth path, then it is rectifiable and its length is given by:

$$L(\gamma) = \int_a^b |\gamma'|$$

Proof. Please refer to the real analysis for the proof. □

Theorem 0.27. Jordan Curve Theorem.

Suppose that $\gamma : [0,1] \to \mathbb{C}$ is a simple closed path. Then $\mathbb{C} \setminus \gamma^*$ has two connected components, one bounded and one unbounded. Each of the components has γ^* as its boundary.

A simple closed curve on the plane is also called a **Jordan curve**.

Proof. See Section 1.A for a complete proof. □

Remark. Without Jordan Curve Theorem we can still define the interior and exterior of a piecewise-smooth closed path. But we do not know whether the interior is non-empty, nor do we know the connectivity of it. See Section 1.4 for detail.

Definition 0.28. Oriented Curves.

Let $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [c,d] \to \mathbb{C}$ be two paths. We define $\gamma_1 \sim \gamma_2$, if there exists a **diffeomorphism** (continuously differentiable bijection with continuously differentiable inverse) $\varphi : [a,b] \to [c,d]$ such that $\forall\, t \in [a,b] :\ \varphi'(t) > 0$ and $\gamma_1 = \gamma_2 \circ \varphi$.

It is easy to check that this defines an equivalence relation. We denote the equivalence class of γ to be $[\gamma]$ and call it an oriented curve.

Remark. The condition that $\varphi'(t) > 0$ ensures that the curve is traversed in the same direction for every parametrisation.

Definition 0.29. Circles.

For a circle centered at a with radius r, the parametrisation $\gamma(t) = a + r\,\mathrm{e}^{2\pi\mathrm{i}t}$ is said to be positively oriented and is denoted $\gamma(a,r)$. We denote by $\gamma^+(a,r)$ the upper semicircle of $\gamma(a,r)$ and $\gamma^-(a,r)$ the lower semicircle of $\gamma(a,r)$. On the other hand, the parametrisation $\gamma(t) = a + r\,\mathrm{e}^{-2\pi\mathrm{i}t}$ is said to be negatively oriented.

0.4.2 Integration.

Next we define integration on the complex place and investigate the line integral along paths.

Definition 0.30. Integrability of Complex-Valued Functions.

Complex-valued function $f = u + v\mathrm{i} : [a,b] \to \mathbb{C}$ is Riemann (*resp.* Lebesgue) integrable if and only if u and v are Riemann (*resp.* Lebesgue) integrable. Moreover we define:

$$\int_a^b f := \int_a^b u + \mathrm{i}\int_a^b v$$

Proposition 0.31. Triangular Inequality.

Suppose $f : [a,b] \to \mathbb{C}$, then $\left|\int_a^b f\right| \leqslant \int_a^b |f|$.

Proof. Suppose $\int_a^b f = z = |z|\,\mathrm{e}^{\mathrm{i}\theta}$ where $\theta = \mathrm{Arg}(z)$. Decompose f with respect to the orthonormal basis $\{\mathrm{e}^{\mathrm{i}\theta}, \mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}\}$:

$$f(t) = u(t)\,\mathrm{e}^{\mathrm{i}\theta} + \mathrm{i}v(t)\,\mathrm{e}^{\mathrm{i}\theta}$$

where $u, v : [a,b] \to \mathbb{R}$. Integrate:

$$\int_a^b f = \mathrm{e}^{\mathrm{i}\theta}\left(\int_a^b u + \mathrm{i}\int_a^b v\right) = |z|\,\mathrm{e}^{\mathrm{i}\theta}$$

Then we have

$$\int_a^b v = 0,\ \int_a^b u = |z| = \left|\int_a^b f\right| \implies \left|\int_a^b f\right| = \int_a^b u \leqslant \int_a^b \sqrt{u^2+v^2} = \int_a^b |f|$$ □

Remark. We shall give a definition of line integral, which is a special case of the Riemann-Stieltjes integral.

Definition 0.32. (*Riemann-Stieltjes*) Line Integral.

Suppose $U \subseteq \mathbb{C}$. $f : U \to \mathbb{C}$ is a complex function and $\gamma : [a,b] \to U$ is a path. Given a partition of $[a,b]$, $\mathscr{P} : a = t_0 < t_1 < \ldots < t_n = b$, we consider the following Riemann-Stieltjes sum:

$$\Sigma(f,\gamma;\boldsymbol{\xi},\mathscr{P}) := \sum_{i=1}^{n} f\circ\gamma(\xi_i)\big(\gamma(t_i) - \gamma(t_{i-1})\big)$$

where $\boldsymbol{\xi} := (\xi_1,\ldots,\xi_n)$ and $\xi_i \in [t_{i-1}, t_i]$ for each $i = 1,\ldots,n$.

If for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all partitions $\mathscr{P}$ with $\max_{1\leqslant i\leqslant n}|t_i - t_{i-1}| < \delta$ and all intermediate points $\boldsymbol{\xi}$, we have $|\Sigma(f,\gamma;\boldsymbol{\xi},\mathscr{P}) - A| < \varepsilon$, then we say that the line integral of f along γ exists. The limit $A \in \mathbb{C}$ is the value of the line integral of f along γ. We denote:

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma f := A = \lim_{\mathrm{mesh}\mathscr{P}\to 0}\Sigma(f,\gamma;\boldsymbol{\xi},\mathscr{P})$$

Proposition 0.33

If $f : U \to \mathbb{C}$ is continuous and $\gamma : [a,b] \to U$ is rectifiable, then the line integral $\int_\gamma f$ exists.

Proof. Please refer to the real analysis for the proof. □

Remark. The definition of line integral is purely conceptual and we shall never use it in these notes. We are only interested in smooth or piecewise smooth paths. In such case the line integral can be computed as follows:

Proposition 0.34

Suppose $\gamma : [a,b] \to \mathbb{C}$ is a smooth path and $f : \mathbb{C} \to \mathbb{C}$ is integrable. Then we have:

$$\int_\gamma f := \int_a^b (f \circ \gamma) \cdot \gamma'$$

Remark. If γ is only piecewise smooth, we can find a partition $a = x_0 < x_1 < ... < x_n = b$ such that each $\gamma|_{(x_{i-1}, x_i)}$ is smooth. Then the line integral is given by

$$\int_\gamma f := \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (f \circ \gamma) \cdot \gamma'$$

Proposition 0.35

If $\gamma : [a,b] \to \mathbb{C}$ and $\tilde{\gamma} : [c,d] \to \mathbb{C}$ are two piecewise smooth equivalent paths, then for continuous function $f : \mathbb{C} \to \mathbb{C}$, we have $\int_\gamma f = \int_{\tilde{\gamma}} f$. That is, the line integral only depends on the oriented curve $[\gamma]$.

Proof. Suppose φ is a diffeomorphism such that $\gamma = \tilde{\gamma} \circ \varphi$. Then:

$$\begin{aligned}
\int_\gamma f &= \int_a^b (f \circ \gamma) \cdot \gamma' \\
&= \int_a^b (f \circ \tilde{\gamma} \circ \varphi) \cdot (\tilde{\gamma} \circ \varphi)' \\
&= \int_a^b (f \circ \tilde{\gamma} \circ \varphi) \cdot (\tilde{\gamma}' \circ \varphi) \cdot \varphi' \\
&= \int_c^d (f \circ \tilde{\gamma}) \cdot \tilde{\gamma}' && \text{(change of variable)} \\
&= \int_{\tilde{\gamma}} f
\end{aligned}$$

□

Proposition 0.36. Properties of Line Integral.

Suppose $f, g : \mathbb{C} \to \mathbb{C}$ are continuous functions. $\gamma, \eta : [a,b] \to \mathbb{C}$ are piecewise smooth paths. Then:

(i) **Linearity.** $\forall\, \alpha, \beta \in \mathbb{C}$: $\int_\gamma (\alpha f + \beta g) = \alpha \int_\gamma f + \beta \int_\gamma g$

(ii) **Opposite Path.** The opposite path of γ is defined by $\gamma^- : [a,b] \to \mathbb{C}$, $\gamma^-(t) = \gamma(a+b-t)$. Then we have:

$$\int_{\gamma^-} f = -\int_\gamma f$$

(iii) **Additivity.** Suppose $\gamma(b) = \eta(a)$. We define the **concatenation** of γ and η to be: $\gamma \star \eta : [a,b] \to \mathbb{C}$,

$$\gamma \star \eta(t) = \begin{cases} \gamma(2t-a), & t \in [a, (a+b)/2] \\ \eta(2t-b), & t \in [(a+b)/2, b] \end{cases}$$

We have:

$$\int_{\gamma\star\eta} f = \int_\gamma f + \int_\eta f$$

(iv) **Estimation Lemma.**

$$\left|\int_\gamma f\right| \leqslant \sup_{z\in\gamma^*} |f(z)| \cdot L(\gamma)$$

where $L(\gamma)$ is the length of γ.

Proof. (i) follows from the linearity of Riemann integral.

$$\begin{aligned}
\text{(ii)}: \int_{\gamma^-} f &= \int_a^b f\circ\gamma^-(t)(\gamma^-)'(t)\,\mathrm{d}t \\
&= \int_a^b -f\circ\gamma(a+b-t)\cdot\gamma'(a+b-t)\,\mathrm{d}t \\
&= \int_b^a f\circ\gamma(t)\cdot\gamma'(t)\,\mathrm{d}t = -\int_a^b f\circ\gamma(t)\cdot\gamma'(t)\,\mathrm{d}t \\
&= -\int_\gamma f
\end{aligned}$$

$$\begin{aligned}
\text{(iii)}: \int_{\gamma\star\eta} f &= \int_a^b f\circ(\gamma\star\eta)(t)\cdot(\gamma\star\eta)'(t)\,\mathrm{d}t \\
&= \int_a^{\frac{a+b}{2}} f\circ\gamma(2t-a)\cdot 2\gamma'(2t-a)\,\mathrm{d}t + \int_{\frac{a+b}{2}}^b f\circ\eta(2t-b)\cdot 2\eta'(2t-b)\,\mathrm{d}t \\
&= \int_a^b f\circ\gamma(t)\cdot\gamma'(t)\,\mathrm{d}t + \int_a^b f\circ\eta(t)\cdot\eta'(t)\,\mathrm{d}t \\
&= \int_\gamma f + \int_\eta f
\end{aligned}$$

(iv) Since $[a,b]$ is compact and γ is continuous, γ^* is compact. f is continuous so that $|f|(\gamma^*)$ is also compact. Hence $\sup_{z\in\gamma^*}|f(z)|$ exists. We have:

$$\begin{aligned}
\left|\int_\gamma f\right| &= \left|\int_a^b (f\circ\gamma)\cdot\gamma'\right| \leqslant \int_a^b \left|(f\circ\gamma)\cdot\gamma'\right| && \text{(by Proposition 0.31)} \\
&= \int_a^b \left|(f\circ\gamma)\right|\cdot\left|\gamma'\right| \leqslant \sup_{t\in[a,b]}\left|f\circ\gamma(t)\right|\cdot\int_a^b\left|\gamma'\right| = \sup_{z\in\gamma^*}|f(z)|\cdot L(\gamma)
\end{aligned}$$

□

Corollary 0.37

If a sequence of continuous functions $\{f_n\}$ converge uniformly to f on γ^*, then we have:

$$\lim_{n\to\infty}\int_\gamma f_n = \int_\gamma f$$

Proof. By uniform convergence, we have $\sup_{z\in\gamma^*}(f_n(z)-f(z)) \to 0$. By the Estimation Lemma, we have:

$$\left|\int_\gamma f_n - \int_\gamma f\right| = \left|\int_\gamma (f_n - f)\right| \leqslant \sup_{z\in\gamma^*}(f_n(z)-f(z))\cdot L(\gamma) \to 0$$

as $n\to\infty$. Then the result follows. □

Definition 0.38. Primitive.

Suppose $U\subseteq\mathbb{C}$ is a domain. $f: U\to\mathbb{C}$ is a complex function. If there exists a holomorphic function $F: U\to\mathbb{C}$ such that $F' = f$, then we say that F is a primitive of f.

Remark. The existence of primitive on $\mathbb{C}$ is analogous to the existence of scalar potential on $\mathbb{R}^2$.

Theorem 0.39. Fundamental Theorem of Line Integral.

Suppose $U \subseteq \mathbb{C}$ is a domain. $f : U \to \mathbb{C}$ is continuous and $F : U \to \mathbb{C}$ is a primitive of f. $\gamma : [a,b] \to U$ is a piecewise smooth path. Then we have:

$$\int_\gamma f = F \circ \gamma(b) - F \circ \gamma(a)$$

Proof. Suppose there exists a partition $a = x_0 < x_1 < \ldots < x_n = b$ such that each $\gamma\big|_{(x_{i-1},x_i)}$ is smooth. Then:

$$\int_\gamma f = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (f \circ \gamma) \cdot \gamma' = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (F' \circ \gamma) \cdot \gamma' = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (F \circ \gamma)' \text{ by chain rule.}$$

By the second Fundamental Theorem of Calculus, we have:

$$\int_\gamma f = \sum_{i=1}^n \big(F \circ \gamma(x_i) - F \circ \gamma(x_{i-1})\big) = F \circ \gamma(b) - F \circ \gamma(a)$$ □

Corollary 0.40

Suppose $U \subseteq \mathbb{C}$ is a domain. $f : U \to \mathbb{C}$ is integrable and has a primitive. $\gamma : [a,b] \to U$ is a piecewise smooth closed path. Then we have $\oint_\gamma f = 0$

Example 0.41

$f(z) = 1/z$ does not have a primitive on $\mathbb{C} \setminus \{0\}$.

Proof. Let the unit circle be parametrised as $\gamma : [0,1] \to \mathbb{C},\ t \mapsto \mathrm{e}^{2\pi i t}$. Then

$$\int_\gamma f = \int_0^1 \mathrm{e}^{-2\pi i t} \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{2\pi i t})\,\mathrm{d}t = \int_0^1 2\pi \mathrm{i}\,\mathrm{d}t = 2\pi\mathrm{i} \neq 0$$

Hence $1/z$ cannot have a primitive. □

Corollary 0.42

Suppose $U \subseteq \mathbb{C}$ is path-connected. $f : U \to \mathbb{C}$ is holomorphic and satisfies $f' = 0$ on U. Then f is constant on U.

Theorem 0.43

Suppose $U \subseteq \mathbb{C}$ is a domain. $f : U \to \mathbb{C}$ is a continuous function. The following statements are equivalent:

(i) f has a primitive on U;

(ii) $\oint_\gamma f = 0$ along any piecewise smooth closed path $\gamma : [a,b] \to \mathbb{C}$.

(iii) The value of $\int_\gamma f = 0$ only depends on the endpoints of γ.

Proof. (i)$\Longrightarrow$(ii): This is just Corollary 0.40.

(ii)$\Longrightarrow$(iii): Suppose $\gamma, \eta : [a,b] \to$ are piecewise smooth paths such that $\gamma(a) = \eta(a)$ and $\gamma(b) = \eta(b)$. Then $\gamma \star \eta^-$ is a closed path. Moreover by (ii) we have:

$$\int_\gamma f - \int_\eta f = \oint_{\gamma\star\eta^-} f = 0$$

Hence $\int_\gamma f = \int_\eta f$. The value of the integral only depends on the endpoints of the path.

(iii)$\Longrightarrow$(i): Fix $z_0 \in U$. Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth path such that $\gamma(a) = z_0$ and $\gamma(b) = z$. Define $F(z) := \int_\gamma f$. We shall show that $F' = f$.

Fix $z \in U$. $\exists\, \varepsilon > 0$ ($B(z,\varepsilon) \subseteq U$). For $w \in B(z,\varepsilon)$, consider a line segment γ starting from z to w given by $s : [0,1] \to U$ such that $s(t) = z + t(w-z)$. For a path γ_1 from z_0 to z, $\gamma_2 := \gamma_1 \star s$ is a path from z_0 to w. We have:

$$\begin{aligned} F(w) - F(z) &= \int_{\gamma_2} f - \int_{\gamma_1} f = \int_s f \\ &= \int_0^1 f(z + t(w-z)) \cdot (w-z)\,\mathrm{d}t \end{aligned}$$

$$\begin{aligned} \implies \left| \frac{F(w)-F(z)}{w-z} - f(z) \right| &= \left| \int_0^1 f(z+t(w-z))\,\mathrm{d}t - f(z) \right| \\ &\leqslant \int_0^1 |f(z+t(w-z)) - f(z)|\,\mathrm{d}t \\ &\leqslant \sup_{t\in[0,1]} |f(z+t(w-z)) - f(z)| \to 0 \end{aligned}$$

as $w \to z$, by the continuity of f. Hence by definition $F' = f$. F is a primitive of f. □

0.A Appendix: Background in Metric and Topological Spaces

Definition 0.44. Metric Spaces.

Suppose that X is a set and $d : X \times X \to \mathbb{R}$ is a map satisfying

1. Symmetry: $\forall\, x, y \in X: \quad d(x,y) = d(y,x)$;
2. Positivity: $\forall\, x, y \in X: \quad d(x,y) \geqslant 0; \quad d(x,y) = 0 \iff x = y$;
3. Triangular Inequality: $\forall\, x, y, z \in X: \quad d(x,y) \leqslant d(x,z) + d(y,z)$.

(X,d) is called a metric space.

Definition 0.45. Open Balls, Closed Balls.

Suppose that (X,d) is a metric space. We define

$$B(x_0,r) := \{x \in X : \ d(x,x_0) < r\}, \qquad \overline{B}(x_0,r) := \{x \in X : \ d(x,x_0) \leqslant r\}$$

which are the open ball and the closed ball centered at x_0 with radius r, respectively.

Definition 0.46. Interior Points, Open Sets, Neighbourhoods.

Suppose that (X,d) is a metric space and $U \subseteq X$. We say that $x \in U$ is an interior point of U, if:

$$\exists\, r > 0 : \quad B(x,r) \subseteq U$$

The set of interior points of U is denoted by $\mathring{U}$. U is called an open subset of X, if $U = \mathring{U}$.

For $x \in X$, an open set $U \subseteq X$ such that $x \in U$ is called a(n open) neighbourhood of x.

Proposition 0.47. Properties of Open Sets

Suppose that (X,d) is a metric space.

1. $\varnothing$ and X are open sets in X;
2. If $U_1, U_2 \subseteq X$ are open, then so is $U_1 \cup U_2$;
3. If $\{U_i\}_{i\in I} \subseteq \mathscr{P}(X)$ is a collection of open subsets of X, then $\bigcap_{i\in I} U_i$ is open in X.

Definition 0.48. Limit Points, Isolated Points, Closed Sets.

Suppose that (X,d) is a metric space and $U \subseteq X$. We say that $x \in X$ is a limit point of U, if

$$\forall\, r > 0: \quad B(x,r) \cap U \setminus \{x\} \neq \varnothing$$

The set of limit points of U is denoted by U'. U is called a closed subset of X, if $U' \subseteq U$.

If $x \in U$ is not a limit point of U, then it is called an isolated point of U.

Proposition 0.49. Closed Sets are the Complement of Open Sets.

Suppose that (X,d) is a metric space and $U \subseteq X$. U is open in X if and only if $X \setminus U$ is closed in X.

Proposition 0.50. Open/Closed Balls are Open/Closed Sets

Suppose that (X,d) is a metric space. For $x \in X$ and $r > 0$, $B(x,r)$ is open in X and $\overline{B}(x,r)$ is closed in X.

Definition 0.51. Topology.

Suppose that X is a set and $\mathscr{T} \subseteq \mathscr{P}(X)$ is a collection of subsets of X satisfying

1. $\varnothing, X \in \mathscr{T}$;
2. $U_1, U_2 \in \mathscr{T} \implies U_1 \cup U_2 \in \mathscr{T}$;
3. $\{U_i\}_{i\in I} \subseteq \mathscr{T} \implies \bigcap_{i\in I} U_i \in \mathscr{T}$.

Then $(X,\mathscr{T})$ is called a topological space. $\mathscr{T}$ is called a topology on X. The elements of $\mathscr{T}$ are called open sets in X.

Proposition 0.52. Metric Spaces induce Topology.

Suppose that (X,d) is a metric space. Then the open sets in X form a topology on X.

Definition 0.53. Closure.

Suppose that $(X,\mathscr{T})$ is a topological space and $U \subseteq X$. The closure of X is the intersection of all closed sets in X that contain U:

$$\overline{X} := \bigcap_{\substack{V \text{ closed,} \\ U \subseteq V \subseteq X}} V$$

Proposition 0.54. Closure are Closed Sets.

Suppose that $(X,\mathscr{T})$ is a topological space and $U \subseteq X$. Then $\overline{U}$ is closed. In particular, $U = \overline{U}$ if and only if U is closed.

Definition 0.55. Denseness, Separability.

Suppose that $(X,\mathscr{T})$ is a topological space and $U \subseteq X$. U is called a dense subset of X if $\overline{U} = X$. X is said to be separable, if it has a countable dense subset.

Definition 0.56. Boundary.

Suppose that (X,d) is a metric space and $U \subseteq X$. The boundary of U is defined by $\partial U := \overline{U} \setminus \mathring{U}$.

Definition 0.57. Subspace Topology.

Suppose that $(X,\mathscr{T})$ is a topological space. For $Y\subseteq X$, we define $\mathscr{S}:=\{Y\cap U:\ U\in\mathscr{T}\}$. Then $\mathscr{S}$ is a topology on Y, called the subspace topology.

Definition 0.58. Convergence in Topological Spaces

Suppose that $(X,\mathscr{T})$ is a topological space and $\{a_i\}_{i\in\mathbb{N}}$ is a sequence in X. We say that $\{a_i\}$ converges to a, if for any open neighbourhood U of a, there exists $N\in\mathbb{N}$ such that $\{a_i\}_{i=N}^{\infty}$ is in U.

Definition 0.59. Continuity.

Suppose that (X,d_X) and (Y,d_Y) are metric spaces. $f:X\to Y$ is a map. The following statements are equivalent:

1. $\forall x_0\in X\ \ \forall\varepsilon>0\ \exists\delta>0\ \ \forall x\in X:\quad d_X(x,x_0)<\delta\implies d_Y(f(x),f(x_0))<\varepsilon;$
2. $\forall U\subseteq Y:\quad U$ is open in $Y\implies f^{-1}(U)$ is open in X.

Any map satisfying the properties are called *continuous functions* from X to Y.

Remark. If $(X,\mathscr{T})$ and $(Y,\mathscr{S})$ are only topological spaces, then we take the second statement as the definition of a continuous function $f:X\to Y$.

Definition 0.60. Uniform Continuity.

Suppose that (X,d_X) and (Y,d_Y) are metric spaces. $f:X\to Y$ is said to be uniform continuous, if

$$\forall\varepsilon>0\ \exists\delta>0\ \ \forall x,y\in X:\quad d_X(x,y)<\delta\implies d_Y(f(x),f(y))<\varepsilon$$

Definition 0.61. Homeomorphisms.

Suppose that $(X,\mathscr{T})$ and $(Y,\mathscr{S})$ are topological spaces. $f:X\to Y$ is called a homeomorphism, if f is continuous and bijective, with a continuous inverse $f^{-1}:Y\to X$. X and Y are said to be homeomorphic if there exists a homeomorphism $f:X\to Y$.

Theorem 0.62. Invariance of Domain.

Suppose that $U\subseteq\mathbb{R}^n$ is open. If $f:U\to\mathbb{R}^n$ is continuous and injective, then $f(U)$ is open, and $f:U\to f(U)$ is a homeomorphism.

Remark. A similar theorem holds for holomorphic functions on $\mathbb{C}$. See Open Mapping Theorem 2.8 and Inverse Function Theorem 2.9.

Definition 0.63. Completeness.

Suppose that (X,d) is a metric space. It is said to be complete if every Cauchy sequence in X converges.

Remark. $\mathbb{C}$ and $\mathbb{R}^n$ are complete metric spaces.

Theorem 0.64. Cantor's Intersection Theorem.

Suppose that (X,d) is a complete metric space. $\{U_i\}_{i\in\mathbb{N}}$ is a descending chain of non-empty closed sets in X. Then $\bigcap_{i\in\mathbb{N}} U_i$ is non-empty.

Theorem 0.65. Contraction Mapping Theorem.

Suppose that (X,d) is a non-empty complete metric space. Let $f: X \to X$ be a map such that

$$\exists K \in [0,1) \;\; \forall x, y \in X: \quad d(f(x), f(y)) \leqslant K d(x,y)$$

f is called a contraction mapping. There exists a unique $x_0 \in X$ such that $f(x_0) = x_0$.

Definition 0.66. Compactness.

Suppose that $(X,\mathcal{T})$ is a topological space. X is said to be compact if for any open cover $\{U_i\}_{i\in I}$ (a collection of open sets that covers X), there exists a finite subcover $\{U_{i_1},\dots,U_{i_n}\}$.

Definition 0.67. Sequential Compactness.

Suppose that $(X,\mathcal{T})$ is a topological space. X is said to be sequentially compact if any sequence in X has a convergent subsequence.

Definition 0.68. Boundedness, Total Boundedness.

Suppose that (X,d) is a metric space. X is said to be bounded if $\{d(x,y) : x,y \in X\}$ is bounded. X is said to be totally bounded if

$$\forall \varepsilon > 0 \;\; \exists x_1,\dots,x_n \in X: \quad X = \bigcup_{i=1}^{n} B(x_i,\varepsilon)$$

Theorem 0.69

Suppose that (X,d) is a metric space. The following statements are equivalent:

1. X is compact;
2. X is sequentially compact;
3. X is complete and totally bounded.

Theorem 0.70. Heine-Borel Theorem.

Suppose that $X \subseteq \mathbb{R}^n$ is equipped with the Euclidean metric. Then X is compact if and only if X is closed and bounded.

Theorem 0.71. Continuity Functions preserve Compactness.

Suppose that $(X,\mathcal{T})$ and $(Y,\mathcal{S})$ are topological spaces. Let $f: X \to Y$ be a continuous function. If X is compact, then $f(X)$ is compact.

Theorem 0.72. Heine-Cantor Theorem.

Suppose that (X,d_X) and (Y,d_Y) are metric spaces. Let $f: X \to Y$ be a continuous function. If X is compact, then f is uniformly continuous.

Definition 0.73. Equicontinuity.

Suppose that (X,d) is a metric space and $\mathscr{F}$ is a collection of functions from X to $\mathbb{R}$. $\mathscr{F}$ is said to be equicontinuous on X, if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall f \in \mathscr{F} \ \forall x_1, x_2 \in X : \quad d(x_1, x_2) < \delta \implies |f(x_1) - f(x_2)| < \varepsilon$$

Definition 0.74. Uniform Boundedness.

Suppose that (X,d) is a metric space and $\mathscr{F}$ is a collection of functions from X to $\mathbb{R}$. $\mathscr{F}$ is said to be uniformly bounded on X, if there exists $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in X$ and $f \in \mathscr{F}$.

Theorem 0.75. Arzelà-Ascoli Theorem.

Suppose that (X,d) is a **compact** metric space and $\mathscr{F}$ is a collection of continuous functions from X to $\mathbb{R}$ which is **equicontinuous** and **uniformly bounded**. Then any sequence $\{f_i\}_{i\in\mathbb{N}}$ in $\mathscr{F}$ contains a subsequence $\{f_{i_k}\}_{k\in\mathbb{N}}$ that **converges uniformly** on X.

Remark. The theorem can be generalized to complex-valued functions without difficulty.

Definition 0.76. Connectivity.

Suppose that $(X,\mathscr{T})$ is a topological space. X is said to be disconnected if there exists non-empty open subsets $A, B \subseteq X$ such that $A \cup B = X$ and $A \cap B = \varnothing$.

X is said to be connected if it is not disconnected.

Proposition 0.77

Suppose that $(X,\mathscr{T})$ is a topological space. The following statements are equivalent:

1. X is connected;
2. Any continuous function $f : X \to \{0,1\}$ (with the discrete topology) is constant;
3. The only subsets that are both open and closed are $\varnothing$ and X.

Proposition 0.78. Connected Subsets of $\mathbb{R}$.

$I \subseteq \mathbb{R}$ is connected if and only if I is an interval.

Theorem 0.79. Continuity Functions preserve Connectedness.

Suppose that $(X,\mathscr{T})$ and $(Y,\mathscr{S})$ are topological spaces. Let $f : X \to Y$ be a continuous function. If X is connected, then $f(X)$ is connected.

Definition 0.80. Path-Connectivity.

Suppose that $(X,\mathscr{T})$ is a topological space. X is said to be path-connected, if for any $x, y \in X$, there exists a continuous path $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Theorem 0.81

Suppose that $(X, \mathscr{T})$ is a topological space.

1. If X is path-connected, then it is connected;
2. If X is a normed vector space, then X is path-connected if and only if it is connected.

Theorem 0.82. Continuity Functions preserve Path-Connectedness.

Suppose that $(X, \mathscr{T})$ and $(Y, \mathscr{S})$ are topological spaces. Let $f : X \to Y$ be a continuous function. If X is path-connected, then $f(X)$ is path-connected.

Definition 0.83. Connected Components, Path Components.

Suppose that $(X, \mathscr{T})$ is a topological space.

The equivalence relation given by

$$x \sim y \iff \exists \text{connected subset } U \subseteq X : \quad x, y \in U$$

partitions X. The equivalence classes are called connected components of X.

The equivalence relation given by

$$x \sim y \iff \exists \text{continuous path } \gamma : [0,1] \to X \quad \gamma(0) = x,\ \gamma(1) = y$$

partitions X. The equivalence classes are called path components of X.

Proposition 0.84

Suppose that $(X, \mathscr{T})$ is a topological space.

1. The connected components of X are connected;
2. The path components of X are path-connected.

Definition 0.85. Local Connectivity, Local Path-Connectivity.

Suppose that $(X, \mathscr{T})$ is a topological space. X is said to be locally connected (*resp.* locally path-connected), if any point $x \in X$ is contained in a connected (*resp.* locally path-connected) neighbourhood of x.

Proposition 0.86

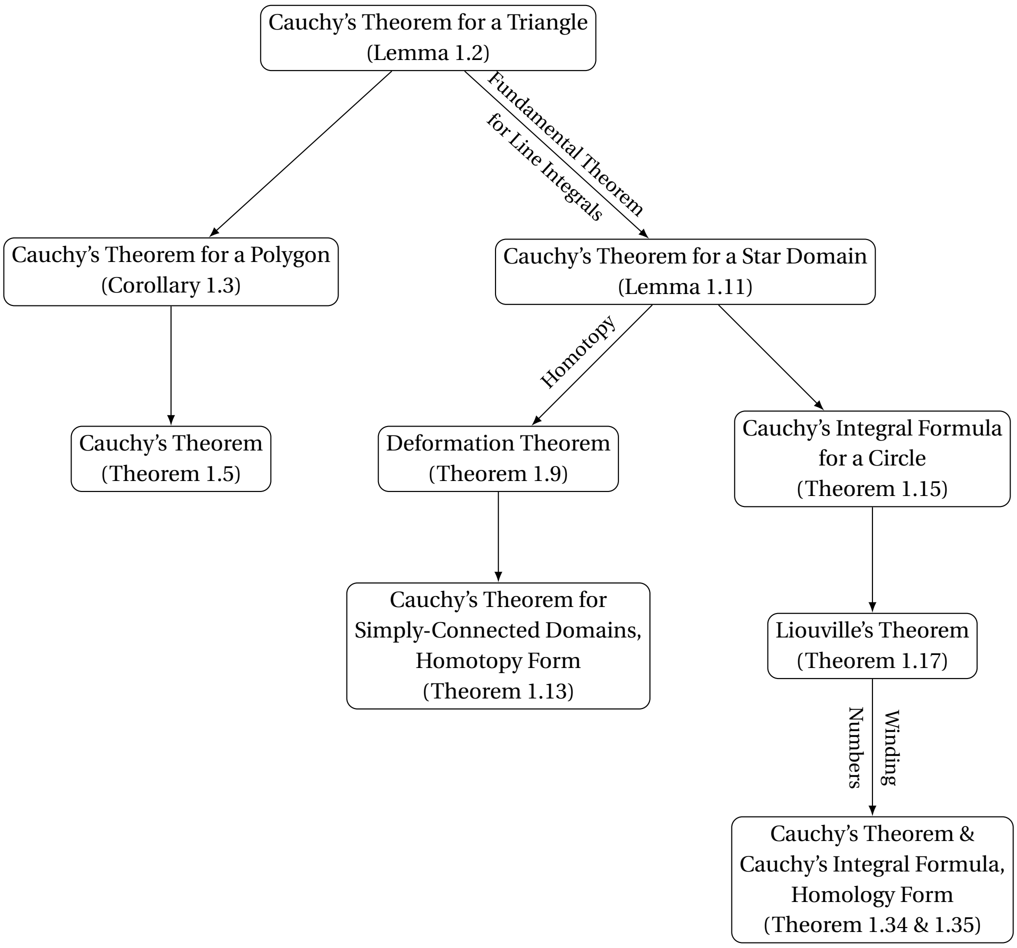
Suppose that $(X, \mathscr{T})$ is a topological space.

1. X is locally connected if and only if for any open subset $U \subseteq X$, the connected components of U are open in X;
2. X is locally path-connected if and only if for any open subset $U \subseteq X$, the path components of U are open in X;

Chapter 1

Cauchy's Integral Theorem

Informally stated, Cauchy's Theorem says that, if $f : U \to \mathbb{C}$ is holomorphic on U and γ is a simple closed curve contained in U, then $\oint_\gamma f = 0$ under some conditions. We shall build up this theorem by a sequence of lemmata and propositions. There are three tracks of proving the theorem. We will develop them respectively in Section 1.1, 1.2 and 1.4. Here is a dendrogram showing the interdependence of each form Cauchy's Theorem.



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1.1 Proof of Cauchy's Theorem, Basic Track

Theorem 1.1. Cauchy's Theorem, 1846

Suppose $U \subseteq \mathbb{C}$ is simply-connected. $f : U \to \mathbb{C}$ is holomorphic on U and its derivative f' is continuous on U. γ is a piecewise-smooth simple closed curve contained in U. Then we have:

$$\oint_\gamma f = 0$$

Remark. This is the original form of Cauchy's Theorem when it was first proposed. The extra condition that f' is continuous on U make it a direct corollary of Green's Theorem for the plane.

Proof. Consider f as a mapping in $\mathbb{R}^2$. $f : (x, y) \mapsto (u, v)$. The line integral on the complex plane has a corresponding form:

$$\oint_\gamma f(z)\,\mathrm{d}z = \oint_\gamma (u + \mathrm{i}v)(\mathrm{d}x + \mathrm{i}\mathrm{d}y) = \oint_\gamma (u\mathrm{d}x - v\mathrm{d}y) + \mathrm{i}\oint_\gamma (v\mathrm{d}x + u\mathrm{d}y)$$

Apply Green's Theorem to f along γ:

$$\iint_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\mathrm{d}x\mathrm{d}y = \oint_\gamma (v\mathrm{d}x + u\mathrm{d}y)$$
$$\iint_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathrm{d}x\mathrm{d}y = \oint_\gamma (u\mathrm{d}x - v\mathrm{d}y)$$

where S is the region enclosed by γ (this is well-defined by Jordan Curve Theorem).

But from Cauchy-Riemann Equations we know that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. Hence $\oint_\gamma f(z)\,\mathrm{d}z = 0$ as claimed. □

Remark. Next we shall loosen the condition on f, of which the holomorphicity on U is sufficient. The result is given by Goursat in 1900. The following proof is adapted from Pringsheim's work published a year after Goursat's.

Lemma 1.2. Cauchy's Theorem for a triangle.

Suppose $U \subseteq \mathbb{C}$ is a domain. $f : U \to \mathbb{C}$ is holomorphic on U. Let T be a triangular path whose interior is contained in U. Then we have:

$$\oint_T f = 0$$

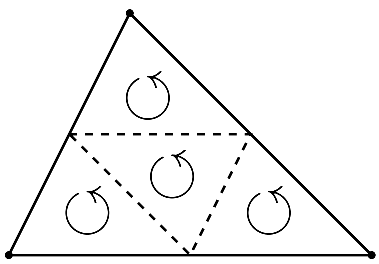

Figure 1.1: Bisecting a triangle.

Proof. "Divide and Conquer". Suppose that $\left|\oint_T f\right| = M$. We are going to prove that $M = 0$.

As shown in Figure 1.1, the image of T is a triangle, which can be divided into four congruent sub-triangles, denoted by T_1, T_2, T_3, T_4. Notice that the integral along the boundary of the inner sub-triangle is cancelled. Hence we have:

$$\oint_T f = \oint_{T_1} f + \oint_{T_2} f + \oint_{T_3} f + \oint_{T_4} f$$

or:

$$\left|\oint_T f\right| \leqslant \sum_{i=1}^{4} \left|\oint_{T_i} f\right|$$

Then $\exists\, i \in \{1,2,3,4\} : \left|\oint_{T_i} f\right| \geqslant \frac{M}{4}$. We denote this sub-triangle $T^{(1)}$ and the original triangle $T^{(0)}$. Start from $T^{(1)}$ and repeat the process, we will inductively obtain a sequence of triangles $T^{(n)}$ such that:

1. $\Delta^{(0)} \subseteq \Delta^{(1)} \subseteq \ldots \subseteq \Delta^{(n)}$ where $\Delta^{(n)}$ denotes the region enclosed by $T^{(n)}$;
2. $\operatorname{diam}(\Delta^{(n)}) = 2^{-n}\operatorname{diam}(\Delta^{(0)})$;
3. $L(T^{(n)}) = 2^{-n} L(T^{(0)})$;
4. $\left|\oint_{T^{(n)}} f\right| \geqslant 4^{-n} M$.

By Cantor Intersection Theorem, the first and the second properties imply that there exists a unique point $z_0 \in \bigcap_{n=0}^{\infty} \Delta^{(n)}$. Since U is simply-connected, $z_0 \in U$. Hence f is differentiable at z_0. For $\varepsilon > 0$ there exists $\delta > 0$ such that for $z \in B(z_0,\delta) \cap U$:

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \varepsilon$$

Choose $n \in \mathbb{N}$ such that $\Delta^{(n)} \in B(z_0,\delta) \cap U$. Then:

$$|f(z)-f(z_0)-f'(z_0)(z-z_0)| < \varepsilon(z-z_0) < \varepsilon \cdot \operatorname{diam}(\Delta^{(n)})$$

Now perform the line integral along $T^{(n)}$:

$$\begin{aligned}\oint_{T^{(n)}} (f(z)-f(z_0)-f'(z_0)(z-z_0))\,\mathrm{d}z &\leqslant \varepsilon \cdot \operatorname{diam}(\Delta^{(n)}) \cdot L(T^{(n)}) \\ &= 4^{-n}\varepsilon \cdot \operatorname{diam}(\Delta^{(0)}) \cdot L(T^{(0)})\end{aligned}$$

But notice that:

$$\oint_{T^{(n)}} (f(z)-f(z_0)-f'(z_0)(z-z_0))\,\mathrm{d}z = \oint_{T^{(n)}} f(z)\,\mathrm{d}z \geqslant 4^{-n} M$$

Hence

$$M \leqslant 4^n \oint_{T^{(n)}} f \leqslant \varepsilon \cdot \operatorname{diam}(\Delta^{(0)}) \cdot L(T^{(0)})$$

Since ε is arbitrary, we conclude that $M = 0$ as claimed. □

Corollary 1.3. Cauchy's Theorem for a polygon.

Suppose $U \subseteq \mathbb{C}$ is a domain. $f : U \to \mathbb{C}$ is holomorphic on U. Let P be a piecewise linear closed path whose interior is contained in U. Then we have:

$$\oint_P f = 0$$

Proof. First we can prove that every simple polygon admits a triangulation by induction on the number of vertices, as shown in Figure 1.2. If P^* is not simple (i.e. it intersects itself), it can be expressed as a finite union of simple polygons, the union of interior of which is the interior of P^*.

After triangulation, the integral of f along the paths in the interior of P^* will be cancelled. What is left is the integral along P. Hence $\oint_P f = \sum \oint_T f = 0$. □

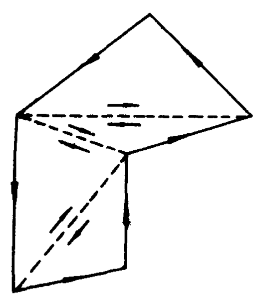

Figure 1.2: Triangulation of a polygon.

Remark. We use the interior of a (not necessarily simple) polygon without defining this concept properly. One way is to invoke the Jordan's Curve Theorem for polygons, which is an elementary fact. The other way is using the winding number introduced in Section 1.4.

Lemma 1.4

Suppose $U \subseteq \mathbb{C}$ is a domain. $f : U \to \mathbb{C}$ is continuous on U and γ is a piecewise smooth path in U. Then for $\varepsilon > 0$ there exists a polygonal (piecewise linear) path P in U such that:

(i) The vertices of P^* are on γ^*;

(ii) $\left|\int_\gamma f - \int_P f\right| < \varepsilon$

Proof. We shall make use of the uniform continuity of f. Since U is open and γ^* is compact, we can find a compact set G such that $\gamma^* \subseteq G \subseteq U$. Then f is uniformly continuous on G by the Heine-Cantor Theorem. Hence:

$$\forall \varepsilon > 0\ \exists \eta > 0\ \forall z, w \in G: \quad |z - w| < \eta \Longrightarrow |f(z) - f(w)| < \frac{\varepsilon}{2L(\gamma)}$$

Let $\rho := \operatorname{dist}(\gamma^*, \partial U) > 0$ and $\delta := \min\{\eta, \rho\}$. Since γ^* is compact, there is a finite open covering: $\gamma^* \subseteq \bigcup_{i=1}^{n} B(z_i, \delta)$ where $z_1, ..., z_n \in \gamma^*$. Let the endpoints of the path be z_0 and z_{n+1}. Now $\{z_0, z_1, ..., z_{n+1}\}$ partitions the curve γ into $n+1$ parts. We connect these points by line segments and denote the corresponding path by P. Since $|z_{i-1} - z_i| < \delta < \rho$, the line segment P_i between z_{i-1} and z_i is contained in U. For the curve γ_i and line segment P_i, we estimate the difference of the integral:

$$\begin{aligned}
\left|\int_{\gamma_i} f - \int_{P_i} f\right| &\leqslant \left|\int_{\gamma_i} f(z)\,\mathrm{d}z - f(z_i)(z_i - z_{i-1})\right| + \left|\int_{P_i} f(z)\,\mathrm{d}z - f(z_i)(z_i - z_{i-1})\right| \\
&= \left|\int_{\gamma_i} (f(z) - f(z_i))\,\mathrm{d}z\right| + \left|\int_{P_i} (f(z) - f(z_i))\,\mathrm{d}z\right| \\
&< \frac{\varepsilon}{2L(\gamma)} \cdot L(\gamma_i) + \frac{\varepsilon}{2L(\gamma)} \cdot |z_i - z_{i-1}| \\
&\leqslant \frac{\varepsilon}{L(\gamma)} \cdot L(\gamma_i)
\end{aligned}$$

Hence we have:

$$\left|\int_\gamma f - \int_P f\right| \leqslant \sum_{i=1}^{n+1} \left|\int_{\gamma_i} f - \int_{P_i} f\right| < \sum_{i=1}^{n+1} \frac{\varepsilon}{L(\gamma)} \cdot L(\gamma_i) = \varepsilon$$ □

Remark. Combining Corollary 1.3 and Lemma 1.4 we finally reach the landmark theorem:

Theorem 1.5. Cauchy-Goursat Theorem.

Suppose $U \subseteq \mathbb{C}$ is a domain. $f : U \to \mathbb{C}$ is holomorphic on U. γ is a piecewise smooth simple closed curve whose interior is contained in U. Then we have:

$$\oint_\gamma f = 0$$

Proof. Fix $\varepsilon > 0$. By Lemma 1.4 we can find a closed piecewise linear path P such that $\left|\int_\gamma f - \int_P f\right| < \varepsilon$. By Corollary 1.3 we have $\oint_P f = 0$, Hence $\left|\oint_\gamma f\right| < \varepsilon$. But ε is arbitrary, we can conclude that $\oint_\gamma f = 0$ as claimed. □

Remark. The Cauchy-Goursat Theorem can be also stated using the concept of primitives: a holomorphic function on a simply-connected domain has a primitive on the domain.

Remark. Again we use the concept of the interior of a simple closed path. In general, Jordan Curve Theorem addresses this problem (see Section 1.A for a complete proof). However, as we are only interested in piecewise smooth paths, we may use winding numbers to define the interior of such closed paths. The discussions are in Section 1.4, where another proof of Cauchy's Theorem is presented.

1.2 Homotopy and Cauchy's Theorem

In this section, we shall develop an alternative way of formulating Cauchy's Theorem. First we introduce the homotopy of curves. Informally, we say that two paths are homotopic in a domain, if one can continuously deform to another.

Definition 1.6. Homotopy.

Suppose $U \subseteq \mathbb{C}$ is a domain and $\gamma, \eta : [0,1] \to U$ are two paths in U with the same endpoints, i.e. $\gamma(0) = \eta(0) = z_0$, $\gamma(1) = \eta(1) = z_1$. We say that γ and η are homotopic, if there exists a continuous function $h : [0,1]^2 \to U$, $(t,s) \mapsto z$, such that:

$$\forall t \in [0,1] : \ h(0,s) = z_0, \ h(1,s) = z_1$$
$$\forall s \in [0,1] : \ h(t,0) = \gamma(t), \ h(t,1) = \eta(t)$$

One should think of h as a family of paths in U indexed by the second variable s which continuously deform γ into η.

Remark. It follows immediately from the definition that homotopy is an equivalence relation. We call the equivalence classes the homotopic classes.

Definition 1.7. Constant Path, Null Homotopy.

If a path $\gamma : [0,1] \to U$ is a constant function, then the image γ^* is just a point and we call this a constant path, denoted c_a as its image is $a \in U$.

A closed path γ starting and ending at $a \in U$ is said to be null homotopic, if it is homotopic to the constant path c_a.

Definition 1.8. Simple-Connectivity

A domain $U \subseteq \mathbb{C}$ is simply-connected, if $\forall z, w \in U$, any two paths starting at z and ending at w are homotopic.

Remark. Equivalently, a domain U is simply-connected if all closed paths starting and ending at a given point $z_0 \in U$ are null-homotopic.

Remark. In the next theorem we shall prove that the line integral only depends on the homotopic class given that the function is holomorphic.

Theorem 1.9. Deformation Theorem.

Suppose $U \subseteq \mathbb{C}$ is a domain. $\gamma, \eta : [0,1] \to U$ are piecewise-smooth paths in U which are homotopic. $f : U \to \mathbb{C}$ is holomorphic on U. Then we have:

$$\int_\gamma f = \int_\eta f$$

Remark. We need some form of Cauchy's Theorem before we prove Theorem 1.9. For now we forget Corollary 1.3 and Theorem 1.5, which depend on some forms of Jordan Curve Theorem. We shall begin with Lemma 1.2 and generalize it to a broader class of domains.

Definition 1.10. Star Domain.

A domain $U \subseteq \mathbb{C}$ is a star domain, if there exists $z \in U$ such that for all $w \in U$, the line segment $[z, w]$ is entirely contained in U.

Remark. Notice the following inclusion relation for concepts:

Convex Domain $\subseteq$ Star Domain $\subseteq$ Simply-Connected Domain $\subseteq$ Domain.

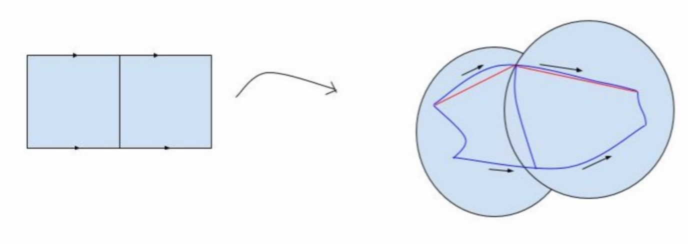

Figure 1.3: Dissecting the homotopy.

Lemma 1.11. Cauchy's Theorem for a star domain.

Suppose $U \subseteq \mathbb{C}$ is a star domain. $f : U \to \mathbb{C}$ is holomorphic on U. $\gamma : [a, b] \to U$ is a piecewise-smooth closed path. Then we have:

$$\oint_\gamma f = 0$$

Proof. It suffices to prove the existence of the primitive of f on U. Suppose $z_0 \in U$ is th point that satisfies the definition of star domain. For every $z \in U$, consider the line segment parametrised by $\gamma_z(t) = z_0 + t(z - z_0)$. We claim that $F(z) = \int_{\gamma_z} f$ is a primitive of $f(z)$. To show this, we fix $z \in U$. $\exists\, \varepsilon > 0$ $(B(z, \varepsilon) \subseteq U)$. For $w \in B(z, \varepsilon)$, consider the line segment parametrised by $\eta(t) = z + t(w - z)$. Notice that the interior of the triangle T with vertices z_0, z, w is entirely contained in U. By Lemma 1.2 we have $\oint_T f = 0$. But T is the concatenation $\gamma_z \star \eta \star \gamma_w^-$. Hence:

$$\int_\eta f = \int_{\gamma_w} f - \int_{\gamma_z} f = F(w) - F(z)$$

And we have:

$$\begin{aligned}\left|\frac{F(w) - F(z)}{w - z} - f(z)\right| &= \left|\frac{1}{w - z}\int_\eta f(\zeta)\,\mathrm{d}\zeta - f(z)\right| \\ &= \left|\frac{1}{w - z}\int_0^1 f(z + t(w - z)) \cdot (w - z)\,\mathrm{d}t - f(z)\right| \\ &\leqslant \sup_{t \in [0,1]} \left|(f(z + t(w - z)) - f(z))\right| \to 0\end{aligned}$$

as $w \to z$ by the continuity of f. Hence $F' = f$ on U. By Corollary 0.40, we have $\oint_\gamma f = 0$ for any piecewise-smooth closed path γ in U. □

Proof of Theorem 1.9. Let $h : [0, 1] \times [0, 1] \to U$ be a homotopy of γ and η. Since $[0, 1]^2$ is compact and h is continuous, h is also uniformly continuous so that we have cover the image $h([0, 1]^2)$ with finitely many disks.

$$\forall\, \varepsilon > 0\, \exists\, \delta > 0\, \forall (t_1, s_1), (t_2, s_2) \in [0, 1]^2 : \|(t_1, s_1) - (t_2, s_2)\| < \delta \Longrightarrow |h(t_1, s_1) - h(t_2, s_2)| < \varepsilon$$

Choose $N \in \mathbb{N}$ such that $N > 1/\delta$. Let us dissect $[0, 1]^2$ into N^2 squares. In order to deform $\gamma(t) = h(t, 0)$ to $\eta(t) = h(t, 1)$, we connect the image of these vertices by piecewise linear paths. More specifically, for $k \in \{0, 1, ..., N\}$, let μ_k be the piecewise linear path that connects the points $h\left(0, \frac{k}{N}\right), h\left(\frac{1}{N}, \frac{k}{N}\right), ..., h\left(1, \frac{k}{N}\right)$. We claim that:

$$\int_\gamma f = \int_{\mu_0} f = \int_{\mu_1} f = ... = \int_{\mu_N} f = \int_\eta f$$

It suffices to prove the second equality and the proof for others are similar. Consider two adjacent squares whose vertices are

$$\left(\frac{j-1}{N}, 0\right), \left(\frac{j}{N}, 0\right), \left(\frac{j+1}{N}, 0\right), \left(\frac{j-1}{N}, \frac{1}{N}\right), \left(\frac{j}{N}, \frac{1}{N}\right), \left(\frac{j+1}{N}, \frac{1}{N}\right)$$

For the left square, by the compactness condition, its image can be covered by a disc $B(h(p_j), \varepsilon)$ where p_j is the center of the square. Therefore by Lemma 1.11, f has a primitive F_j on the disc (disks are star domains). Similarly we cover the image of the right square with another disc $B(h(p_{j+1}), \varepsilon)$ and find a primitive F_{j+1} of f. Since the two disks intersect, F_j and F_{j+1} only differ by a constant. In particular, since $\mu_0(j/N), \mu_1(j/N) \in B(h(p_j), \varepsilon) \cap B(h(p_{j+1}), \varepsilon)$, we have:

$$F_j \circ \mu_0(j/N) - F_j \circ \mu_1(j/N) = F_{j+1} \circ \mu_0(j/N) - F_{j+1} \circ \mu_1(j/N) \tag{1.1}$$

By the Fundamental Theorem for Line Integral, we have:

$$\int_{\mu_0|_{[j-1,j]}} f = F_j \circ \mu_0\left(\frac{j}{N}\right) - F_j \circ \mu_0\left(\frac{j-1}{N}\right)$$
$$\int_{\mu_1|_{[j-1,j]}} f = F_j \circ \mu_1\left(\frac{j}{N}\right) - F_j \circ \mu_1\left(\frac{j-1}{N}\right)$$

After we have covered two paths with N disks, we have:

$$\begin{aligned}
\int_{\mu_0} f &= \sum_{j=1}^{N} \int_{\mu_0|_{[j-1,j]}} f \\
&= \sum_{j=1}^{N} \left(F_j \circ \mu_0\left(\frac{j}{N}\right) - F_j \circ \mu_0\left(\frac{j-1}{N}\right)\right) \\
&= F_N \circ \mu_0(1) - F_1 \circ \mu_0(0) + \sum_{j=1}^{N-1} \left(F_j \circ \mu_0\left(\frac{j}{N}\right) - F_{j+1} \circ \mu_0\left(\frac{j+1}{N}\right)\right) \\
&= F_N \circ \mu_1(1) - F_1 \circ \mu_1(0) + \sum_{j=1}^{N-1} \left(F_j \circ \mu_1\left(\frac{j}{N}\right) - F_{j+1} \circ \mu_1\left(\frac{j+1}{N}\right)\right) \\
&= \int_{\mu_1} f
\end{aligned}$$

where the fourth equality follows from Equation 1.1 and that μ_0 and μ_1 have the same endpoints. □

Remark. We use this cumbersome piecewise linear approximation because we only know the continuity of h, and the integrability of f along $\gamma_k(t) = h(t, k/N)$ is not assumed. [1]

Corollary 1.12

Suppose $U \subseteq \mathbb{C}$ is a domain. γ is a piecewise-smooth closed path which is null homotopic. $f : U \to \mathbb{C}$ is holomorphic on U. Then we have:

$$\oint_\gamma f = 0$$

Theorem 1.13. Cauchy-Goursat Theorem.

Suppose $U \subseteq \mathbb{C}$ is simply-connected. $f : U \to \mathbb{C}$ is holomorphic on U. γ is a piecewise-smooth closed curve contained in U. Then we have:

$$\oint_\gamma f = 0$$

Proof. It immediately follows from the previous corollary and the definition of a simply-connected domain. □

1.3 Cauchy's Integral Formulae

1.3.1 Cauchy's Integral Formulae.

We are now ready to present some important consequences of Cauchy's Theorem. All results in this section are based on Cauchy's Theorem for star domain (Lemma 1.11). The most general form of the theorem via the winding number will be postponed after we present Liouville's theorem and Riemann's Removable Singularity Theorem.

Lemma 1.14

Let $\gamma(a, r)$ be a positively oriented circle. Then for $w \in B(a, r)$ we have:

$$\oint_{\gamma(a,r)} \frac{1}{z-w}\,\mathrm{d}z = 2\pi \mathrm{i}$$

[1] See `https://math.stackexchange.com/questions/44306` for detail.

Proof.

$$\frac{1}{z-w} = \frac{1}{(z-a)-(w-a)} = \frac{1}{z-a}\left(1-\frac{w-a}{z-a}\right)^{-1} = \frac{1}{z-a}\sum_{n=0}^{\infty}\left(\frac{w-a}{z-a}\right)^n$$

Since $(w-a) < (z-a)$ for $z \in \gamma^*$, the series converges uniformly on the image of the circle. Hence by Corollary 0.37 we can integrate term by term:

$$\begin{aligned}
\oint_{\gamma(a,r)} \frac{1}{z-w}\,\mathrm{d}z &= \sum_{n=0}^{\infty}\oint_{\gamma(a,r)} \frac{1}{z-a}\left(\frac{w-a}{z-a}\right)^n \mathrm{d}z \\
&= \sum_{n=0}^{\infty}(w-a)^n \oint_{\gamma(a,r)} \frac{1}{(z-a)^{n+1}}\,\mathrm{d}z \\
&= \sum_{n=0}^{\infty}(w-a)^n \int_0^1 (r\,\mathrm{e}^{2\pi\mathrm{i}t})^{-(n+1)}\cdot 2\pi\mathrm{i}\cdot r\,\mathrm{e}^{2\pi\mathrm{i}t}\,\mathrm{d}t \\
&= 2\pi\mathrm{i}\sum_{n=0}^{\infty}(w-a)^n r^{-n}\int_0^1 \mathrm{e}^{-2n\pi\mathrm{i}t}\,\mathrm{d}t \\
&= 2\pi\mathrm{i} + 2\pi\mathrm{i}\sum_{n=1}^{\infty}(w-a)^n r^{-n}\left(\int_0^1 \cos(-2n\pi t)\,\mathrm{d}t + \mathrm{i}\int_0^1 \sin(-2n\pi t)\,\mathrm{d}t\right) \\
&= 2\pi\mathrm{i}
\end{aligned}$$

The last equality follows from that the integrals $\int_0^1 \cos(-2n\pi t)\mathrm{d}t$ and $\int_0^1 \sin(-2n\pi t)\mathrm{d}t$ are obviously 0 for $n \neq 0$. □

Theorem 1.15. Cauchy's Integral Formula for a Circle.

Suppose $f: U \to \mathbb{C}$ is holomorphic in a domain U that contains $\overline{B}(a,r)$. Then for any $w \in B(a,r)$, we have:

$$f(w) = \frac{1}{2\pi\mathrm{i}}\oint_{\gamma(a,r)} \frac{f(z)}{z-w}\,\mathrm{d}z$$

Proof. Fix $w \in B(a,r)$. Define $g: U \to \mathbb{C}$ by

$$g(z) := \begin{cases} \dfrac{f(z)-f(w)}{z-w} & z \in U \setminus \{w\} \\ f'(w) & z = w \end{cases}$$

Then g is continuous on U and holomorphic on $U \setminus \{w\}$. $U \setminus \{w\}$ is not a star domain. Alternatively, we consider the closed paths Γ_1 and Γ_2 as shown in Figure. Note that Γ_1 is in a star domain which is contained in $U \setminus \{w\}$. By Lemma 1.11 we have $\oint_{\Gamma_1} g = 0$. Similarly, $\oint_{\Gamma_2} g = 0$

The integrals over the linear segments cancel. We have:

$$0 = \oint_{\Gamma_1} g + \oint_{\Gamma_2} g = \oint_{\gamma(a,r)} g + \oint_{\gamma(w,\varepsilon)^-} g$$

. But $\oint_{\gamma(w,\varepsilon)^-} g \to 0$ as $\varepsilon \to 0$, by the continuity of g at w. Hence $\oint_{\gamma(a,r)} g = 0$.

$$\begin{aligned}
&\implies \oint_{\gamma(a,r)} \frac{f(z)-f(w)}{z-w}\,\mathrm{d}z = 0 \\
&\implies \oint_{\gamma(a,r)} \frac{f(z)}{z-w}\,\mathrm{d}z = f(w)\oint_{\gamma(a,r)} \frac{\mathrm{d}z}{z-w} \\
&\implies \oint_{\gamma(a,r)} \frac{f(z)}{z-w}\,\mathrm{d}z = 2\pi\mathrm{i}\cdot f(w) \quad \text{by Lemma 1.14.}
\end{aligned}$$

And the result follows.

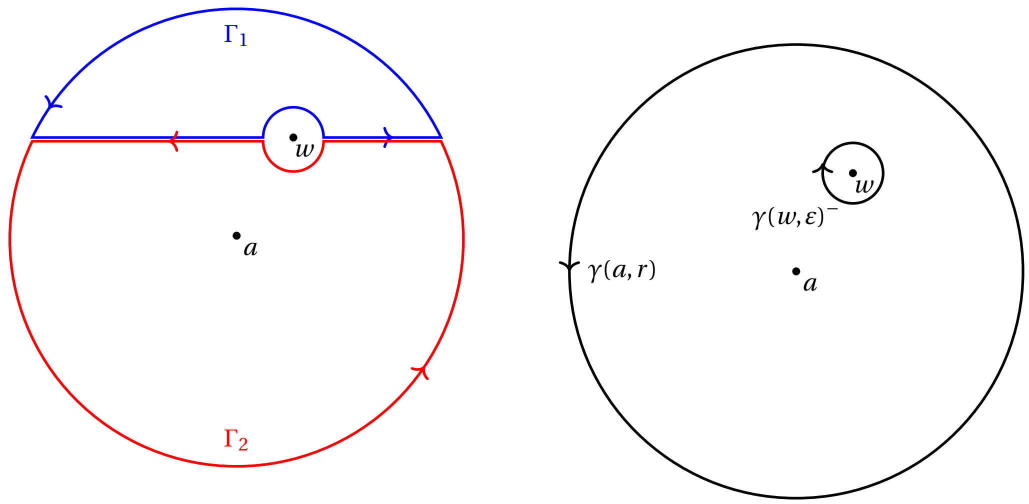

Figure 1.4: Cauchy's Integral Formula.

□

Remark. In fact this result holds for any simple closed positively path. The generalisation of Cauchy's Integral Formula will be presented after we introduce winding number and make sense about orientation and interior in the next section.

Definition 1.16. Entire Functions.

If $f:\mathbb{C}\to\mathbb{C}$ is holomorphic on the whole $\mathbb{C}$, then we call it an entire function.

Theorem 1.17. Liouville's Theorem.

A bounded entire function is constant.

Proof. Suppose $f:\mathbb{C}\to\mathbb{C}$ is entire and bounded by M. For $w\in\mathbb{C}$, choose $r>|w|$ and consider the integral:

$$f(w)=\frac{1}{2\pi\mathrm{i}}\oint_{\gamma(0,r)}\frac{f(z)}{z-w}\,\mathrm{d}z,\qquad f(0)=\frac{1}{2\pi\mathrm{i}}\oint_{\gamma(0,r)}\frac{f(z)}{z}\,\mathrm{d}z$$

Hence

$$\begin{aligned}|f(w)-f(0)|&=\left|\frac{1}{2\pi\mathrm{i}}\oint_{\gamma(a,r)}f(z)\left(\frac{1}{z-w}-\frac{1}{z}\right)\mathrm{d}z\right|\\&=\frac{1}{2\pi}\left|\oint_{\gamma(0,r)}\frac{wf(z)}{z(z-w)}\,\mathrm{d}z\right|\\&\leqslant r\cdot\sup_{z\in\partial B(0,r)}\left|\frac{wf(z)}{z(z-w)}\right|\\&\leqslant r\cdot\frac{|w|M}{r(r-|w|)}=M\cdot\frac{1}{r/|w|-1}\to 0\end{aligned}$$

as $r\to\infty$. Hence $f(w)=f(0)$ for all $w\in\mathbb{C}$. It follows that f is constant. □

Remark. There is a result much stronger than Liouville's Theorem, namely **Picard's Little Theorem**, which states that the image of a non-constant entire function is either $\mathbb{C}$ or $\mathbb{C}\setminus\{z_0\}$ for some point $z_0\in\mathbb{C}$.

Corollary 1.18. Fundamental Theorem of Algebra.

Any non-constant polynomial has a root in $\mathbb{C}$.

Proof. Suppose $p(z) = \sum_{j=0}^{n} a_j z^j \in \mathbb{C}[z]$ has no roots in $\mathbb{C}$. Then $f(z) = 1/p(z)$ is defined on the whole complex plane and is entire. Without loss of generality we may assume that $a_n = 1$. Note that

$$|p(z)| = |z^n| + \left|\sum_{j=0}^{n-1} a_j z^j\right| \geqslant |z^n| \left|1 - \sum_{j=0}^{n-1} \frac{|a_j|}{|z|^{n-j}}\right| \to \infty$$

as $|z| \to \infty$. Hence $f \to 0$ as $|z| \to \infty$. Hence f is bounded. By Liouville's Theorem, f is constant. Then $p(z)$ is also constant. □

Remark. The following theorems shows the powerful aspects of complex analysis. Any holomorphic function is in fact infinitely differentiable and even analytic. This is much more well-behaved than real functions.

Theorem 1.19. Taylor Expansion.

If $f: U \to \mathbb{C}$ is holomorphic on domain $U \subseteq \mathbb{C}$ which contains $\overline{B}(a, r)$, then the power series $\sum_{n=0}^{\infty} c_n (z-a)^n$ converges uniformly to f on $B(a, r)$, where

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi\mathrm{i}} \oint_{\gamma(a,r)} \frac{f(z)}{(z-a)^{n+1}} \,\mathrm{d}z$$

Proof. We will use the same technique as in Lemma 1.14. For $w \in B(a, r)$, by Cauchy's Integral Formula,

$$\begin{aligned} f(w) &= \frac{1}{2\pi\mathrm{i}} \oint_{\gamma(a,r)} \frac{f(z)}{z-w} \,\mathrm{d}z = \frac{1}{2\pi\mathrm{i}} \oint_{\gamma(a,r)} \frac{f(z)}{z-a} \sum_{n=0}^{\infty} \left(\frac{w-a}{z-a}\right)^n \mathrm{d}z \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi\mathrm{i}} \oint_{\gamma(a,r)} \frac{f(z)}{(z-a)^{n+1}} \,\mathrm{d}z\right) (w-a)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (w-a)^n \end{aligned}$$

Hence the Taylor series of f converges to the function on $B(a, r)$. The absolute and uniform convergence follows immediately. □

Remark. The previous theorem demonstrates that all holomorphic functions are analytic. From now on we shall use these two words interchangably. However, some physicists like to call complex differentiable functions "analytic" from the very beginning. From my perspective, this is not appropriate until we prove the Taylor expansion of holomorphic functions.

Remark. We know that any power series is in fact infinitely differentiable. The Taylor expansion of a holomorphic function not only gives a proof that it is analytic, but also gives the explicit formulae for the derivatives of the function.

Corollary 1.20. Infinite Differentiability of Holomorphic Functions.

If $f: U \to \mathbb{C}$ is holomorphic on domain $U \subseteq \mathbb{C}$, then f is infinitely differentiable on U. Moreover, if U contains $\overline{B}(a, r)$, then for any $w \in B(a, r)$, we have:

$$f^{(n)}(w) = \frac{n!}{2\pi\mathrm{i}} \oint_{\gamma(a,r)} \frac{f(z)}{(z-w)^{n+1}} \,\mathrm{d}z$$

Remark. These integral representations for derivatives are also called **Cauchy's Integral Formulae**.

Remark. The next theorem is an immediate corollary of Corollary 1.20 and is a converse to Cauchy's Theorem.

Corollary 1.21. Morera's Theorem.

Suppose $f: U \to \mathbb{C}$ is continuous on a domain $U \subseteq \mathbb{C}$. $\oint_\gamma f = 0$ for any closed path in U. Then f is holomorphic on U.

Proof. By Theorem 0.43, f has a primitive F on U. But by Corollary 1.20, the second derivative $F'' = f'$ exists on U. Hence f is holomorphic on U. □

Remark. In fact the condition for Morera's Theorem can be weakened as follows. Instead of any closed path, $\oint_T f = 0$ for any triangle T is sufficient to deduce that f is holomorphic.

Corollary 1.22. Cauchy's Inequalities.

If $f : U \to \mathbb{C}$ is holomorphic on domain $U \subseteq \mathbb{C}$ which contains $\overline{B}(a,r)$, then the modules of the derivatives of f at a are controlled by:

$$\left|f^{(n)}(a)\right| \leqslant \frac{n!}{r^n} \sup_{z\in\partial B(a,r)} |f(z)|$$

Proof.

$$\begin{aligned}\left|f^{(n)}(a)\right| &= \left|\frac{n!}{2\pi \mathrm{i}} \oint_{\gamma(a,r)} \frac{f(z)}{(z-a)^{n+1}} \,\mathrm{d}z\right| \\ &\leqslant \frac{n!}{2\pi} \cdot 2\pi r \sup_{z\in\partial B(a,r)} \left|\frac{f(z)}{(z-a)^{n+1}}\right| \\ &\leqslant \frac{n!}{r^n} \sup_{z\in\partial B(a,r)} |f(z)|\end{aligned}$$

□

1.3.2 Consequences of Cauchy's Integral Formulae.

Theorem 1.23. Riemann's Removable Singularity Theorem.

Suppose $U \subseteq \mathbb{C}$ is open and $z_0 \in U$. $f : U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic in $U \setminus \{z_0\}$ and is bounded near z_0. Then f can be holomorphically extended on the whole U.

Remark. For this reason, z_0 is called a **removable singularity** of f. We will discuss the classification of isolated singularities in detail in Section 2.2.

Proof. We define $g : U \to \mathbb{C}$ by

$$g(z) := \begin{cases} (z-z_0)^2 f(z), & z \in U \setminus \{z_0\} \\ 0, & z = z_0 \end{cases}$$

Clearly g is holomorphic in $U \setminus \{z_0\}$. Since f is bounded near z_0, we have

$$\frac{g(z) - g(z_0)}{z - z_0} = (z - z_0) f'(z) \to 0$$

as $z \to z_0$. It follows that g is in fact holomorphic on U. Choose $r > 0$ such that $\overline{B}(z_0, r) \subseteq U$. By Theorem 1.19, g has the power series expansion in $B(z_0, r)$:

$$g(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n.$$

Since $g(z_0) = g'(z_0) = 0$, we have $c_0 = c_1 = 0$. Therefore $g(z) = (z-z_0)^2 \sum_{n=0}^{\infty} c_{n-2}(z-z_0)^n$. Now we define $f(z_0) = c_2$. Then we have

$$f(z) = \sum_{n=0}^{\infty} c_{n-2}(z - z_0)^n.$$

which implies that f is holomorphic on $B(z_0, r)$. We conclude that f is holomorphic on $U = (U \setminus \{z_0\}) \cup B(z_0, r)$. □

Remark. The next theorem suggests that the uniform limit of holomorphic functions is holomorphic. This is a very strong result which has no analogy in real analysis.

Theorem 1.24. Weierstrass' Theorem.

Suppose $U \subseteq \mathbb{C}$ is a domain. $f_n : U \to \mathbb{C}$ is a sequence of holomorphic functions on U. If $f_n \to f$ uniformly on any compact subset of U, then f is holomorphic on U. Moreover, $f_n^{(k)} \to f^{(k)}$ uniformly on every compact subset of U for any $k \in \mathbb{N}$.

Proof. Fix $z_0 \in U$. It suffice to prove that f is holomorphic on a neighbourhood of z_0. Let $r > 0$ such that $B(z_0, r) \subseteq U$. For any piecewise-smooth closed path $\gamma : [a, b] \to \overline{B}(z_0, r)$, since f_n is holomorphic in $B(z_0, r)$, we have:

$$\oint_{\gamma} f_n = 0$$

Since γ^* is compact, by the assumption, $f_n \to f$ uniformly on γ^*. Hence:

$$\oint_\gamma f = \lim_{n\to\infty} \oint_\gamma f_n = 0$$

By Morera's Theorem, f is holomorphic in $B(z_0, r)$. Hence f is holomorphic on U.

To prove the second part, note that by Corollary 1.20 f is infinitely differentiable. Fix compact subset $K \subseteq U$. Since U is open and K is compact, we can find a open subset G such that $K \subseteq G \subseteq \overline{G} \subseteq U$. More explicitly, we define $G := \bigcup_{z\in K} B(z, \frac{\rho}{2})$ where $\rho := \inf_{z\in K, w\in\partial U} |z-w|$. By Cauchy's Inequalities, for $z \in K$ we have:

$$f_n^{(k)}(z) \leqslant \frac{k!}{(\rho/2)^k} \sup_{w\in\partial B(z,\rho/2)} |f_n(w)|, \quad f^{(k)}(z) \leqslant \frac{k!}{(\rho/2)^k} \sup_{w\in\partial B(z,\rho/2)} |f(w)|$$

Since $\overline{G}$ is compact, $f_n \to f$ uniformly on $\overline{G}$:

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N: \sup_{z\in\overline{G}} |f_n(z) - f(z)| < \varepsilon$$

Hence:

$$\sup_{z\in K} \left| f_n^{(k)}(z) - f^{(k)}(z) \right| \leqslant \frac{k!}{(\rho/2)^k} \sup_{w\in\overline{G}} \left| f_n(z) - f(z) \right| \leqslant \frac{k!}{(\rho/2)^k} \varepsilon$$

Therefore we conclude that $f_n^{(k)} \to f^{(k)}$ uniformly on every compact $K \subseteq U$. □

Remark. The property "uniformly convergent on any compact subset" is an important property that we shall exploit in Section 4.6. In the language of Definition 4.52, we say that $\{f_n\}$ is a normal family and it converges normally.

Proposition 1.25. Holomorphic Function defined in terms of Integrals.

Suppose $U \subseteq \mathbb{C}$ is a domain. $F : U \times [a, b] \to \mathbb{C}$ is a continuous function. Suppose $z \mapsto F(z, s)$ is holomorphic on U for every $s \in [a, b]$. Then the function defined by:

$$f(z) := \int_a^b F(z, s)\, \mathrm{d}s$$

is holomorphic on U.

Proof using Fubini's Theorem. For any triangle T contained in U,

$$\begin{aligned}
\oint_T f(z)\,\mathrm{d}z &= \oint_T \left(\int_a^b F(z, s)\, \mathrm{d}s \right) \mathrm{d}z \\
&= \int_a^b \left(\oint_T F(z, s)\, \mathrm{d}z \right) \mathrm{d}s \quad \text{(by Fubini's Theorem)} \\
&= \int_a^b 0\, \mathrm{d}s = 0 \quad \text{(since } F(z,s) \text{ is holomorphic in } z\text{)}
\end{aligned}$$

Then by Morera's Theorem, f is holomorphic on U. □

A more rigorous proof. We shall find a sequence of holomorphic functions f_n that converges uniformly to f, so that by the previous theorem we can assert that f is holomorphic. To construct f_n we use partitions of $[a, b]$, which is analgous to the way we approximate the Riemann integral of continuous functions.

Without loss of generality we shift $[a, b]$ to $[0, 1]$. Let $f_n(z) := \frac{1}{n} \sum_{j=1}^n F(z, \frac{j}{n})$. We claim that $f_n(z) \to f(z)$ uniformly on every compact subsets $K \subseteq U$. For compact K, note that F is continuous on compact set $K \times [0, 1]$, and hence is uniformly continuous. Therefore:

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall z \in K\ \forall s, t \in [0, 1]: |s - t| < \delta \Longrightarrow |F(z, s) - F(z, t)| < \varepsilon$$

For $n \in \mathbb{N}$ such that $n > 1/\delta$, we have:

$$|f_n(z) - f(z)| = \left| \frac{1}{n} \sum_{j=1}^n F(z, \frac{j}{n}) - \int_0^1 F(z, s)\, \mathrm{d}s \right|$$

$$= \left| \sum_{j=1}^{n} \int_{(j-1)/n}^{j/n} \left(F(z, \frac{j}{n}) - F(z,s) \right) \mathrm{d}s \right|$$
$$\leqslant \sum_{j=1}^{n} \int_{(j-1)/n}^{j/n} \left| F(z, \frac{j}{n}) - F(z,s) \right| \mathrm{d}s$$
$$\leqslant \sum_{j=1}^{n} \varepsilon/n = \varepsilon$$

Hence $f_n \to f$ on every compact subset $K \subseteq U$. By Weierstrass Theorem f is holomorphic on U. □

1.4 Winding Numbers and Cauchy's Theorem

In this section we wish to define the interior of a simple closed curve by introducing the **winding number** of a closed curve with respect to a certain point. This will allow us to present the ultimate form of Cauchy's theorem, as well as many consequences of it. Informally stated, the winding number is the number of anti-clockwise rotations that a path goes around a point.

1.4.1 Winding Numbers.

Lemma 1.26. Continuous Choices of Argument.

Suppose $\gamma : [0,1] \to \mathbb{C}$ is a path and $z_0 \in \mathbb{C} \setminus \gamma^*$. Then there exists a continuous function $\theta : [0,1] \to \mathbb{R}$ such that:

$$\gamma(t) = z_0 + |\gamma(t) - z_0| \, \mathrm{e}^{\mathrm{i}\theta(t)}$$

Moreover, if θ and φ are two such functions, then

$$\exists\, k \in \mathbb{Z}\ \forall\, t \in [0, 2\pi] :\ \theta(t) - \varphi(t) = 2\pi k$$

Proof. Let $\eta : [0,1] \to S^1 \subseteq \mathbb{C}$ be the path such that $\eta(t) = (\gamma(t) - z_0)/|\gamma(t) - z_0|$. Again we appeal to the uniform continuity of the path. Let $\delta > 0$ such that

$$\forall\, s, t \in [0,1] :\ |s - t| < \delta \Longrightarrow |\eta(s) - \eta(t)| < \sqrt{3}$$

Let $n \in \mathbb{N}$ such that $n > 1/\delta$. Note that for $|z| = |w| = 1$ and $|z - w| < \sqrt{3}$, we must have $|\mathrm{Arg}(z) - \mathrm{Arg}(w)| < 2\pi/3$. Therefore we can define a holomorphic branch of the argument function on this subinterval. That is, for $1 \leqslant j \leqslant n$, there is a continuous function $\theta_j : [\frac{j-1}{n}, \frac{j}{n}] \to \mathbb{R}$ such that $\eta|_{[\frac{j-1}{n}, \frac{j}{n}]}(t) = \mathrm{e}^{\mathrm{i}\theta_j(t)}$.

But for $t = j/n$, we must have $\eta(j/n) = \mathrm{e}^{\mathrm{i}\theta_j(j/n)} = \mathrm{e}^{\mathrm{i}\theta_{j+1}(j/n)}$, which means that $|\theta_j(j/n) - \theta_{j+1}(j/n)| = 2\pi k_j$ for some $k_j \in \mathbb{Z}$. We may choose each k_j such that $\theta_j(j/n) = \theta_{j+1}(j/n)$ for $j \in \{1, \ldots, n-1\}$ and obtain a continuous function $\theta : [0,1] \to \mathbb{R}$ such that $\eta(t) = \mathrm{e}^{\mathrm{i}\theta(t)}$ as claimed.

Moreover, if θ and φ are two such functions, then we have

$$\mathrm{e}^{\mathrm{i}(\theta(t) - \varphi(t))} = 1 \Longrightarrow \theta(t) - \varphi(t) \in 2\pi\mathbb{Z}$$

But $\theta(t) - \varphi(t)$ is continuous, it must take constant value on each connected domain. $[0,1]$ is connected, so $\theta(t) - \varphi(t)$ is a constant integer. □

Remark. The Lemma shows that, if γ is a closed path, then $\mathrm{e}^{\mathrm{i}(\theta(1) - \theta(0))} = 1$. It follows that $\theta(1) - \theta(0) \in 2\pi\mathbb{Z}$. We shall demonstrate that this integer is an important parameter of the path, which is called the winding number.

Definition 1.27. Winding Number.

Suppose $\gamma : [0,1] \to \mathbb{C}$ is a closed path and $z_0 \in \mathbb{C} \setminus \gamma^*$. $\theta : [0,1] \to \mathbb{R}$ is a continuous function such that $\gamma(t) = z_0 + |\gamma(t) - z_0| \, \mathrm{e}^{\mathrm{i}\theta(t)}$. Then we define $I(\gamma, z_0) := \dfrac{\theta(1) - \theta(0)}{2\pi}$ to be the winding number or index of γ about z_0.

By the previous lemma, we know that $I(\gamma, z_0) \in \mathbb{Z}$ and is independent of the choice of $\theta(t)$.

Remark. The winding number can also be interpreted by logarithm. Since $\log(\eta(t)) = \ln|\eta(t)| + \mathrm{i}\theta(t)$, we can define $\theta(t)$ locally by choosing holomorphic branches of $[\mathrm{Log}(\eta(t))]$. Hence we have the line integral form of the winding number:

Proposition 1.28

Suppose γ is a piecewise-smooth closed path and $z_0 \in \mathbb{C} \setminus \gamma^*$. The winding number of γ about z_0 is given by:

$$I(\gamma, z_0) := \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{\mathrm{d}z}{z - z_0}$$

Proof. By Lemma 1.26, we may write $\gamma(t) = z_0 + r(t)\mathrm{e}^{\mathrm{i}\theta(t)}$ where $r(t) = |\gamma(t) - z_0|$. Then $\gamma'(t) = (r'(t) + \mathrm{i}r(t)\theta'(t))\mathrm{e}^{\mathrm{i}\theta(t)}$. Compute the integral:

$$\begin{aligned} \oint_\gamma \frac{\mathrm{d}z}{z - z_0} &= \int_0^1 \frac{(r'(t) + \mathrm{i}r(t)\theta'(t))\mathrm{e}^{\mathrm{i}\theta(t)}}{r(t)\mathrm{e}^{\mathrm{i}\theta(t)}} \mathrm{d}t \\ &= \int_0^1 \frac{r'(t)}{r(t)} \mathrm{d}t + \mathrm{i}\int_0^1 \theta'(t)\mathrm{d}t \\ &= (\ln r(t) + \mathrm{i}\theta(t))_0^1 = 0 + \mathrm{i}(\theta(1) - \theta(0)) \\ &= 2\pi\mathrm{i} \cdot I(\gamma, z_0) \quad \text{(by Definition 1.27.)} \end{aligned}$$

and the result follows. □

Remark. The next corollary reveals the connection between homotopy and winding number.

Corollary 1.29

If γ and η are homotopic piecewise-smooth paths via the homotopy $h : [0,1]^2 \to \mathbb{C}$ and $z_0 \notin h([0,1]^2)$. Then $I(\gamma, z_0) = I(\eta, z_0)$.

Proof.

$$I(\gamma, z_0) = \frac{1}{2\pi\mathrm{i}} \oint_\gamma \frac{\mathrm{d}z}{z - z_0} = \frac{1}{2\pi\mathrm{i}} \oint_\eta \frac{\mathrm{d}z}{z - z_0} = I(\eta, z_0)$$

□

Proposition 1.30

Suppose $U \subseteq \mathbb{C}$ is a domain. $\gamma : [0,1] \to U$ is a piecewise-smooth closed path and f is a function continuous on γ^*. Then the function defined by

$$I_f(\gamma, w) := \oint_\gamma \frac{f(z)}{z - w} \mathrm{d}z$$

is holomorphic on U.

Proof. The proof is similar to the one in Taylor Expansion. Since $\mathbb{C} \setminus \gamma^*$ is open and holomorphicity is a local property, it suffices to show that $I_f(\gamma, w)$ is holomorphic in $B(z_0, r)$ for each $z_0 \in \mathbb{C} \setminus \gamma^*$ and some $r > 0$.

Now fix $z_0 \in \mathbb{C} \setminus \gamma^*$. Let $r = \frac{1}{2} \inf_{t \in [0,1]} |\gamma(t) - z_0|$. Then for $w \in B(z_0, r)$ and $z \in \gamma^*$ we have $\left|\frac{w - z_0}{z - z_0}\right| < \frac{1}{2}$. Moreover, since γ^* is compact, $M = \sup_{z \in \gamma^*} |f(z)|$ exists. Hence:

$$\left| f(z) \frac{(w - z_0)^n}{(z - z_0)^{n+1}} \right| \leqslant \frac{M}{2r} \left(\frac{1}{2}\right)^n$$

By Weierstrass M-test, $\sum_{n=0}^{\infty} f(z) \frac{(w - z_0)^n}{(z - z_0)^{n+1}}$ converges uniformly to $\frac{f(z)}{z - w}$ on γ^*. Hence we have:

$$\oint_\gamma \frac{f(z)}{z - w} \mathrm{d}z = \sum_{n=0}^{\infty} \left(\oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}} \mathrm{d}z \right) (w - z_0)^n$$

Since $I_f(\gamma, w)$ is given by a power series, it is analytic and of course holomorphic. □

Corollary 1.31

Fix the piecewise-smooth closed path γ. The winding number as a function $z \mapsto I(\gamma, z)$ is continuous on $\mathbb{C} \setminus \gamma^*$. Therefore it takes constant value on connected components of $\mathbb{C} \setminus \gamma^*$.

Definition 1.32. Interior of Closed Path.

Suppose γ is a piecewise-smooth closed path. We define the interior of γ to be $\{z \in \mathbb{C} \setminus \gamma^* : I(\gamma, z) \neq 0\}$.

By the previous lemma, it is the union of bounded connected components of $\mathbb{C} \setminus \gamma^*$ (if it is not empty).

We define the exterior of γ to be $\{z \in \mathbb{C} \setminus \gamma^* : I(\gamma, z) = 0\}$. Since $\lim_{z \to \infty} I(\gamma, z) = 0$, the exterior is made of exactly one unbounded connected component of $\mathbb{C} \setminus \gamma^*$.

Definition 1.33. Orientation.

A closed path γ is said to be positively oriented, if $I(\gamma, z_0) > 0$ for z_0 in the interior of γ; γ is said to be negatively oriented, if $I(\gamma, z_0) < 0$ for z_0 in the interior of γ.

Remark. We can see that the definition of orientation above is consistent with our definition of the orientation of circles. In fact, for a simple closed positively oriented curve, $I(\gamma, z) = 1$ for z in the interior and $I(\gamma, z) = 0$ for z in the exterior.

1.4.2 Dixon's Proof of Cauchy's Theorem.

Remark. We shall end this chapter with our final goal: Cauchy's Theorem and Cauchy's Integral Formula in the form of winding numbers. The proof uses Liouville's Theorem, Riemann's Removable Singularity Theorem, and Proposition 1.25. These are consequences of Cauchy's Integral Formula for a circle, which is based on Cauchy's Theorem for star domains. We can see that it is independent of the homotopy form of Cauchy's Theorem.

Theorem 1.34. Cauchy-Goursat Theorem, Homology Form.

Suppose $U \subseteq \mathbb{C}$ is open. $f : U \to \mathbb{C}$ is holomorphic on U. $\gamma : [a, b] \to U$ is a piecewise-smooth closed path whose interior is entirely contained in U, i.e. $I(\gamma, z) = 0$ for all $z \notin U$. Then we have:

$$\oint_\gamma f = 0$$

Theorem 1.35. Cauchy's Integral Formula, Homology Form.

Suppose $U \subseteq \mathbb{C}$ is open. $f : U \to \mathbb{C}$ is holomorphic on U. γ is a piecewise-smooth closed curve whose interior is entirely contained in U. Then for all $w \in U \setminus \gamma^*$:

$$\frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{f(z)}{z - w} \, \mathrm{d}z = I(\gamma, w) \cdot f(w)$$

Proof of Theorem 1.34 and 1.35. We only prove the Cauchy's Integral Formula.

Since $I(\gamma, w) = \dfrac{1}{2\pi \mathrm{i}} \displaystyle\oint_\gamma \frac{f(w)}{z - w} \, \mathrm{d}z$, the formula to be proved can be written as:

$$\oint_\gamma \frac{f(z) - f(w)}{z - w} \, \mathrm{d}z = 0$$

Define $g : U \times U \to \mathbb{C}$ by:

$$g(z, w) := \begin{cases} \dfrac{f(z) - f(w)}{z - w} & z \neq w; \\ f'(z) & z = w. \end{cases}$$

Then g is continuous on $U \times U$. Fix $z \in U$, then $w \mapsto g(z, w)$ is holomorphic on $U \setminus \{z\}$. But by continuity, $w \mapsto g(z, w)$ is bounded near z. Hence by Riemann's Removable Singularity Theorem, $w \mapsto g(z, w)$ is actually holomorphic on the whole U. Then the line integral:

$$F(w) := \oint_\gamma g(z, w) \, \mathrm{d}z = \int_a^b g(\gamma(t), w) \gamma'(t) \, \mathrm{d}t$$

is holomorphic on U by Proposition 1.25.

Let V denote the exterior of γ, i.e. $I(\gamma,w)=0$ for $w\in V$. Define $G:V\to\mathbb{C}$ by $G(w):=\oint_\gamma \frac{f(z)}{z-w}\,\mathrm{d}z$. Then $F(w)$ and $G(w)$ agrees on $U\cap V$:

$$F(w)=\oint_\gamma \frac{f(z)-f(w)}{z-w}\,\mathrm{d}z=G(w)-f(w)I(\gamma,w)=G(w)$$

Hence the function $H:\mathbb{C}\to\mathbb{C}$ defined by:

$$H(w):=\begin{cases}F(w), & w\in U;\\ G(w), & w\in V.\end{cases}$$

is entire. But

$$|H(w)|=\left|\oint_\gamma \frac{f(z)}{z-w}\,\mathrm{d}z\right|\leqslant \frac{L(\gamma)\sup\limits_{z\in\gamma^*}|f(z)|}{|z|-\sup\limits_{t\in[a,b]}|\gamma(t)|}\to 0$$

as $|z|\to\infty$. Then by Liouville's Theorem, H is constant on $\mathbb{C}$. That is $H(w)=0$ for $w\in\mathbb{C}$. And the formula follows immediately. □

Definition 1.36. Cycles.

Suppose $\gamma_1,\cdots,\gamma_n$ are closed paths in $\mathbb{C}$ and $a_1,\cdots,a_n\in\mathbb{C}$. We define a cycle to be the formal sum $\Gamma:=\sum_{i=1}^n a_i\gamma_i$. The line integral along a cycle is defined by:

$$\oint_\Gamma f=\sum_{i=1}^n a_i\oint_{\gamma_i} f$$

Since winding numbers can be expressed as integrals, we can naturally define the winding number of Γ to be $I(\Gamma,z):=\sum_{i=1}^n a_i I(\gamma_i,z)$, which is well-defined for $z\notin\Gamma^*:=\bigcup_{i=1}^n\gamma_i^*$. We define the interior of Γ to be the set of $z\in\mathbb{C}$ such that $I(\Gamma,z)\neq 0$.

Corollary 1.37

Theorem 1.34 and 1.35 also holds for cycles.

Remark. Recall that we first define simple-connectivity in terms of the interior of a closed path. After introducing homotopy, we redefine simply-connected domains to be domains such that paths with the same endpoints are homotopic. We summarise the properties related to simply-connectivity in the next proposition. To complete the whole proof, we have to use the Riemann's Mapping Theorem in Chapter 3, which states that simply-connected domains are not only homeomorphic, but also conformally equivalent to the unit disk.

Proposition 1.38. Equivalent Formulations of Simple-Connectivity.

Suppose $U\subseteq\mathbb{C}$ is a domain. Then the following statements are equivalent:

(i) U is homeomorphic to the unit disk $\mathbb{D}$;

(ii) U is simply-connected (all paths with the same endpoints are homotopic);

(iii) Any piecewise-smooth closed path in U is null-homotopic;

(iv) The interior of any piecewise-smooth closed path is contained in U;

(v) For any piecewise-smooth closed path γ in U and holomorphic function f on U, we have: $\oint_\gamma f=0$;

(vi) Any holomorphic function f on U has a primitive.

(vii) If f and $1/f$ are both holomorphic on U, then there exists a holomorphic function g on U, such that $f=\exp\circ g$.

Proof. (i)$\Longrightarrow$(iii): By homeomorphism, there exists a continuous bijection $\varphi:U\to\mathbb{D}$ such that φ^{-1} is also continuous. For any piecewise-smooth closed path $\gamma:[0,1]\to U$, suppose $\gamma(0)=\gamma(1)=z_0$ and define the homotopy by:

$$H(t,s):=\varphi^{-1}(s\cdot\varphi\circ\gamma(t)+(1-s)\varphi(z_0))$$

Then H is obviously continuous. Moreover, $H(t,0) = z_0$; $H(t,1) = \gamma(t)$; and $H(0,s) = H(1,s) = z_0$. We claim that the image $H([0,1]^2) \subseteq U$. To show this, it suffices to show that the line segment $s\cdot\varphi\circ\gamma(t)+(1-s)\varphi(z_0)$, $s \in [0,1]$ is contained in $\mathbb{D}$ for all $t \in [0,1]$. This is true because $\varphi(\gamma^*) \subseteq \mathbb{D}$ and $\mathbb{D}$ is convex. Hence we conclude that H is a homotopy between γ and z_0. γ is null-homotopic.

(ii)$\Leftrightarrow$(iii): Trivial.

(iii)$\Longrightarrow$(iv): Suppose γ is a closed path in U. We have to assume that γ is piecewise-smooth.

For $w \in \mathbb{C}\setminus U$, note that $\dfrac{1}{z-w}$ is holomorphic in U. Then by Cauchy's Theorem,

$$I(\gamma,w) = \oint_\gamma \frac{1}{z-w}\,\mathrm{d}z = 0$$

Hence the interior of γ $\{z \in \mathbb{C}\setminus\gamma^* : I(\gamma,z) \neq 0\} \subseteq U$.

(iv)$\Longrightarrow$(v): This is Theorem 1.34.

(v)$\Longrightarrow$(vi): This is Theorem 0.43.

(vi)$\Longrightarrow$(vii): Since f has no roots in U, f'/f is holomorphic on U. By (vi), there exists a function $g : U \to \mathbb{C}$ such that $g' = f'/f$. By adding a constant to g, we can have $f(z_0) = \exp(g(z_0))$ for some $z_0 \in U$. But

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(f\,\mathrm{e}^{-g}\right) = f'\,\mathrm{e}^{-g} - fg'\,\mathrm{e}^{-g} = 0$$

Hence $f\,\mathrm{e}^{-g} = const$ on U. The condition that $f(z_0) = \mathrm{e}^{g(z_0)}$ implies that the constant is 1. Hence $f = \mathrm{e}^g$ on U as claimed.

(vii)$\Longrightarrow$(i): This is a direct corollary of Riemann's Mapping Theorem 4.51. See the remark after the proof of the theorem. ☐

Corollary 1.39. Global Existence of Holomorphic Logarithm.

Suppose $U \subseteq \mathbb{C}$ is simply connected and $0 \notin U$, then there exists a holomorphic branch of complex logarithm on U.

Proof. Let $f(z) = z$ in part (vii) of the previous theorem. Then there exists a holomorphic function g on U such that $\mathrm{e}^{g(z)} = z$. Hence g is a holomorphic branch of the complex logarithm on U. ☐

1.A Appendix: Proof of Jordan Curve Theorem*

We shall formulate a quick proof of the full Jordan Curve Theorem using the tools from fundamental groups and covering spaces in algebraic topology.[1] We shall quote a few topological theorems without proof:

Theorem 1.40. Seifert-van Kampen Theorem.

Suppose that X is a topological space. Let $X = X_1 \cup X_2$, where X_1 and X_2 are path-connected open subsets of X such that $X_1 \cap X_2$ is also path-connected. For $b \in X_1 \cap X_2$, the push-out of the based set

$$(X_1,b) \xleftarrow{\iota_1} (X_1\cap X_2,b) \xrightarrow{\iota_2} (X_2,b)$$

induces the push-out of the fundamental group

$$\pi_1(X_1,b) \xleftarrow{\iota_{1*}} \pi_1(X_1\cap X_2,b) \xrightarrow{\iota_{2*}} \pi_1(X_2,b)$$

which is isomorphic to $\pi_1(X,b)$.

Theorem 1.41. Homotopy Extension Lemma.

Suppose that X is a topological space such that $X\times[0,1]$ satisfies the T_4 axiom. Let A be a closed subset of X, and Y be an open subset of $\mathbb{R}^n$. If $f : A \to Y$ is a null-homotopic continuous map, then it can be extended to a continuous map $\tilde{f} : X \to Y$ which is also null-homotopic.

[1] The proof is adapted from Munkres' *Topology*.

We state the Jordan Curve Theorem for $\mathbb{S}^2$:

Theorem 1.42. Jordan Curve Theorem for $\mathbb{S}^2$.

Suppose that $\gamma : [0,1] \to \mathbb{S}^2$ is a simple closed path. Then $\mathbb{S}^2 \setminus \gamma^*$ has two connected components, each of which has γ^* as its boundary.

We break down the proof of Jordan Curve Theorem into a sequence of lemmata.

Lemma 1.43

Suppose that $\gamma : [0,1] \to \mathbb{S}^2$ is an injective path. Then $\mathbb{S}^2 \setminus \gamma^*$ is connected.

Proof. Suppose that $a, b \in \mathbb{S}^2 \setminus \gamma^*$. We shall show that a and b lie in the same connected component.

We first apply a stereographic projection $\mathbb{S}^2 \setminus \{b\} \to \mathbb{C}$ (see Section 4.1). Let $z_0 \in \mathbb{C}$ be the image of a, and C be the image of γ^*. Since C is compact, there is a unique unbounded component of $\mathbb{C} \setminus C$. Let U be the connected component of $\mathbb{C} \setminus C$ containing z_0. It suffices to prove that U is unbounded.

Suppose for contradiction that U is bounded. Let V be the union of other connected components of $\mathbb{C} \setminus C$. Then V is unbounded. Consider the identity map $\mathrm{id}_C : C \to C$. Since C is the image of an injective path, it is contractible. So id_C is null-homotopic. Since $C \times [0,1]$ is a metric space, it satisfies the T_4 axiom. C is a closed subset of $U \cup C$. By Homotopy Extension Lemma, $\mathrm{id}_C : C \to C$ extends to a continuous map $\alpha : U \cup C \to C$, which is also null-homotopic. We extend α to $\beta : \mathbb{C} \to V \cup C$ by setting $\beta|_V = \mathrm{id}_V$.

Since $U \cup C$ is bounded, there exists $r > 0$ such that $U \cup C \subseteq B(z_0, r)$. By restricting β on $\overline{B}(z_0, r)$, we obtain a continuous map $\delta : \overline{B}(z_0, r) \to V \cup C$ where $\delta|_{\partial B(z_0,r)} = \mathrm{id}_{\partial B(z_0,r)}$. Note that $z_0 \notin V \cup C$. Consider the retraction $f : \mathbb{C} \setminus \{z_0\} \to \partial B(z_0, r)$, $f(z) = z_0 + r\dfrac{z - z_0}{|z - z_0|}$. It is easy to see that $f \circ \delta$ is a deformation retraction from $\overline{B}(z_0, r)$ to $\partial B(z_0, r)$. But it is impossible, as $\pi_1(\overline{B}(z_0, r)) = \{e\}$ whereas $\pi_1(\partial B(z_0, r)) = \mathbb{Z}$. □

Lemma 1.44

Suppose that $\gamma : [0,1] \to \mathbb{S}^2$ is a simple closed path. Then $\mathbb{S}^2 \setminus \gamma^*$ has **at least** two connected components.

Proof. $\mathbb{S}^2 \setminus \gamma^*$ is an open subset of a normed vector space. The path components of $\mathbb{S}^2 \setminus \gamma^*$ are exactly the connected components.

Let x_1 and x_2 be two distinct points on γ^*. Let C_1 and C_2 be the arcs between x_1 and x_2. That is, $\gamma^* = C_1 \cup C_2$ and $C_1 \cap C_2 = \{x_1, x_2\}$. Let $X_1 = \mathbb{S}^2 \setminus C_1$ and $X_2 = S^2 \setminus C_2$. By the previous lemma, X_1 and X_2 are both connected open subsets of $\mathbb{S}^2$. $X_1 \cup X_2 = \mathbb{S}^2 \setminus \{x_1, x_2\}$.

Suppose for contradiction that $\mathbb{S}^2 \setminus \gamma^*$ is connected. Then $X_1 \cap X_2 = \mathbb{S}^2 \setminus \gamma^*$ is path-connected. Fix $x_0 \in \mathbb{S}^2 \setminus \gamma^*$, By Seifert-van Kampen Theorem, $\pi_1(X)$ is isomorphic to the push-out

$$\pi_1(X_1, x_0) \xleftarrow{\iota_{1*}} \pi_1(\mathbb{S}^2 \setminus \gamma^*, x_0) \xrightarrow{\iota_{2*}} \pi_1(X_2, x_0)$$

We consider the group homomorphism $j_{1*} : \pi_1(X_1, x_0) \to \pi_1(X_1 \cup X_2, x_0)$ induced by the inclusion map $j_1 : X_1 \hookrightarrow X_1 \cup X_2$. We claim that it is a trivial homomorphism.

Consider a closed path $\eta : [0,1] \to X_1$ with $\eta(0) = \eta(1) = x_0$. We shall show that $j_1 \circ \eta$ is null-homotopic in $\mathbb{S}^2 \setminus \{x_1, x_2\}$. As in the previous lemma we consider the stereographic projection $p : \mathbb{S}^2 \setminus \{x_1\} \to \mathbb{C}$. Let $z_0 := p(x_2)$. Then $p \circ j_1 \circ \eta$ is a closed path in $\mathbb{C} \setminus \{z_0\}$, whose image is denoted by C. Note that x_1 and x_2 are path-connected in $\mathbb{S}^2$ via C_1, away from η^*. Since p maps x_1 to infinity, z_0 lies in the unbounded connected component of $\mathbb{C} \setminus C$. C is bounded on $\mathbb{C}$. There exists $r > 0$ such that $C \subseteq B(z_0, r)$. Pick $z_1 \in \mathbb{C} \setminus B(z_0, r)$. Then z_0 and z_1 are path-connected in $\mathbb{C} \setminus C$. Let $\xi : [0,1] \to \mathbb{C} \setminus C$ be a path connecting z_0 and z_1. We can define a homotopy $G : [0,1]^2 \to \mathbb{C} \setminus \{z_0\}$

$$G(t, s) = p \circ j_1 \circ \eta(t) - \xi(s) + z_0.$$

It is clear that $G(t, s) \neq z_0$ since $p \circ j_{1*} \circ \eta$ and ξ are disjoint paths.

Finally we define a homotopy $H : [0,1]^2 \to \mathbb{C} \setminus \{z_0\}$

$$H(t, s) = (1 - s)(p \circ j_{1*} \circ \eta(t)) - z_1 + z_0.$$

It is clear that $H(t,s) \neq z_0$ since $|p \circ j_1 \circ \eta(t) - z_0| < r < |z_1 - z_0|$.

Since $p \circ j_1 \circ \eta(t) = G(t,0)$, $G(t,1) = H(t,0)$, and $H(t,1) = z_0 - z_1$, we deduce that $p \circ j_{1*} \circ \eta$ is homotopic to a constant path in $\mathbb{C} \setminus \{z_0\}$. Hence $j_1 \circ \eta$ is null-homotopic in $\mathbb{S}^2 \setminus \{x_1, x_2\}$. It follows that j_{1*} is a trivial homomorphism. Similarly, $j_{2*} : \pi_1(X_2, x_0) \to \pi_2(X_1 \cup X_2, x_0)$ is also a trivial homomorphism.

Next we claim that $\pi_1(X_1 \cup X_2)$ is a trivial group. Consider the universal property of the push-out of fundamental groups:

$$\begin{array}{ccccc} & & \pi_1(X_1) & & \\ & \overset{\iota_{1*}}{\nearrow} & \downarrow \theta_1 & \overset{j_{1*}}{\searrow} & \\ \pi_1(X_1 \cap X_2) & & \pi_1(X_1) *_{\pi_1(X_1 \cap X_2)} \pi_1(X_2) & \overset{\exists! \sigma}{\dashrightarrow} & \pi_1(X_1 \cup X_2) \\ & \underset{\iota_{2*}}{\searrow} & \uparrow \theta_2 & \underset{j_{2*}}{\nearrow} & \\ & & \pi_1(X_2) & & \end{array}$$

By Seifert-van Kampen Theorem, $\pi_1(X_1) *_{\pi_1(X_1 \cap X_2)} \pi_1(X_2) \cong \pi_1(X_1 \cup X_2)$. Since j_{1*} and j_{2*} are trivial, the maps θ_1 and θ_2 are also trivial. The unique induced map σ has to be both a trivial homomorphism and an isomorphism. We conclude that $\pi_1(X_1 \cup X_2) = \{e\}$.

But this leads to a contradiction, because $\pi_1(X_1 \cup X_2) = \pi_1(\mathbb{S}^2 \setminus \{x_1, x_2\}) \cong \mathbb{Z}$. We conclude that $\mathbb{S}^2 \setminus \gamma^*$ is not path-connected. □

Lemma 1.45

Suppose that $\gamma : [0,1] \to \mathbb{S}^2$ is a simple closed path. Then $S^2 \setminus \gamma^*$ has **at most** two connected components.

Proof. As in the previous lemma, let x_1 and x_2 be two distinct points on γ^*. Let C_1 and C_2 be the arcs between x_1 and x_2. Let $U = \mathbb{S}^2 \setminus C_1$ and $V = S^2 \setminus C_2$. $U \cup V = \mathbb{S}^2 \setminus \{x_1, x_2\}$ and $U \cap V = \mathbb{S}^2 \gamma^*$. Suppose for contradiction that $U \cap V = \mathbb{S}^2 \setminus \gamma^*$ has more than two connected components. Let X_1, X_2 be two of them and W be the union of the rest.

First we shall construct a covering space for $U \cup V$. Let $Y := (U \times 2\mathbb{Z}) \sqcup (V \times (2\mathbb{Z}+1))$. Define an equivalence relation:

$$\forall n \in \mathbb{Z} \ \ \forall x \in X_1 \cup X_2 : \quad (x, 2n) \sim (x, 2n-1)$$
$$\forall n \in \mathbb{Z} \ \ \forall x \in W : \quad (x, 2n) \sim (x, 2n+1)$$

Define the quotient space $E := Y / \sim$. Let $\pi : Y \twoheadrightarrow E$ be the quotient map. Let $\rho : Y \twoheadrightarrow U \cup V$, $\rho(x,n) = x$ induces the map $p : E \to U \cup V$ by $\rho = p \circ \pi$. It is not hard to verify that p is a covering map. Therefore E is a covering space of $U \cup V$.

We fix $a_1 \in X_1$, $a_2 \in X_2$ and $b \in B$. Construct the following paths:

$$\begin{aligned} \alpha : [0,1] \to U, &\quad \alpha(0) = a_1,\ \alpha(1) = b; \\ \beta : [0,1] \to V, &\quad \beta(0) = b,\ \beta(1) = a_1; \\ \delta : [0,1] \to U, &\quad \delta(0) = a_1,\ \delta(1) = a_2; \\ \lambda : [0,1] \to V, &\quad \lambda(0) = a_2,\ \lambda(1) = a_1. \end{aligned}$$

Let $f := \alpha \star \beta$ and $g := \delta \star \lambda$. They are loops in $U \cup V$ based at a_1. Since E is connected, f and g has unique based liftings in E, which we are going to construct.

Now we consider the lifting of f. Let $\tilde{\alpha}_n(t) = \pi(\alpha(t), 2n)$ and $\tilde{\beta}_n(t) = \pi(\beta(t), 2n+1)$. They are liftings of α and β repsectively. Let $\tilde{f}_n := \tilde{\alpha}_n \star \tilde{\beta}_n$. Then $\tilde{f}_n$ is a path in E such that $\tilde{f}_n(0) = \pi(a_1, 2n)$ and $\tilde{f}_n(1) = \pi(a_1, 2(n+1))$. If we fix the based point $\pi(a_1, 0) \in E$, the path f^m has the unique lifting in E

$$\tilde{f^m} := \tilde{f}_0 \star \cdots \star \tilde{f}_{m-1}$$

where $\tilde{f^m}(0) = \pi(a_1, 0)$ and $\tilde{f^m}(1) = \pi(a_1, 2m)$.

Now we consider the lifting of g. Let $\tilde{\delta}_n(t) = \pi(\delta(t), 2n)$ and $\tilde{\lambda}_n(t) = \pi(\lambda(t), 2n-1)$. Since $\pi(a_1, 2n) = \pi(a_1, 2n-1)$ and $\pi(a_2, 2n) = \pi(a_2, 2n-1)$, $\tilde{g}_n$ is a loop in E such that $\tilde{g}_n(0) = \tilde{g}_n(1) = (a_1, 2n)$ The path g^k has the unique lifting in E

$$\tilde{g^k} := \tilde{g}_0 \star \cdots \star \tilde{g}_0$$

where $\tilde{g^k}(0) = \tilde{g^k}(1) = \pi(a_1, 0)$.

Since $\tilde{g^k}(1) = \tilde{f^m}(1)$ if and only if $m = 0$, it follows that $[f]^m \neq [g]^k$ for all $m, k \in \mathbb{Z} \setminus \{0\}$. $[f] \in \pi_1(U \cup V)$ is non-trivial as we have shown. But $[g] \in \pi_1(U \cup V)$ is also non-trivial because $X_1 \cup X_2$ is disconnected: construct another covering space of $U \cup V$, where the roles of $X_1 \cup X_2$ and W are replaced by $X_1 \cup W$ and X_2.

In summary, $[f], [g] \in \pi_1(U \cup V)$ are non-trivial elements such that $[f]^m \neq [g]^k$ for all $m, k \in \mathbb{Z} \setminus \{0\}$. This is impossible because $\pi_1(U \cup V) = \pi_1(\mathbb{S}^2 \setminus \{x_1, x_2\}) = \mathbb{Z}$. □

Lemma 1.46

Suppose that $\gamma : [0,1] \to \mathbb{S}^2$ is a simple closed path. Let X_1 and X_2 be the two connected components of $\mathbb{S}^2 \setminus \gamma^*$. Then $\partial X_1 = \partial X_2 = \gamma^*$.

Proof. Since $\mathbb{S}^2$ is locally connected, X_1 and X_2 are open. In particular $\partial X_1 \cap X_1 = \varnothing$ and $\partial X_2 \cap X_2 = \varnothing$. It is clear that $\partial X_1 \cap X_2 = \varnothing$ and $\partial X_2 \cap X_1 = \varnothing$. Therefore $\partial X_1, \partial X_2 \subseteq \gamma^*$.

For the reverse inclusion, suppose that $x \in \gamma^*$. We shall show that $x \in \partial X_1$. Let U be an open neighbourhood of x. Choose $x_1, x_2 \in U$ such that one of the arcs connecting x_1 and x_2 is entirely in U. Denote it by C_1 and denote the other arc by C_2 Choose $a \in X_1$ and $b \in X_2$. Since $\mathbb{S}^2 \setminus C_2$ is path-connected, there exists a path $\alpha : [0,1] \to S^2 \setminus C_2$ such that $\alpha(0) = a$ and $\alpha(1) = b$. Since X_1 and X_2 are open and disjoint, and α^* is connected, then $\alpha^* \cap \partial X_1 \neq \varnothing$. But $\partial X_1 \subseteq \gamma^*$ and $\alpha^* \cap C_2 = \varnothing$. Hence $\alpha^* \cap C_1 \neq \varnothing$. In other words, there exists $y \in C_1 \subseteq U$ such that $y \in \partial X_1$. It follows that $x \in \overline{paX_1} = \partial X_1$. Similarly, $x \in \partial X_2$.

In conclusion, we have $\partial X_1 = \partial X_2 = \gamma^*$. □

Theorem 0.27. Jordan Curve Theorem for $\mathbb{C}$.

Suppose that $\gamma : [0,1] \to \mathbb{C}$ is a simple closed path. Then $\mathbb{C} \setminus \gamma^*$ has two connected components, one bounded and one unbounded. Each of the components has γ^* as its boundary.

Proof. Choose $x_0 \in \mathbb{S}^2 \setminus \gamma^*$ and consider the stereographic projection $\mathbb{S}^2 \setminus \{x_0\} \to \mathbb{C}$. □

Chapter 2

Series Representation of Functions

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2.1 Taylor Series

We have shown in Section 1.3 that holomorphic functions on a disk can be expanded into Taylor Series:

Theorem 1.19. Taylor Expansion.

If $f: U \to \mathbb{C}$ is holomorphic on domain $U \subseteq \mathbb{C}$ which contains $\overline{B}(a,r)$, then the power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ converges to f absolutely on $B(a,r)$ and uniformly on any compact subset of $B(a,r)$, where

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi \mathrm{i}} \oint_{\gamma(a,r)} \frac{f(z)}{(z-a)^{n+1}} \, \mathrm{d}z$$

2.1.1 Identity Theorem.

Lemma 2.1

Suppose f is holomorphic in a domain $U \subseteq \mathbb{C}$. Let $S := f^{-1}(\{0\})$ be the set of zeros of f. $z_0 \in S$ is an isolated point in S, then there exists a unique $k \in \mathbb{N}$ and a holomorphic function $g: U \to \mathbb{C}$ such that $f(z) = (z-z_0)^k g(z)$ for all $z \in U$ and $g(z_0) \neq 0$.

Proof. f is analytic at z_0. So $f(z) = \sum_{i=0}^{\infty} c_n (z-z_0)^n$ for all $z \in B(z_0.r) \subseteq U$. Since z_0 is an isolated zero of f, not all c_n are zero. Let k be the smallest integer such that $c_n \neq 0$. Clearly $c_0 = 0$ so $k \geqslant 1$.

Define $g(z) := (z-z_0)^{-k} f(z) = \sum_{i=0}^{\infty} c_{n+k}(z-z_0)^n$, which is holomorphic on $U \setminus \{z_0\}$ and continuous on U. $g(z_0) = c_k \neq 0$, and by continuity there exists $\varepsilon > 0$ such that $g(z) \neq 0$ on $B(z_0, \varepsilon)$. Hence $f(z) = (z-z_0)^k g(z)$ is non-zero on $B(z_0, \varepsilon) \setminus \{z_0\}$.

To prove the uniqueness of k, let $f(z) = (z-z_0)^k g(z) = (z-z_0)^l h(z)$. If $k < l$, then $g(z) = f(z)(z-z_0)^{-k} = (z-z_0)^{l-k} h(z)$. Since $h(z_0) \neq 0$, letting $z \to z_0$ we have $g(z) \to 0$, which contradicts that $g(z_0) \neq 0$. Similarly we cannot have $k > l$. Hence $k = l$. □

Definition 2.2. Multiplicity of zeros.

Suppose f is holomorphic in a domain $U \subseteq \mathbb{C}$ and $z_0 \in f^{-1}(\{z_0\})$ is an isolated. Then we define the multiplicity of z_0 to be the unique integer k in the previous lemma.

Theorem 2.3. Identity Theorem.

Suppose f is holomorphic in a domain $U \subseteq \mathbb{C}$. The following statements are equivalent:

(i) $f(z) = 0$ for all $z \in U$;

(ii) $\exists\, a \in U\ \forall\, k \in \mathbb{N}:\ f^{(k)}(a) = 0$;

(iii) The set $f^{-1}(\{0\})$ has a limit point in U.

Proof. (i)$\Longrightarrow$(ii): Trivial.

(ii)$\Longrightarrow$(iii): There exists $r > 0$ such that $\overline{B}(a,r) \subseteq U$. By Theorem 1.19, we have:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n = 0$$

for all $z \in B(a,r)$. Then $f^{-1}(\{0\})$ contains $B(a,r)$ and hence has a limit point.

(iii)$\Longrightarrow$(i): Let S be the set of limit points of $f^{-1}(\{0\})$ in U. Since f is continuous on U, $f^{-1}(\{0\})$ is closed. Therefore we have $S \subseteq f^{-1}(\{0\})$. Suppose z_0 is a limit point of $f^{-1}(\{0\})$. f is analytic at z_0. So $f(z) = \sum_{i=0}^{\infty} c_n (z-z_0)^n$ for all $z \in B(z_0.r) \subseteq U$. If there exists a non-zero coefficient c_n, then by the previous lemma f is non-zero on some deleted neighbourhood of z_0, contradicting that z_0 is a limit point. Hence $f(z) = 0$ on $B(a,r)$. Hence z_0 is an interior point of S and S is open in U. But by definition S is closed in U. Since U is connected and $S \neq \varnothing$, we must have $S = U \subseteq f^{-1}(\{0\})$. That is, $f(z) = 0$ for all $z \in U$. □

Corollary 2.4

Suppose f and g are holomorphic in a domain $U \subseteq \mathbb{C}$. The following statements are equivalent:

(i) $f(z) = g(z)$ for all $z \in U$;

(ii) $\exists\, a \in U\ \forall\, k \in \mathbb{N}:\ f^{(k)}(a) = g^{(k)}(a)$;

(iii) The set $S := \{z \in U : f(z) = g(z)\}$ has a limit point in U.

Proof. Simply apply the Identity Theorem to $f - g$. □

2.1.2 Argument Principle & Rouché's Theorem.

Theorem 2.5. Argument Principle for Holomorphic Functions.

Suppose $U \subseteq \mathbb{C}$ is a domain and $f : U \to \mathbb{C}$ is holomorphic. Let γ be a piecewise-smooth simple closed path in U and f is non-zero on γ. Then we have:

$$N = \frac{1}{2\pi \mathrm{i}} \oint_{\gamma} \frac{f'(z)}{f(z)}\,\mathrm{d}z$$

where N is the number of zeros of f inside γ (counting multiplicity).

Proof. Suppose f has zeros $a_1, \cdots, a_n$ with multiplicity $m_1, \cdots, m_n$ in the interior of γ. Without loss of generality, suppose that γ is positively oriented. Since $a_1, \cdots, a_n$ are isolated zeros, we can choose $r_1, \cdots, r_n$ such that $B(a_i, r_i)$ are mutually disjoint and that $\bigcup_{i=1}^{n} B(a_i, r_i)$ is in the interior of γ. Now consider the cycle $\Gamma := \gamma - \sum_{i=1}^{n} \gamma(a_i, r_i)$, we observe that all the zeros are in the exterior of Γ. By Cauchy's Theorem:

$$0 = \oint_{\Gamma} \frac{f'(z)}{f(z)}\,\mathrm{d}z = \oint_{\gamma} \frac{f'(z)}{f(z)}\,\mathrm{d}z - \sum_{i=1}^{n} \oint_{\gamma(a_i, r_i)} \frac{f'(z)}{f(z)}\,\mathrm{d}z$$

For each $i \in \{1, \cdots, n\}$, $g_i(z) = (z - a_i)^{-m_i} f(z)$ is holomorphic and non-zero on U. Moreover, we have:

$$\frac{f'(z)}{f(z)} = \frac{m_i (z-a_i)^{m_i - 1} g_i(z) + (z-a_i)^{m_i} g_i'(z)}{(z-a_i)^{m_i} g_i(z)} = \frac{g_i'(z)}{g_i(z)} + \frac{m_i}{z - a_i}$$

But g_i'/g_i is holomorphic in $B(a_i, r_i)$, we have:

$$0 = \oint_{\gamma(a_i, r_i)} \frac{g_i'(z)}{g_i(z)} \mathrm{d}z = \oint_{\gamma(a_i, r_i)} \frac{f'(z)}{f(z)} \mathrm{d}z - m_i \oint_{\gamma(a_i, r_i)} \frac{1}{z - a_i} \mathrm{d}z$$
$$\Longrightarrow \oint_{\gamma(a_i, r_i)} \frac{f'(z)}{f(z)} \mathrm{d}z = 2\pi \mathrm{i} m_i$$

Hence
$$\oint_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z = \sum_{i=1}^n \oint_{\gamma(a_i, r_i)} \frac{f'(z)}{f(z)} \mathrm{d}z = 2\pi \mathrm{i} \sum_{i=1}^n m_i$$

which completes the proof. □

Remark. To see why the theorem is called "argument principle", observe that the winding number of $f \circ \gamma$ about the origin is just:

$$I(f \circ \gamma, 0) = \oint_{f \circ \gamma} \frac{\mathrm{d}w}{w} = \int_a^b \frac{f' \circ \gamma(t) \cdot \gamma'(t)}{f \circ \gamma(t)} \mathrm{d}t = \oint_\gamma \frac{f'(z)}{f(z)} \mathrm{d}z$$

So the number of zeros of f inside γ is the same as the winding number of $f \circ \gamma$ about the origin.

Remark. Theorem 2.5 can be generalized to meromorphic functions. See Theorem 2.23.

Theorem 2.6. Rouché's Theorem.

Suppose $U \subseteq \mathbb{C}$ is a domain and f, g are holomorphic functions on U. Suppose γ is a piecewise-smooth closed path in U. If $|f(z)| > |g(z)|$ for all $z \in \gamma^*$, then f and $f + g$ have same number of zeros (counting multiplicity) in the interior of γ.

Proof. Since $|f(z)| > |g(z)|$ for all $z \in \gamma^*$, we can see that $f + tg$ have no zeros on γ^* for all $t \in [0, 1]$. Define:

$$F(z, t) = \frac{f'(z) + tg'(z)}{f(z) + tg(z)}$$

Then $F(z, t)$ is continuous on $\gamma^* \times [0, 1]$ and $z \mapsto F(z, t)$ is holomorphic for all $t \in [0, 1]$. Hence the function $n(t)$ defined by the integral:

$$n(t) := \oint_\gamma F(z, t) \, \mathrm{d}z$$

is continuous on $[0, 1]$.

But $n(t)$ is the number of zeros of $f + tg$ in the interior of γ, by argument principle, and hence is integer-valued. Since $[0, 1]$ is connected, $n(t)$ is constant. $n(0) = n(1)$. Therefore f and $f + g$ have the same number of zeros. □

Remark. As an application of the argument principle, Rouché's Theorem implies that a holomorphic function can be slightly perturbed without changing the number of its zeros.

2.1.3 Maximum Modulus Principle.

Definition 2.7. Open Mapping.

A mapping is said to be open if it maps open sets to open sets.

Theorem 2.8. Open Mapping Theorem.

A holomorphic and non-constant function on a domain $U \subseteq \mathbb{C}$ is an open mapping.

Proof. Suppose $f : U \subseteq \mathbb{C}$ is holomorphic and non-constant. To show that $f(U)$ is open, it suffices to show that

$$\forall\, w_0 \in f(U)\ \exists \varepsilon > 0 :\ B(w_0, \varepsilon) \subseteq f(U).$$

Let $z_0 \in U$ such that $w_0 = f(z_0)$. Then the function $g(z) := f(z) - w_0$ has an isolated zero at z_0. Then there exists $r > 0$ such that $g(z) \neq 0$ on $\overline{B}(z_0, r) \setminus \{z_0\}$. Since $\partial B(z_0, r)$ is compact, there exists $\varepsilon > 0$ such that $|g(z)| > \varepsilon$ on $\partial B(z_0, r)$. Then for all $w \in B(w_0, \varepsilon)$, we have $|g(z)| > \varepsilon > |w_0 - w|$ on $\partial B(z_0, r)$. By Rouché's Theorem, $h(z) := g(z) + (w_0 - w) = f(z) - w$ also has a zero in $B(z_0, r)$. Hence $w \in f(U)$. Hence $B(w_0, \varepsilon) \subseteq f(U)$ as claimed. □

Theorem 2.9. Inverse Function Theorem.

Suppose $f : U \to \mathbb{C}$ is holomorphic.

1. If f is injective, then $f'(z) \neq 0$ for all $z \in U$;
2. If $f'(z) \neq 0$ for all $z \in U$, then f is locally injective.

For the first case, the inverse function $g : f(U) \to U$ is holomorphic on $f(U)$. Moreover, we have $g' = 1/(f' \circ g)$. If γ is a piecewise smooth closed path in U, then

$$g(w) = \oint_\gamma \frac{z f'(z)}{f(z) - w}\,\mathrm{d}z$$

for w in the interior of γ.

Proof. The proof of (1) and (2) are exactly the same as in real analysis. But there are simple proofs for holomorphic functions that make use of Rouché's Theorem.

(1). Suppose for contradiction that there exists $a \in U$ with $f'(a) = 0$. We claim that there exists $R > 0$ such that $f(z) \neq f(a)$ and $f'(z) \neq 0$ in the deleted neighbourhood $B(a, R) \setminus \{a\}$. Indeed, if the first condition does not hold, then for each sufficiently large $n \in \mathbb{N}$ there exists $z_n \in B(a, 1/n) \setminus \{a\}$ such that $f(a) = f(z_n)$. Then a is a limit point in the set $\{z \in \mathbb{C} : f(z) = f(a)\}$. By identity theorem $f(z) = f(a)$ for all $z \in U$, contradicting that f is injective. If the second condition does not hold, for similar reason we must have $f'(z) = 0$ for all $z \in U$, again contradicting that f is injective.

Let $\varepsilon = \inf\limits_{z \in \partial B(a,R)} |f(z) - f(a)| > 0$ and $w \in B(f(a), \varepsilon) \setminus \{f(a)\}$. For $z \in \partial B(a, R)$:

$$|f(z) - f(a)| \geqslant \varepsilon > |w - f(a)| = |(f(z) - w) - (f(z) - f(a))|$$

By Rouché's Theorem, $f(z) - f(a)$ and $f(z) - w$ have the same number of zeros in $B(a, R)$. However, $f(z) - f(a)$ has exactly at least two zeros in $B(a, R)$ because $f'(a) = 0$, whereas $f(z) - w$ has exactly one zero because f is injective and $f'(z) \neq 0$. This is a contradiction. Hence $f'(a) \neq 0$.

(2). For the beginning the proof is the same as the previous part. The continuity of f at a implies that there exists $\delta > 0$ such that $f(B(a, \delta)) \subseteq B(f(a), \varepsilon)$, where ε is defined as above. Let $r := \min\{R, \delta\}$. We claim that f is injective in $B(a, r)$. Suppose for contradiction that there exists $z_1, z_2 \in B(a, r)$ such that $w = f(z_1) = f(z_2)$. By Rouché's Theorem, $f(z) - f(a)$ and $f(z) - w$ have the same number of zeros in $B(a, R)$. However, $f(z) - f(a)$ has exactly one simple zero in $B(a, R)$ because $f'(a) \neq 0$, whereas $f(z) - w$ has at least two zeros, contradiction. Hence f is injective in $B(a, r)$.

The continuity of the inverse function follows from the open mapping theorem. By open mapping theorem, for any open set $V \subseteq U$, the pre-image of V under g is $f(V)$, which is open. Hence g is continuous.

$g' = 1/(f' \circ g)$ follows directly from the definition of derivatives.

Let $\Gamma := f \circ \gamma$. Γ is a closed path. By Cauchy's Integral Formula:

$$g(w) = \oint_\Gamma \frac{g(\zeta)}{\zeta - w}\,\mathrm{d}\zeta$$

As $\zeta = f(z)$, $\mathrm{d}\zeta = f'(z)\mathrm{d}z$, we have:

$$g(w) = \oint_\gamma \frac{g \circ f(z)}{f(z) - w} f'(z)\,\mathrm{d}z = \frac{z f'(z)}{f(z) - w}\,\mathrm{d}z$$ □

Remark. Sometimes we call an injective holomorphic function a biholomorphism or a univalent function. In Corollary 4.13 we shall show that biholomorphisms are angle-preserving mappings.

Theorem 2.10. Maximum Modulus Principle.

Suppose $U \subseteq \mathbb{C}$ is a domain and $f : U \to \mathbb{C}$ is holomorphic. Then $|f|$ cannot attain a maximum value in U.

Proof. Suppose there exists $z_0 \in U$ such that $|f(z_0)|$ attains a maximum value. But $f(U)$ is open by the open mapping theorem, so $f(z_0)$ is an interior point. There exists $z \in U$ such that $|f(z)| > |f(z_0)|$, which is a contradiction. □

Corollary 2.11

Suppose $U \subseteq \mathbb{C}$ is a domain and $\overline{U}$ is compact. f is holomorphic on U and continuous on $\overline{U}$. Then $|f|$ attains a maximum value on ∂U.

Proof. Since $\overline{U}$ is compact and $|f|$ is continuous, $f(\overline{U})$ is compact. $|f|$ attains maximum on $\overline{U}$. But by maximum modulus principle, $|f|$ cannot attain maximum in U, so it attains maximum on $\overline{U} \setminus U = \partial U$. □

2.2 Laurent Series and Isolated Singularities

2.2.1 Laurent Series.

A Laurent series is a generalisation of the power series. We called a series of the form $\sum_{n=-\infty}^{+\infty} c_n(z-z_0)^n$ a **Laurent series**. The power series part $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ is called the **holomorphic part** and the negative part $\sum_{n=-1}^{-\infty} c_n(z-z_0)^n$ is called the **principal part**. We shall see that many properties of the Laurent series depend on the principal part. Just as holomorphic functions can be expanded into Taylor series in an open disk, we shall prove that they can be expanded into Laurent series in an open annulus.

Definition 2.12. Open Annulus.

A open annulus $A(z_0, r, R)$ is a open set on the complex plane defined by

$$A(z_0, r, R) = B(z_0, R) \setminus \overline{B}(z_0, r) = \{z \in \mathbb{C} : r < |z - z_0| < R\}$$

Theorem 2.13. Laurent Expansion.

If f is holomorphic on a domain $U \subseteq \mathbb{C}$ which contains the annulus $A(z_0, r, R)$, then the Laurent series $\sum_{n=-\infty}^{+\infty} c_n(z-a)^n$ converges to f uniformly on any compact subset of $A(z_0, r, R)$, where

$$c_n = \frac{1}{2\pi \mathrm{i}} \oint_{\gamma(z_0,\rho)} \frac{f(z)}{(z-z_0)^{n+1}} \,\mathrm{d}z$$

for any $\rho \in (r, R)$. Moreover, the expansion is unique.

Proof. For $w \in A(z_0, r, R)$, we apply the homology form of the Cauchy's Integral Formula to the cycle $\Gamma := \gamma(z_0, R) - \gamma(z_0, r)$:

$$2\pi \mathrm{i} \cdot f(w) = \oint_{\gamma(z_0,R)} \frac{f(z)}{z-w} \,\mathrm{d}z - \oint_{\gamma(z_0,r)} \frac{f(z)}{z-w} \,\mathrm{d}z$$

On the circle $\partial B(z_0, R)$, since $|w| < |z|$, the series $\frac{1}{z-w} = \frac{1}{z-z_0} \sum_{n=0}^{\infty} \left(\frac{w-z_0}{z-z_0}\right)^n$ converges uniformly. We can integrate term by term:

$$\oint_{\gamma(z_0,R)} \frac{f(z)}{z-w} \,\mathrm{d}z = \sum_{n=0}^{\infty} \left(\oint_{\gamma(z_0,R)} \frac{f(z)}{(z-z_0)^{n+1}} \,\mathrm{d}z \right) (w-z_0)^n$$

On the circle $\partial B(z_0, r)$, since $|w| > |z|$, the series $\frac{1}{w-z} = \frac{1}{w-z_0} \sum_{m=0}^{\infty} \left(\frac{z-z_0}{w-z_0}\right)^m$ converges uniformly. We can integrate term by term:

$$\begin{aligned}
\oint_{\gamma(z_0,r)} \frac{f(z)}{z-w} \,\mathrm{d}z &= -\oint_{\gamma(z_0,r)} f(z) \sum_{m=0}^{\infty} \frac{(z-z_0)^m}{(w-z_0)^{m+1}} \,\mathrm{d}z \\
&= -\oint_{\gamma(z_0,r)} f(z) \sum_{n=-1}^{-\infty} \frac{(w-z_0)^n}{(z-z_0)^{n+1}} \,\mathrm{d}z \qquad \text{(let } n = -m-1\text{)}
\end{aligned}$$

$$= \sum_{n=-1}^{-\infty} \left(\oint_{\gamma(z_0,r)} \frac{f(z)}{(z-z_0)^{n+1}} \mathrm{d}z \right) (w-z_0)^n$$

If we define

$$c_n := \begin{cases} \dfrac{1}{2\pi \mathrm{i}} \displaystyle\oint_{\gamma(z_0,R)} \frac{f(z)}{(z-z_0)^{n+1}} \mathrm{d}z, & n \geqslant 0 \\ \dfrac{1}{2\pi \mathrm{i}} \displaystyle\oint_{\gamma(z_0,r)} \frac{f(z)}{(z-z_0)^{n+1}} \mathrm{d}z, & n < 0 \end{cases}$$

Then the above equations implies that $f(w) = \sum_{n=-\infty}^{+\infty} c_n (w-z_0)^n$. To see that the coefficients can be expressed by integral along $\gamma(z_0,\rho)$, notice that $f(z)/(z-z_0)^{n+1}$ is holomorphic on tha annulus $A(z_0,r,R)$. So when we apply Cauchy's Theorem to the cycle $\Gamma := \gamma(z_0,\rho) - \gamma(z_0,r)$, we have:

$$0 = \oint_{\gamma(z_0,\rho)} \frac{f(z)}{(z-z_0)^{n+1}} \mathrm{d}z - \oint_{\gamma(z_0,r)} \frac{f(z)}{(z-z_0)^{n+1}} \mathrm{d}z$$

And similar for the cycle $\gamma(z_0,r) - \gamma(z_0,R)$. Hence we have $c_n = \dfrac{1}{2\pi \mathrm{i}} \displaystyle\oint_{\gamma(z_0,\rho)} \frac{f(z)}{(z-z_0)^{n+1}} \mathrm{d}z$ as claimed.

To prove uniqueness, suppose f has another Laurent series $\sum_{n=-\infty}^{+\infty} d_n (z-z_0)^n$ on $A(z_0,r,R)$. Then:

$$\begin{aligned} 2\pi \mathrm{i} c_n &= \oint_{\gamma(z_0,\rho)} \frac{f(z)}{(z-z_0)^{n+1}} \mathrm{d}z \\ &= \oint_{\gamma(z_0,\rho)} \sum_{k=-\infty}^{+\infty} \frac{d_k (z-z_0)^k}{(z-z_0)^{n+1}} \mathrm{d}z \\ &= \oint_{\gamma(z_0,\rho)} \sum_{k=0}^{+\infty} d_k (z-z_0)^{k-n-1} \mathrm{d}z + \oint_{\gamma(z_0,\rho)} \sum_{k=1}^{+\infty} d_{-k} (z-z_0)^{-k-n-1} \mathrm{d}z \\ &= \sum_{k=-\infty}^{+\infty} d_k \oint_{\gamma(z_0,\rho)} (z-z_0)^{k-n-1} \mathrm{d}z \qquad \text{(since both power series converges uniformly)} \\ &= \sum_{k=-\infty}^{+\infty} d_k \cdot 2\pi \mathrm{i} \delta_{n,k} = 2\pi \mathrm{i} d_n \end{aligned}$$

The fifth equality follows from that the integral $\oint_{\gamma(z_0,\rho)} (z-z_0)^n \mathrm{d}z$ is zero except for $n = -1$, where the value is given in Lemma 1.14. □

Remark. The following examples illustrates some techniques in calculating Laurent series. For a specific function, using the integrals given in the previous theorem is not economical or even feasible. One shall exploit the properties of some known expansions.

Example 2.14

Compute the Laurent series of $f(z) = \dfrac{1}{(z-1)^2(z+2)}$ in the annulus $A(0,1,2) := \{z \in \mathbb{C} : 1 < |z| < 2\}$.

Solution. We know that $1/(1-z)$ can be expanded into Taylor series for $|z| < 1$:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

For $|z| > 1$, let $w := 1/z$. Then $\dfrac{1}{z-1} = -\dfrac{1}{z}\dfrac{1}{1-z^{-1}} = -\dfrac{w}{1-w}$ can be expanded into Taylor series:

$$\frac{1}{1-z} = -\frac{w}{1-w} = -w \sum_{n=0}^{\infty} w^n = -\sum_{n=1}^{\infty} z^{-n}$$

To expand f, we first split it into partial fractions:

$$f(z) = \frac{1}{(z-1)^2(z+2)} = \frac{1}{9(z+2)} - \frac{1}{9(z-1)} + \frac{1}{3(z-1)^2}$$

f has singularities at $z=1$ and $z=-2$. Since $1<|z|<2$ we expand $1/z$ and $z/2$ respectively:

$$\begin{aligned} f(z) &= \frac{1}{18(1+z/2)} - \frac{1}{9z(1-z^{-1})} - \frac{1}{3z^2(1-z^{-1})^2} \\ &= \frac{1}{18}\sum_{n=0}^{\infty}(-\frac{1}{2})^n z^n - \frac{1}{9z}\sum_{n=0}^{\infty} z^{-n} + \frac{1}{3z^2}\sum_{n=0}^{\infty}(n+1)z^{-n} \\ &= \sum_{n=-\infty}^{\infty} c_n z^n \end{aligned}$$

where

$$c_n = \begin{cases} \frac{1}{18}\left(-\frac{1}{2}\right)^n & n \geqslant 0 \\ -(3n+4)/9 & n<0 \end{cases}$$

This gives the desired Laurent series. □

2.2.2 Isolated Singularities.

Definition 2.15. Singularities.

Suppose $U \subseteq \mathbb{C}$ is a domain and $D \subseteq U$ is the set of points at which $f: U \to \mathbb{C}$ are holomorphic. We say that z_0 is a singularity of f, if $z_0 \notin D$ and z_0 is a limit point of D. Especially, z_0 is said to be an isolated singularity, if there exists a deleted neighbourhood $B(z_0, r) \setminus \{z_0\} \subseteq D$.

Definition 2.16. Classification of Isolated Singularities.

Suppose $f: U \to \mathbb{C}$ has an isolated singularity at z_0. We say that z_0 is a:

1. **removable singularity**, if $\lim_{z\to z_0} f(z)$ exists and finite;
2. **pole**, if $\lim_{z\to z_0} f(z) = \infty$;
3. **essential singularity**, if $\lim_{z\to z_0} f(z)$ does not exists in $\mathbb{C}_\infty$.

Remark. Often we concern the behavior of a function at infinity. We say that ∞ is a removable singularity (*resp.* pole/essential singularity) of f, if 0 is a removable singularity (*resp.* pole/essential singularity) of g, where g is defined by $g(z) = f(1/z)$.

Definition 2.17. Meromorphic Functions.

Suppose U is open in $\mathbb{C}$ and $S \subseteq U$ is at most countable with no limit points in U. Then $f: U \setminus S \to \mathbb{C}$ is said to be a meromorphic function, if f is holomorphic on $U \setminus S$ and has poles at the points in S.

Remark. It is easy to prove that f has the same Laurent expansion at $A(z_0, 0, r)$ regardless of the choice of r. So we can safely say the Laurent expansion "near the singularity". We can now show the connection between the isolated singularities and the Laurent expansion near them. The general result is as follows:

1. **removable singularity**: no principal part;
2. **pole**: finitely many terms in the principal part;
3. **essential singularity**: infinitely many terms in the principal part.

Remark. First, for removable singularities, we already have the Riemann's Removable Singularity Theorem (1.23), which will be restated below.

Theorem 2.18. Riemann's Removable Singularity Theorem.

Suppose $f : U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic. The following statements are equivalent:

(i) z_0 is a removable singularity of f;

(ii) f is bounded near z_0;

(iii) f can be extended to a holomorphic function on U;

(iv) If $\sum_{n=-\infty}^{+\infty} c_n(z-z_0)^n$ is a Laurent expansion of f in a deleted neighbourhood of z_0, then $c_n = 0$ for all $n < 0$. That is, the Laurent expansion coincides with the Taylor expansion.

Remark. Next we turn to poles. We will see that the characterisation of poles plays an important role in computing integrals in the next section.

Proposition 2.19

z_0 is a pole of f if and only if z_0 is a zero of $1/f$.

Proof. Trivial by algebra of limits. □

Definition 2.20. Multiplicity of Poles.

Suppose z_0 is a pole of f. The multiplicity or order of z_0 of f is defined to be the multiplicity of z_0 as a zero of $1/f$.

A pole of order 1 is called a simple pole.

Proposition 2.21

f has a pole of order k at z_0 if and only if the Laurent expansion of f in a deleted neighbourhood of z_0 is

$$\sum_{n=-k}^{\infty} c_n(z-z_0)^n \quad (c_{-k} \neq 0).$$

Proof. "$\Longrightarrow$": By definition, z_0 is a zero of $1/f$ with multiplicity k. That is, $1/f(z) = (z-z_0)^k g(z)$ with g holomorphic in a neighbourhood $B(z_0, r)$ and $g(z_0) \neq 0$. Hence $1/g$ is also holomorphic in $B(z_0, r)$. Suppose its Taylor expansion is $\frac{1}{g(z)} = \sum_{n=0}^{\infty} a_n(z-z_0)^n$. Then in $B(z_0, r) \setminus \{z_0\}$ we have:

$$\frac{1}{f(z)} = \frac{(z-z_0)^k}{\sum_{n=0}^{\infty} a_n(z-z_0)^n} \Longrightarrow f(z) = \sum_{n=-k}^{\infty} a_{n+k}(z-z_0)^n$$

"$\Longleftarrow$": Suppose f has a Laurent series $\sum_{n=-k}^{\infty} c_n(z-z_0)^n$ in $B(z_0, r) \setminus \{z_0\}$ with $c_{-k} \neq 0$. Then

$$f(z) = (z-z_0)^{-k} \sum_{n=0}^{\infty} c_{n-k}(z-z_0)^n = (z-z_0)^{-k} g(z)$$

$g(z)$ is defined by a power series and hence is holomorphic in $B(z_0, r)$. Moreover, $g(z_0) = c_{-k} \neq 0$. Therefore $1/g(z)$ is also holomorphic in $B(z_0, r)$ and non-zero at z_0. We have:

$$\frac{1}{f(z)} = (z-z_0)^k \frac{1}{g(z)}$$

Hence z_0 is a zero of $1/f$ of multiplicity k. □

Remark. The next proposition is helpful in classifying the singularities of some known functions.

Proposition 2.22

Suppose $f : U \to \mathbb{C}$ and $g : U \to \mathbb{C}$ are holomorphic functions. If f has a zero of multiplicity m at $z_0 \in U$, and g has a zero of multiplicity n at $z_0 \in U$, then the function $h := f/g$ has a

1. removable singularity at z_0, if $m \geqslant n$;
2. pole of order $n-m$ at z_0, if $m < n$.

Proof. It follows immediately from Proposition 2.21. □

Theorem 2.23. Argument Principle for Meromorphic Functions.

Suppose $U \subseteq \mathbb{C}$ is a domain and $S \subseteq U$ is an at most countable subset. $f : U \setminus S \to \mathbb{C}$ is meromorphic function with poles at the points of S. Let γ be a piecewise-smooth simple closed path in U and f is non-zero on γ. Then we have:

$$N - P = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{f'(z)}{f(z)} \,\mathrm{d}z$$

where N is the number of zeros of f inside γ and P is the number of poles of f inside γ (both counting multiplicity).

Proof. The proof is essential the same as 2.5. Suppose f has zeros $a_1, \cdots, a_n$ with multiplicity $m_1, \cdots, m_n$ and poles $b_1, \cdots, b_k$ with multiplicity $p_1, \cdots, p_k$ in the interior of γ. We have:

$$\oint_\gamma \frac{f'(z)}{f(z)} \,\mathrm{d}z = \sum_{i=1}^{n} \oint_{\gamma(a_i, r_i)} \frac{f'(z)}{f(z)} \,\mathrm{d}z + \sum_{i=1}^{k} \oint_{\gamma(b_i, s_i)} \frac{f'(z)}{f(z)} \,\mathrm{d}z$$

At each pole b_i, $g_i(z) = (z - b_i)^{p_i} f(z)$ is a power series and hence is holomorphic on U. Moreover, $g_i(b_i) = c_{-p_i} \neq 0$. We have:

$$\frac{f'(z)}{f(z)} = \frac{g_i'(z)}{g_i(z)} - \frac{p_i}{z - b_i}$$

Therefore:

$$\oint_{\gamma(b_i, s_i)} \frac{f'(z)}{f(z)} \,\mathrm{d}z = \oint_{\gamma(b_i, s_i)} \frac{g_i'(z)}{g_i(z)} \,\mathrm{d}z - \oint_{\gamma(b_i, s_i)} \frac{p_i}{z - b_i} \,\mathrm{d}z = 0 - 2\pi \mathrm{i} p_i$$

We have already known the behavior of the integral near the zeros. Finally,

$$\frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{f'(z)}{f(z)} \,\mathrm{d}z = \sum_{i=1}^{n} m_i - \sum_{i=1}^{k} p_i = N - P$$

□

Remark. We have proven in the preceding theorems that the Laurent series of f has no principal part near a removable singularity and finite terms of principal part near a pole. The case of essential singularity is more complicated. The Laurent series has infinite terms of principal parts. In fact we have a deeper result:

Theorem 2.24. Casorati-Weierstrass Theorem.

Suppose $U \subseteq \mathbb{C}$ is a domain and $z_0 \in U$. $f : U \setminus \{z_0\} \to \mathbb{C}$ is a holomorphic function with an essential singularity at z_0. Then for all $r > 0$ with $B(z_0, r) \subseteq U$, the image set $f(B(z_0, r) \setminus \{z_0\})$ is dense in $\mathbb{C}$.

Proof. We argue by contradiction. Suppose:

$$\exists\, r > 0\ \exists\, w \in \mathbb{C}\ \exists\, \varepsilon > 0\ \forall\, z \in B(z_0, r) \setminus \{z_0\} : |f(z) - w| > \varepsilon$$

Then $g(z) = 1/(f(z) - w)$ is holomorphic in $B(z_0, r) \setminus \{z_0\}$ and bounded near z_0. Therefore z_0 is a removable singularity of g and $\lim_{z \to z_0} g(z)$ exists. Hence

$$\lim_{z \to z_0} f(z) = w + \frac{1}{\lim_{z \to z_0} g(z)}$$

exists in $\mathbb{C}$ or is equal to ∞, contradicting that z_0 is an essential singularity of f. □

Remark. A significant generalisation of Casorati-Weierstrass Theorem is the **Picard's Great Theorem**, which states that, if z_0 is an essential singularity of f, then for any deleted neighbourhood of z_0, $f(z)$ assumes all possible complex values, with at most one exception, infinitely many times.

2.3 Weierstrass Factorisation Theorem*

2.3.1 Weierstrass Factorisation Theorem.

Now we begin with discussion of the zeros of an entire function. Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function with zeros at $0, a_1, \cdots, a_n$ with multiplicities $m, m_1, \cdots, m_n$. (If $f(0) \neq 0$, then we put $m = 0$.) Consider the polynomial

$$p(z) = z^m \left(1 - \frac{z}{a_1}\right)^{m_1} \cdots \left(1 - \frac{z}{a_n}\right)^{m_n}$$

Then the function $g(z) = f(z)/p(z)$ has removable singularities at $0, a_1, \cdots, a_n$, so we can extend g to an entire function with no zeros in $\mathbb{C}$. By Proposition 1.38 (vii), since $\mathbb{C}$ is simply-connected, there exists an entire function h such that $g(z) = \mathrm{e}^{h(z)}$. Hence we have expressed f as a product of a polynomial, whose zeros and multiplicities are the same as f, and an entire function without zeros.

$$f(z) = z^m \mathrm{e}^{h(z)} \left(1 - \frac{z}{a_1}\right)^{m_1} \cdots \left(1 - \frac{z}{a_n}\right)^{m_n}$$

If f has infinitely many zeros in $\mathbb{C}$, we can generalize the above formula to an infinite product:

Theorem 2.25. Weierstrass Factorisation Theorem.

Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function with zeros at $a_1, a_2, \cdots$ (counting multiplicity). $\{a_n\} \in \mathbb{C} \setminus \{0\}$ forms an infinite set which has no limit points. Suppose furthermore that 0 is a zero of order m of f. (If $f(0) \neq 0$, then we put $m = 0$.) Then there exists an entire function $h : \mathbb{C} \to \mathbb{C}$ such that

$$f(z) = z^m \mathrm{e}^{h(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left(\frac{z}{a_n} + \cdots + \frac{1}{n-1}\left(\frac{z}{a_n}\right)^{n-1}\right)$$

Before giving the formal proof, we first introduce the so-called elementary factors, named by Weierstrass.

Definition 2.26. Elementary Factors.

Let $E_0(z) = 1 - z$. For $p \in \mathbb{Z}_+$, we define

$$E_p(z) = (1 - z) \exp\left(z + \cdots + \frac{z^p}{p}\right) = (1 - z) \exp\left(\sum_{i=1}^{p} \frac{z^i}{i}\right)$$

Then Weierstrass Factorisation Theorem can be written as

$$f(z) = z^m \mathrm{e}^{h(z)} \prod_{n=1}^{\infty} E_{n-1}(z/a_n)$$

Lemma 2.27

$$\exists c > 0 \ \forall p \in \mathbb{N} \ \forall z \in \mathbb{C} : \quad |z| \leqslant \frac{1}{2} \implies |1 - E_p(z)| \leqslant c|z|^{p+1}$$

Proof. Observe that

$$E_p(z) = \exp\left(\log(1 - z) + \sum_{n=1}^{p} \frac{z^n}{n}\right) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} + \sum_{n=1}^{p} \frac{z^n}{n}\right) = \exp\left(-\sum_{n=p+1}^{\infty} \frac{z^n}{n}\right)$$
$$=: \exp(w)$$

For $|z| \leqslant 1/2$, we have

$$|w| = |z|^{p+1} \left|\sum_{n=0}^{\infty} \frac{z^n}{n + p + 1}\right| \leqslant |z|^{p+1} \left|\sum_{n=0}^{\infty} \frac{1}{2^n}\right| = 2|z|^{p+1}$$

In particular, $|w| \leqslant 2|z|^{p+1} \leqslant 1/2^p \leqslant 1$. Hence

$$|1 - E_p(z)| = |1 - \mathrm{e}^w| = |w| \left|\sum_{n=1}^{\infty} \frac{w^{n-1}}{n!}\right| \leqslant |w| \sum_{n=1}^{\infty} \frac{|w|^{n-1}}{n!}$$

$$\leqslant |w| \sum_{n=1}^{\infty} \frac{1}{n!} = (e-1)|w| \leqslant 2(e-1)|z|^{p+1}$$ □

Proof of Theorem 2.25. We shall show that the infinite product converges. Since $\{a_n\}$ is an infinite set in $\mathbb{C}$ without limit points, we have $\lim_{n\to\infty} |a_n| = +\infty$. Fix any $z \in \mathbb{C}$,

$$\{n \in \mathbb{Z}_+ : |z/a_n| > 1/2\}$$

is a finite set. By Lemma 2.27,

$$\sum_{n:\ |z/a_n|\leqslant 1/2} |1 - E_{n-1}(z/a_n)| \leqslant c \sum_{n:\ |z/a_n|\leqslant 1/2} |z/a_n|^n \leqslant c \sum_{n=1}^{\infty} |1/2|^n = c < +\infty$$

Hence the series $\sum_{n=1}^{\infty} |1 - E_{n-1}(z/a_n)|$ converges.

For $x > 0$, we have $1 + x < e^x$. Hence

$$\begin{aligned}\prod_{n=1}^{\infty} |E_{n-1}(z/a_n)| &\leqslant \prod_{n=1}^{\infty} (1 + |1 - E_{n-1}(z/a_n)|) \leqslant \prod_{n=1}^{\infty} \exp(|1 - E_{n-1}(z/a_n)|) \\ &= \exp\left(\sum_{n=1}^{\infty} |1 - E_{n-1}(z/a_n)|\right) < +\infty\end{aligned}$$

Hence the infinite product $P(z) := \prod_{n=1}^{\infty} E_{n-1}(z/a_n)$ converges absolutely on $\mathbb{C}$ and uniformly on any compact subset of $\mathbb{C}$. Now we observe that $z^m P(z)$ is an entire function with the same zeros and multiplicities as $f(z)$. Therefore $g(z) := \frac{f(z)}{z^m P(z)}$ has only removable singularities, and can be extended to an entire function without zeros. By Proposition 1.38 (vii), there exists an entire function $h(z)$ such that $g(z) = e^{h(z)}$. Hence we have

$$f(z) = z^m e^{h(z)} \prod_{n=1}^{\infty} E_{n-1}(z/a_n) = z^m e^{h(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left(\frac{z}{a_n} + \cdots + \frac{1}{n-1}\left(\frac{z}{a_n}\right)^{n-1}\right)$$

as claimed. □

Remark. Weierstrass Factorisation Theorem can also be stated as follows: we can always find an entire function with prescribed zeros and multiplicities.

Remark. In the proof of Theorem 2.25, a key step is the convergence of $\sum_{n=1}^{\infty} \left|\frac{z}{a_n}\right|^n$. This motivates us to make the following generalisation:

Corollary 2.28

Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function with zeros at $a_1, a_2, \cdots$ (counting multiplicity). $\{a_n\} \in \mathbb{C} \setminus \{0\}$ forms an infinite set which has no limit points. Suppose furthermore that 0 is a zero of order m of f. (If $f(0) \neq 0$, then we put $m = 0$.) If $\{k_n\} \subseteq \mathbb{Z}$ is a integer sequence such that

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|}\right)^{k_n}$$

converges for all $r > 0$. Then there exists an entire function $h : \mathbb{C} \to \mathbb{C}$ such that

$$f(z) = z^m e^{h(z)} \prod_{n=1}^{\infty} E_{k_n-1}(z/a_n) = z^m e^{h(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) \exp\left(\frac{z}{a_n} + \cdots + \frac{1}{n-1}\left(\frac{z}{a_n}\right)^{k_n-1}\right)$$

Proof. Trivial. □

Corollary 2.29

Any meromorphic function on $\mathbb{C}$ is the quotient of two entire functions.

Proof. Suppose f is a meromorphic function on $\mathbb{C}$ with poles at $a_1, a_2, \cdots$ of order $m_1, m_2 \cdots$. By Weierstrass Factorsation Theorem, we can construct an entire function g such that g has zeros at $a_1, a_2, \cdots$ with multiplicities $m_1, m_2 \cdots$.

Then near each a_n, $\lim_{z \to a_n} (z - a_n)^{m_n} f(z)$ exists and is finite, and $\lim_{z \to a_n} \frac{g(z)}{(z - a_n)^{m_n}} \neq 0$. Let $h(z) = f(z)g(z)$. The limit

$$\lim_{z \to a_n} h(z) = \lim_{z \to a_n} (z - a_n)^{m_n} f(z) \cdot \lim_{z \to a_n} \frac{g(z)}{(z - a_n)^{m_n}}$$

exists and is finite. Hence h has only removable singularities and can be extended to an entire function. Hence $f = h/g$ is the quotient of two entire functions. □

Next we shall use the Weierstrass Factorisation Theorem to prove some famous identities:

Example 2.30

1. $\pi \cot \pi z = \sum_{n \in \mathbb{Z}} \frac{1}{z+n} \quad (z \notin \mathbb{Z})$;
2. $\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2 \pi^2}\right)$;
3. $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Remark. Note that both $\sum_{n=1}^{\infty} \frac{1}{z+n}$ and $\sum_{n=-1}^{-\infty} \frac{1}{z+n}$ diverges. The sum $\sum_{n \in \mathbb{Z}} \frac{1}{z+n}$ should be interpreted as $\lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{z+n}$.

Proof. 1. Let $f(z) = \pi \cot \pi z - \sum_{n \in \mathbb{Z}} \frac{1}{z+n}$ defined on $\mathbb{C} \setminus \mathbb{Z}$. First we observe that f is periodic:

$$f(z+1) = \pi \cot \pi(z+1) - \sum_{n \in \mathbb{Z}} \frac{1}{z+n+1} = \pi \cot \pi z - \sum_{n \in \mathbb{Z}} \frac{1}{z+n} = f(z)$$

Second, note that

$$\lim_{z \to 0} \pi z \cot \pi z = 1 \quad \text{and} \quad \lim_{z \to 0} z \sum_{n \in \mathbb{Z}} \frac{1}{z+n} = 1$$

We have $\lim_{z \to 0} z f(z) = 0$. Therefore $z = 0$ is a removable singularity of f. By periodicity, $\mathbb{Z}$ are removable singularities of f. Hence f can be extended to an entire function on $\mathbb{C}$.

We claim that f is bounded in $\{z \in \mathbb{C} : |\operatorname{Re} z| \leqslant 1/2\}$. Since f is holomorphic in that domain, it suffices to show that $f(x + y\mathrm{i})$ is bounded as $|y| \to \infty$ for $|x| \leqslant 1/2$. We have

$$|\cot \pi z| = \left| \mathrm{i} \frac{\mathrm{e}^{-2\pi y} + \mathrm{e}^{-2\pi \mathrm{i} x}}{\mathrm{e}^{-2\pi y} - \mathrm{e}^{-2\pi \mathrm{i} x}} \right| \to 1 \quad \text{as } |y| \to \infty$$

Also, for $|x| \leqslant 1/2$ and $|y| > 1$:

$$\begin{aligned}
\left| \sum_{n \in \mathbb{Z}} \frac{1}{z+n} \right| &= \left| \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \right| = \left| \frac{1}{x + y\mathrm{i}} + \sum_{n=1}^{\infty} \frac{2(x + y\mathrm{i})}{x^2 - y^2 - n^2 + 2xy\mathrm{i}} \right| \\
&\leqslant \left| \frac{1}{x + y\mathrm{i}} \right| + \sum_{n=1}^{\infty} \left| \frac{2(x + y\mathrm{i})}{x^2 - y^2 - n^2 + 2xy\mathrm{i}} \right| \leqslant A + B \sum_{n=1}^{\infty} \frac{y}{y^2 + n^2} \\
&\leqslant A + B \int_0^{\infty} \frac{y}{x^2 + y^2} \, \mathrm{d}x \qquad \left(\frac{y}{x^2 + y^2} \text{ is decreasing in } x \right) \\
&= A + B \arctan \left(\frac{x}{y} \right) \Bigg|_{x=0}^{x=\infty} = A \pm \frac{1}{2} \pi B
\end{aligned}$$

where $A, B > 0$ are some constants. Hence f is bounded in $\{z \in \mathbb{C} : |\operatorname{Re} z| \leqslant 1/2\}$.

By periodicity, f is bounded on $\mathbb{C}$. By Liouville's Theorem, f is constant on $\mathbb{C}$. But

$$f(-z) = \pi \cot(-\pi z) - \sum_{n \in \mathbb{Z}} \frac{1}{-z+n} = -\left(\pi \cot(\pi z) - \sum_{n \in \mathbb{Z}} \frac{1}{z+n} \right) = -f(z)$$

for $z \in \mathbb{C}$. We conclude that $f(z) = 0$ for all $z \in \mathbb{C}$. Hence $\pi \cot(-\pi z) = \sum_{n\in\mathbb{Z}} \frac{1}{z+n}$ for $z \in \mathbb{C}\setminus\mathbb{Z}$.

2. Note that $\sin z$ has zeros at $z \in \pi\mathbb{Z}$. If we fix $k_n = 2$, we see that

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} \left(\frac{r}{|n\pi|}\right)^{k_n} = \frac{2r}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

converges for all $r > 0$. Hence by Corollary 2.28, we have

$$\sin z = z\mathrm{e}^{h(z)} \prod_{n\in\mathbb{Z}\setminus\{0\}} E_1\left(\frac{z}{n\pi}\right) = z\mathrm{e}^{h(z)} \prod_{n\in\mathbb{Z}\setminus\{0\}} \left(1-\frac{z}{n\pi}\right)\mathrm{e}^{z/n\pi}$$

for some entire function h. Note that the infinite product is absolutely convergent, we have

$$\sin z = z\mathrm{e}^{h(z)} \prod_{n=1}^{\infty}\left(1-\frac{z}{n\pi}\right)\mathrm{e}^{z/n\pi}\left(1+\frac{z}{n\pi}\right)\mathrm{e}^{-z/n\pi} = z\mathrm{e}^{h(z)}\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)$$

It remains to determine the function h. Taking logarithm on both sides:

$$\log\sin z = \log z + h(z) + \sum_{n=1}^{\infty}\log\left(1-\frac{z^2}{n^2\pi^2}\right)$$

Taking derivative:

$$\cot z = \frac{1}{z} + h'(z) + \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2\pi^2}$$

or

$$\pi\cot\pi z = \frac{1}{z} + h'(\pi z) + \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}$$

Combining with the first part, we deduce that $h'(\pi z) = 0$ for $z \in \mathbb{C}\setminus\mathbb{Z}$. Hence $h(z) = \text{const}$. By evaluating at $z = 0$ we deduce that $h(z) = 0$. Hence $\sin z = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)$.

3. By expanding $\sin z$ into Taylor series at $z = 0$, we obtain:

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1} = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)$$

By comparing the coefficient of z^3, we obtain

$$-\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2} = -\frac{1}{6}$$

Thus the result follows. □

Remark. $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$ can also be proven by calculus of residue in the next chapter. See Example 3.18 for detail.

2.3.2 Mittag-Leffler Theorem.

Theorem 2.31. Mittag-Leffler Theorem.

Suppose $\{a_n\}$ is a sequence of complex numbers in $\mathbb{C}$ without limit points. $\varphi_n := \sum_{j=1}^{k_n}\frac{c_{n,j}}{(z-a_n)^j}$ is a sequence of rational functions with $c_{n,j} \in \mathbb{C}$. Then there exists a meromorphic function $f : U \to \mathbb{C}$ such that f has poles at $\{a_n\}$ and principal parts φ_n at each pole.

Remark. Mittag-Leffler Theorem states that we can always find a meromorphic function with prescribed poles and principal parts near each pole.

Proof. For each $n \in \mathbb{Z}_+$, φ_n is a polynomial of $1/(z-a_n)$. Hence φ_n is holomorphic in $B(0,|a_n|)$ and has Taylor series $\sum_{k=0}^{\infty} \frac{\varphi_n^{(k)}(0)}{k!} z^k$, which converges uniformly to φ_n in $B(0,|a_n|/2)$. Hence there exists $s_n \in \mathbb{N}$ such that

$$\left|\varphi_n - \sum_{k=0}^{s_n} \frac{\varphi_n^{(k)}(0)}{k!} z^k\right| < \frac{1}{2^n}$$

for $z \in B(0,|a_n|/2)$. We denote $\sum_{k=0}^{s_n} \varphi_n^{(k)}(0) z^k / k!$ by $p_n(z)$.

Fix $R > 0$. Since $\{a_n\}$ is an infinite set in $\mathbb{C}$ without limit points, we have $\lim_{n\to\infty} |a_n| = +\infty$. Let

$$N := \max\{n \in \mathbb{Z}_+ : |a_n| \leqslant 2R\}$$

Then for $z \in \overline{B}(0,R)$ and $n > N$, $|z| \leqslant R < |a_n|$, the function defined by $\varphi_n(z) - p_n(z)$ is holomorphic. Moreover,

$$\sum_{n=N+1}^{\infty} \left|\varphi_n(z) - p_n(z)\right| \leqslant \sum_{n=N+1}^{\infty} < 1$$

The series $\sum_{n=N+1}^{\infty} \left(\varphi_n(z) - p_n(z)\right)$ converges absolutely and uniformly in $\overline{B}(0,R)$. Hence it defines a holomorphic function in $B(0,R)$.

Therefore the series

$$\sum_{n=1}^{\infty} \left(\varphi_n(z) - p_n(z)\right) = \sum_{n=1}^{N} \left(\varphi_n(z) - p_n(z)\right) + \sum_{n=N+1}^{\infty} \left(\varphi_n(z) - p_n(z)\right)$$

defines a meromorphic function in $B(0,R)$, with prescribed poles at $\{a_n : |a_n| < R\}$ and corresponding principal parts $\{\varphi_n : |a_n| < R\}$. Since R is arbitrary, $\sum_{n=1}^{\infty} \left(\varphi_n(z) - p_n(z)\right)$ is a meromorphic function on $\mathbb{C}$ with the desired properties. □

2.3.3 Interpolation Theorem.

Mittag-Leffler Theorem may be combined with Weierstrass Factorisation Theorem to give a solution to the interpolation problem: Given a infinite set $\{a_n\} \in \mathbb{C}$ without limit points, we want to find an entire function with prescribed values on the set. In fact the result is much stronger. We can also prescribe finitely many derivatives at each a_n.

Theorem 2.32. Interpolation Theorem.

Suppose $\{a_n\}$ is a sequence of complex numbers in $\mathbb{C}$ without limit points. For each a_n we associate a non-negative integer m_n and a sequence of complex numbers $c_{k,n}$, $(0 \leqslant k \leqslant m_n)$. Then there exists an entire function f such that $f^{(k)}(a_n) = k! c_{k,n}$ for each a_n and $0 \leqslant k \leqslant m_n$.

In other words, f has the prescribed m_n terms of Taylor expansion at each a_n.

Proof. By Weierstrass Factorisation Theorem, we can find an entire function g such that g has zeros at each a_n with multiplicities $m_n + 1$. We claim that we can associate each a_n with a rational function

$$\varphi_n(z) = \sum_{k=1}^{m_n+1} \frac{d_{k,n}}{(z-a_n)^k}$$

such that $g(z)\varphi_n(z)$ has a Taylor expansion near a_n:

$$g(z)\varphi_n(z) = \sum_{k=0}^{m_n} c_{k,n}(z-a_n)^k + O\left((z-a_n)^{m_n+1}\right)$$

Suppose that we are given that $g(z) = \sum_{k=0}^{\infty} b_k (z-a_n)^{k+m_n+1}$ near a_n. Then

$$\begin{aligned} g(z)\varphi_n(z) &= \left(\sum_{k=1}^{m_n+1} \frac{d_{k,n}}{(z-a_n)^k}\right) \cdot \left(\sum_{k=0}^{\infty} b_k (z-a_n)^{k+m_n+1}\right) \\ &= \left(d_{m_n+1,n} + d_{m_n,n}(z-a_n) + \cdots + d_{1,n}(z-a_n)^{m_n}\right) \cdot \left(\sum_{k=0}^{\infty} b_k (z-a_n)^k\right) \end{aligned}$$

Comparing the coefficients, we have:

$$\begin{aligned} b_0 \cdot d_{m_n+1,n} &= c_{0,n} \\ b_1 \cdot d_{m_n+1,n} + b_0 \cdot d_{m_n,n} &= c_{1,n} \\ &\cdots\cdots \\ \sum_{j=0}^{k} b_{k-j} \cdot d_{m_n+1-j,n} &= c_{k,n} \end{aligned}$$

Thus we can determine $d_{m_n+1,n}, \cdots, d_1$ successively, since $b_0 \neq 0$. In this way we have obtained the desired functions φ_n.

By Mittag-Leffler Theorem, we can find a meromorphic function h on $\mathbb{C}$ such that h has poles at each a_n with principal parts φ_n. Now $f(z) := g(z)h(z)$ has only removable singularities and can be extended to an entire function with the desired properties. □

Remark. The Interpolation Theorem can also help determine the structure of the ring of holomorphic functions. Suppose $U \in \mathbb{C}$ is a domain. Let $\mathrm{H}(U)$ denotes the set of all holomorphic functions on U. Then $\mathrm{H}(U)$ is a ring under function additions and multiplications. We have the following proposition:

Proposition 2.33

Every finitely-generated ideal in $\mathrm{H}(U)$ is principal.

Proof. Given $f_1, \cdots, f_n \in \mathrm{H}(U)$. The corollary states that there exists $f \in \mathrm{H}(U)$ such that $\langle f_1, \cdots, f_n \rangle = \langle f \rangle$. But first we shall prove the following lemma: If $g_1, \cdots, g_n \in \mathrm{H}(U)$, if none of them is identically zero in U, and if they have no common zeros in U, then $\langle g_1, \cdots, g_n \rangle = \langle 1 \rangle = \mathrm{H}(U)$.

We use induction on n. Suppose that the lemma holds for any $n-1$ functions with no common zeros. Let $m(g_i; \alpha)$ denotes the multiplicities of zero of g_i at $\alpha \in U$. By Weierstrass Factorisation Theorem (applied to a domain rather than the whole $\mathbb{C}$, which is still valid), we can find $\varphi \in \mathrm{H}(U)$ such that $m(\varphi, \alpha) = \min_{1 \leqslant i \leqslant n-1} m(g_i, \alpha)$ for each $\alpha \in U$. This is practical because the common zeros of $g_1, \cdots, g_{n-1}$ is a discrete point set.

Let $h_i := g_i/\varphi$ for $i = 1, \cdots, n-1$. Then $h_1, \cdots, h_{n-1}$ have no common zeros. By the induction hypothesis, we have $\langle h_1, \cdots, h_{n-1} \rangle = \mathrm{H}(U)$. Hence $\langle g_1, \cdots, g_{n-1}, g_n \rangle = \langle \varphi, g_n \rangle$.

Since $g_1, \cdots, g_n$ have no common zeros, φ and g_n have no common zeros. By Interpolation Theorem (again applied to a domain U), we can find $\psi \in \mathrm{H}(U)$ such that $m(1 - \psi g_n; \alpha) \geqslant m(\varphi; \alpha)$ for all $\alpha \in U$. This could be done by prescribing the value of $\psi^{(k)}(\alpha)$ for $0 \leqslant k \leqslant m(\varphi; \alpha)$ at each $\alpha \in U$ such that $m(\varphi; \alpha) > 0$.

Hence $\xi := (1 - \psi g_n)/\varphi$ has only removable singularities and can be extended to a holomorphic function in U. That is, $\exists \psi, \xi \in \mathrm{H}(U) : 1 = \psi g_n + \varphi \xi$.

Then $1 \in \langle \varphi, g_n \rangle$. $\langle g_1, \cdots, g_n \rangle = \langle \varphi, g_n \rangle = \mathrm{H}(U)$, which completes the proof. □

Chapter 3

Calculus of Residues

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3.1 Residue Theorem

Definition 3.1. Residue.

Suppose f has an isolated singularity at $z_0 \in \mathbb{C}$. The Laurent series of f in a deleted neighbourhood is $\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$. We call the coefficient c_{-1} the residue of f at z_0 and denote it by $\operatorname{Res}(f, z_0)$.

Proposition 3.2. Residue at a Pole.

If f has a pole of order n at z_0, then the residue of f at z_0 is given by:

$$\operatorname{Res}(f, z_0) = \frac{1}{(n-1)!} \lim_{z\to z_0} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n-1} \big((z-z_0)^n f(z)\big)$$

Proof. Suppose the Laurent series of f near z_0 is $\sum_{m=-n}^{\infty} c_m(z-z_0)^m$. Then $(z-z_0)^n f(z) = \sum_{m=0}^{\infty} c_{m-n}(z-z_0)^m$ is can be holomorphically extended to z_0. We have:

$$\begin{aligned}
\lim_{z\to z_0} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n-1} \big((z-z_0)^n f(z)\big) &= \lim_{z\to z_0} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^{n-1} \sum_{m=0}^{\infty} c_{m-n}(z-z_0)^m \\
&= \lim_{z\to z_0} \sum_{m=n-1}^{\infty} \frac{m!}{(n+1)!} c_{m-n}(z-z_0)^{m-n+1} \\
&= (n-1)!\, c_{-1} = (n-1)!\, \operatorname{Res}(f, z_0)
\end{aligned}$$

□

Corollary 3.3. Residue at a Simple Pole.

If f has a simple pole at z_0, then the residue of f at z_0 is given by:

$$\operatorname{Res}(f, z_0) = \lim_{z\to z_0} (z-z_0) f(z)$$

Proposition 3.4

Suppose g, h are holomorphic at $z_0 \in \mathbb{C}$, with $g(z_0) \neq 0$, $h(z_0) = 0$, and $h'(z_0) \neq 0$. Then z_0 is a simple pole of $f(z) := g(z)/h(z)$. Moreover, we have $\operatorname{Res}(f, z_0) = g(z_0)/h'(z_0)$.

Proof. $1/f(z_0) = \dfrac{h(z_0)}{g(z_0)} = 0$ and $(1/f)'(z_0) = \dfrac{h'(z_0)g(z_0) - h(z_0)g'(z_0)}{g(z_0)^2} = \dfrac{h'(z_0)}{g(z_0)} \neq 0$. So z_0 is a simple zero of $1/f$ and hence a simple pole of f. By Corollary 3.3, we have:

$$\operatorname{Res}(f, z_0) = \lim_{z\to z_0} (z - z_0)\frac{g(z)}{h(z)} = \lim_{z\to z_0} \frac{g(z)}{\frac{h(z)-h(z_0)}{z-z_0}} = \frac{g(z_0)}{h'(z_0)}$$ □

Theorem 3.5. Residue Theorem.

Suppose $U \subseteq \mathbb{C}$ is a domain and $S \subseteq U$ is an at most countable subset with no limit points in U. $f : U \setminus S \to \mathbb{C}$ is holomorphic with isolated singularities at the points of S. γ is piecewise-smooth closed path contained in U with $S \cap \gamma^* = \varnothing$. Then we have:

$$\oint_\gamma f = 2\pi \mathrm{i} \sum_{a\in S} I(\gamma, a) \operatorname{Res}(f, a)$$

Proof. Since γ^* is bounded, without loss of generality we can assume that S is finite. For $a \in S$, let p_a be the principal part of f near a. Then $f - p_a$ is holomorphic at $a \in S$. Consequently $f - \sum_{a\in S} p_a$ is holomorphic on S. Apply Cauchy's Theorem:

$$0 = \oint_\gamma f(z)\,\mathrm{d}z - \sum_{a\in S}\oint_\gamma p_a(z)\,\mathrm{d}z$$

But we also have:

$$\begin{aligned}\oint_\gamma p_a(z)\,\mathrm{d}z &= \oint_\gamma \sum_{n=-1}^{-\infty} c_n(z - z_0)^n \mathrm{d}z = \sum_{n=1}^{+\infty} \oint_\gamma \frac{c_{-n}}{(z - z_0)^n}\,\mathrm{d}z \\ &= \oint_\gamma \frac{c_{-1}}{z - z_0}\,\mathrm{d}z = 2\pi\mathrm{i} \cdot I(\gamma, a) \operatorname{Res}(f, a)\end{aligned}$$

The second equality follows from the uniform convergence of the principal part. The third equality follows from that $(z - z_0)^n$ always has a primitive in U except for $n = -1$.

Therefore we have $\oint_\gamma f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{a\in S} I(\gamma, a) \operatorname{Res}(f, a)$ as claimed. □

Remark. Residue Theorem provides a powerful tool that transforms the calculation of integrals into the calculation of the residue of singularities, which, by Proposition 3.2, is essentially doing differentiation.

Now we will give a number of examples to demonstrate the power of Residue Theorem in computing integrals, especially improper integrals on the real line. For more techniques and examples of residue calculus, please refer to **[Priestley]**.

3.2 Semicircular Contour

We first consider improper integrals of the type $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x$:

Theorem 3.6

Suppose f is holomorphic on the upper half plane $\{z \in \mathbb{C} : \operatorname{Im} z \geqslant 0\}$ except at a finite set S. If $\lim_{z\to\infty} z \cdot f(z) = 0$, then we have:

$$\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = 2\pi\mathrm{i} \sum_{a\in S} \operatorname{Res}(f, a)$$

provided the improper integral on the left side exists.

Proof. We will use a semicircular contour in the upper half plane that consists of a line segment $[-R, R]$ and a semicircle $\gamma_R := \gamma^+(0, R)$, as shown in Figure 3.1. By Residue Theorem, for sufficiently large R, we have:

$$\int_{-R}^{R} f(x)\,\mathrm{d}x + \int_{\gamma_R} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{a\in S}\operatorname{Res}(f, a) \tag{3.1}$$

Let $M(R) := \sup_{z\in\gamma_R^*} |f(z)|$. We know that $\lim_{R\to\infty} R\cdot M(R) = 0$. Then:

$$\left|\int_{\gamma_R} f(z)\,\mathrm{d}z\right| = \left|\int_0^\pi f(R\mathrm{e}^{\mathrm{i}\theta})\cdot R\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\ \mathrm{d}\theta\right| \leqslant \pi R M(R) \to 0$$

as $R\to\infty$.

By letting $R\to\infty$ in the Equation (3.1), we have

$$\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = 2\pi\mathrm{i}\sum_{a\in S}\mathrm{Res}(f,a)$$

as claimed. □

Remark. We must prove the existence of the improper integral before we apply the theorem. If the integral diverges, then $2\pi\mathrm{i}\sum_{a\in S}\mathrm{Res}(f,a)$ only gives the **Cauchy principal value** of the integral. We should write:

$$\text{P.V.}\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = 2\pi\mathrm{i}\sum_{a\in S}\mathrm{Res}(f,a)$$

Remark. There is nothing special about our choice of the semicircle in the upper half plane. One gets the same conclusion using the semicircle in the lower half plane, with the corresponding poles and residues.

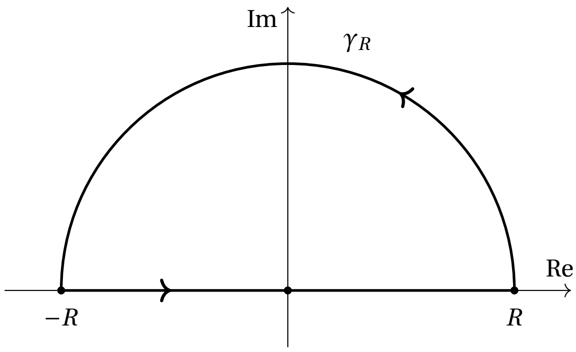

Figure 3.1: A semicircular contour.

Example 3.7

Evaluate $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{1+x^2+x^4}\,\mathrm{d}x$.

Solution. Let us consider the function $f(z) = 1/(1+z^2+z^4)$ defined on the upper half plane. Notice that $1/f(z) = 1+z^2+z^4 = (z^6-1)/(z^2-1)$ has simple zeros at $\mathrm{e}^{\pi\mathrm{i}/3}$, $\mathrm{e}^{2\pi\mathrm{i}/3}$, $\mathrm{e}^{4\pi\mathrm{i}/3}$ and $\mathrm{e}^{5\pi\mathrm{i}/3}$. Then, in the upper half plane, f has simple poles at $\omega = \mathrm{e}^{\pi\mathrm{i}/3}$ and $\omega^2 = \mathrm{e}^{2\pi\mathrm{i}/3}$.

We calculate the residues by using Corollary 3.3:

$$\mathrm{Res}(f,\omega) = \lim_{z\to\omega}\frac{z-\omega}{z^4+z^2+1} = \frac{1}{(\omega-\omega^2)(\omega+\omega)(\omega+\omega^2)} = -\frac{1}{2(1-\omega^2)}$$
$$\mathrm{Res}(f,\omega^2) = \lim_{z\to\omega^2}\frac{z-\omega^2}{z^4+z^2+1} = \frac{1}{(\omega^2-\omega)(\omega^2+\omega)(\omega^2+\omega^2)} = \frac{1}{2\omega(1-\omega^2)}$$

Note that

$$\lim_{z\to\infty} zf(z) = \lim_{z\to\infty}\frac{1}{z^{-1}+z+z^3} = 0$$

Now we can apply Theorem 3.6:

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{1}{1+x^2+x^4}\,\mathrm{d}x &= 2\pi\mathrm{i}\big(\mathrm{Res}(f,\omega)+\mathrm{Res}(f,\omega^2)\big)\\ &= 2\pi\mathrm{i}\cdot\frac{1}{2(1-\omega^2)}\left(\frac{1}{\omega}-1\right) = \frac{\pi\mathrm{i}}{\omega^2+\omega}\\ &= \frac{\pi}{\sqrt{3}}\end{aligned}$$

□

3.3 Jordan's Lemma

When evaluating integrals of the type $\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\alpha x} f(x)\,\mathrm{d}x$, we need the following lemma:

Lemma 3.8. Jordan's Lemma.

Suppose f is continuous on the upper half plane (with possible exceptions at a finite set). If $\lim_{z\to\infty} f(z) = 0$, then:

$$\lim_{R\to\infty} \int_{\gamma_R} \mathrm{e}^{\mathrm{i}\alpha z} f(z)\,\mathrm{d}z = 0$$

for any $\alpha > 0$, where $\gamma_R(t) = \gamma^+(0,R)(t) = R\mathrm{e}^{\mathrm{i}t},\ t \in [0,\pi]$ is a semicircular path.

Proof. Since $\sin x$ is convex on $[0,\pi/2]$, by Jensen's Inequality, we have:

$$\sin x = \sin\left(\left(1 - \frac{2x}{\pi}\right)0 + \frac{2x}{\pi}\cdot\frac{\pi}{2}\right) \geqslant \left(1 - \frac{2}{\pi}\right)\sin 0 + \frac{2x}{\pi}\sin\frac{\pi}{2} = \frac{2}{\pi}x$$

for $x \in [0,\pi/2]$.

Let $M(R) := \sup_{z\in\gamma_R^*} |f(z)|$. We know that $\lim_{R\to\infty} M(R) = 0$. Then:

$$\begin{aligned}
\left|\int_{\gamma_R} \mathrm{e}^{\mathrm{i}\alpha z} f(z)\,\mathrm{d}z\right| &= \left|\int_0^\pi \mathrm{e}^{\mathrm{i}\alpha R\cos\theta}\,\mathrm{e}^{-\alpha R\sin\theta} f(R\mathrm{e}^{\mathrm{i}\theta})R\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}\ \mathrm{d}\theta\right| \\
&\leqslant R\cdot M(R)\int_0^\pi \mathrm{e}^{-\alpha R\sin\theta}\ \mathrm{d}\theta \leqslant 2R\cdot M(R)\int_0^{\pi/2} \mathrm{e}^{-\frac{2\alpha}{\pi}R\theta}\ \mathrm{d}\theta \\
&= 2M(R)\cdot\frac{\pi}{2\alpha}\left(1-\mathrm{e}^{-\alpha R}\right) \to 0
\end{aligned}$$

as $R\to\infty$. □

Corollary 3.9

Suppose f is holomorphic on the upper half plane $\{z\in\mathbb{C} : \operatorname{Im} z \geqslant 0\}$ except at a finite set S. If $\lim_{z\to\infty} f(z) = 0$, then for $\alpha > 0$ we have:

$$\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\alpha x} f(x)\,\mathrm{d}x = 2\pi\mathrm{i}\sum_{a\in S}\operatorname{Res}\left(\mathrm{e}^{\mathrm{i}\alpha z} f(z), a\right)$$

Proof. This is exactly like the proof of Theorem 3.6. We know from Jordan's Lemma that the integral on the upper semicircular path tends to zero, and the result follows. □

Remark. We can also translate the complex exponential into real trigonometrics:

$$\int_{-\infty}^{+\infty} \cos\alpha x f(x)\,\mathrm{d}x = \operatorname{Re}\left(2\pi\mathrm{i}\sum_{a\in S}\operatorname{Res}\left(\mathrm{e}^{\mathrm{i}\alpha z} f(z), a\right)\right)$$

$$\int_{-\infty}^{+\infty} \sin\alpha x f(x)\,\mathrm{d}x = \operatorname{Im}\left(2\pi\mathrm{i}\sum_{a\in S}\operatorname{Res}\left(\mathrm{e}^{\mathrm{i}\alpha z} f(z), a\right)\right)$$

Example 3.10

Evaluate $\int_{-\infty}^{+\infty} \frac{\sin x}{x}\,\mathrm{d}x$.

Solution. Normally we will consider $\mathrm{e}^{\mathrm{i}z}/z$ on the upper half plane. But it has a simple pole at the origin, which is on the contour we are about to integrate along. So we consider $f(z) = (\mathrm{e}^{\mathrm{i}z} - 1)/z$ instead. By expanding the exponential function we can see that

$$\lim_{z\to 0}\frac{\mathrm{e}^{\mathrm{i}z}-1}{z} = \mathrm{i}$$

Therefore $z = 0$ is a removable singularity of f, which can be extended to an entire function on the plane. By Cauchy's Theorem, along the semicircular contour we have:

$$0 = \oint f(z)\,\mathrm{d}z = \int_{-R}^{R} \frac{\mathrm{e}^{\mathrm{i}x} - 1}{x}\,\mathrm{d}x + \int_{\gamma_R} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\,\mathrm{d}z - \int_{\gamma_R} \frac{1}{z}\,\mathrm{d}z \tag{3.2}$$

The third term in the Equation (3.2) is just:

$$\int_{\gamma_R} \frac{1}{z}\,\mathrm{d}z = \int_0^{\pi} \frac{\mathrm{i}R\,\mathrm{e}^{\mathrm{i}\theta}}{R\,\mathrm{e}^{\mathrm{i}\theta}}\,\mathrm{d}\theta = \mathrm{i}\pi$$

Letting $R \to \infty$, the second term in the Equation (3.2) vanishes by Jordan's Lemma. Hence we have:

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x}\,\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} \frac{\mathrm{e}^{\mathrm{i}x}}{x}\,\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} \frac{\mathrm{e}^{\mathrm{i}x} - 1}{x}\,\mathrm{d}x = \mathrm{i}\pi$$

Take the imaginary part:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \pi$$

□

Remark. If the integrand function has singularities on the real line. We often indent the contour by a small circular arc around the singularity. The following lemma shows that it works for simple poles.

Lemma 3.11. Indentation Lemma.

Suppose f is holomorphic on the sector $\{z = z_0 + r\,\mathrm{e}^{\mathrm{i}\theta} : r \in (0, R],\ \theta \in [\theta_1, \theta_2]\}$ and has a simple pole at z_0. Then:

$$\lim_{r\to 0} \int_{\gamma_r} f(z)\,\mathrm{d}z = \mathrm{i}(\theta_2 - \theta_1)\,\mathrm{Res}(f, z_0)$$

where $\gamma_r(t) = z_0 + r\,\mathrm{e}^{\mathrm{i}t},\ t \in [\theta_1, \theta_2]$.

Proof. Suppose $\mathrm{Res}(f, z_0) = A$. Let $g(z) = (z - z_0)f(z) - A$. Then $\lim_{z\to z_0} g(z) = 0$. We have

$$\int_{\gamma_r} f(z)\mathrm{d}z = \int_{\gamma_r} \frac{g(z)}{z - z_0}\,\mathrm{d}z + A\int_{\gamma_r} \frac{1}{z - z_0}\,\mathrm{d}z = \mathrm{i}A(\theta_2 - \theta_1) + \int_{\gamma_r} \frac{g(z)}{z - z_0}\,\mathrm{d}z$$

where

$$\begin{aligned}\left|\int_{\gamma_r} \frac{g(z)}{z - z_0}\,\mathrm{d}z\right| &= \left|\int_{\theta_1}^{\theta_2} \frac{g(z_0 + r\,\mathrm{e}^{\mathrm{i}\theta})}{r\,\mathrm{e}^{\mathrm{i}\theta}} r\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta\right| \\ &= \left|\int_{\theta_1}^{\theta_2} \mathrm{i}g(z_0 + r\,\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta\right| \leqslant \sup_{z\in\gamma_r^*} |g(z)|(\theta_2 - \theta_1) \to 0\end{aligned}$$

as $r \to 0$. Hence $\int_{\gamma_r} f(z)\,\mathrm{d}z \to \mathrm{i}A(\theta_2 - \theta_1)$ as $r \to 0$ as claimed. □

Remark. The indentation only works for simple poles and fails at other types of singularities. This is because the contour has length $O(\varepsilon)$ and the integrand grows as $O(\varepsilon^{-1})$ near a simple pole.

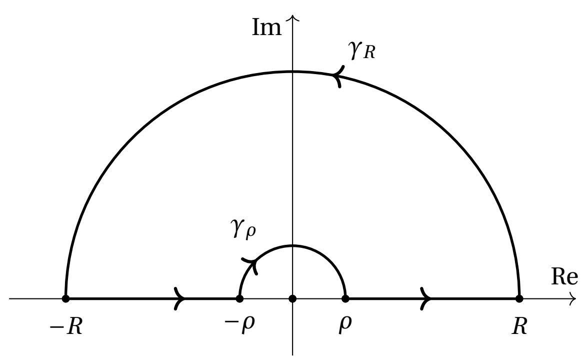

Figure 3.2: A semi-annular contour.

Example 3.12

Use Lemma 3.11 to evaluate $\displaystyle\int_{-\infty}^{+\infty} \frac{\sin x}{x}\,\mathrm{d}x$.

Solution. Let $f(z) = \mathrm{e}^{\mathrm{i}z}/z$. We integrate it along the semi-annular contour as shown in Figure 3.2. Since f is holomorphic inside and on the contour, by Cauchy's Theorem we have:

$$0 = \int_{-R}^{-\rho} \frac{\mathrm{e}^{\mathrm{i}x}}{x}\,\mathrm{d}x - \int_{\gamma_\rho} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\,\mathrm{d}z + \int_{\rho}^{R} \frac{\mathrm{e}^{\mathrm{i}x}}{x}\,\mathrm{d}x + \int_{\gamma_R} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\,\mathrm{d}z$$

where $\gamma_\rho(t) = \rho\,\mathrm{e}^{\mathrm{i}t},\ t \in [0,\pi]$ and $\gamma_R(t) = R\,\mathrm{e}^{\mathrm{i}t},\ t \in [0,\pi]$.

By Jordan's Lemma, the fourth term vanishes as $R \to \infty$.By Lemma 3.11, since $\lim_{z\to 0} z\cdot \mathrm{e}^{\mathrm{i}z}/z \to 1$, the second term

$$\int_{\gamma_\rho} \frac{\mathrm{e}^{\mathrm{i}z}}{z}\,\mathrm{d}z \to \mathrm{i}\pi$$

as $\rho \to 0$. Therefore we have:

$$\int_{-\infty}^{+\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x}\,\mathrm{d}x = \lim_{\substack{R\to\infty\\ \rho\to 0}} \int_{-R}^{-\rho} \frac{\mathrm{e}^{\mathrm{i}x}}{x}\,\mathrm{d}x + \lim_{\substack{R\to\infty\\ \rho\to 0}} \int_{\rho}^{R} \frac{\mathrm{e}^{\mathrm{i}x}}{x}\,\mathrm{d}x = \mathrm{i}\pi$$

We can see that this method produces the same answer as the previous one. □

Example 3.13

Evaluate $\displaystyle\int_0^{\infty} \frac{\ln x}{(1+x^2)^2}\,\mathrm{d}x$.

Solution. Let $f(z) = \log z/(1+z^2)^2$. f has a simple pole at 0 and double poles at i and $-$i. We wish to indent at $z = 0$ and use the semi-annulur contour in Figure 3.2. Since logarithm is multi-valued, we have to choose a holomorphic branch of the logarithm such that the cut line does not cross the contour. We use the principal logarithm here. First let us calculate the residue. By Proposition 3.2,

$$\mathrm{Res}(f,\mathrm{i}) = \lim_{z\to\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}z}(z-\mathrm{i})^2 \frac{\log z}{(1+z^2)^2} = \lim_{z\to\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}z} \frac{\log z}{(z+\mathrm{i})^2} = \lim_{z\to\mathrm{i}} \frac{(z+\mathrm{i})^2/z - 2(z+\mathrm{i})\log z}{(z+\mathrm{i})^4} = \frac{\pi}{8} + \frac{\mathrm{i}}{4}$$

For $x \in \mathbb{R}_-$, we have $\ln x = \ln|x| + \mathrm{i}\pi$. Apply the Residue Theorem:

$$\int_{-R}^{-\rho} \frac{\ln|x| + \mathrm{i}\pi}{(1+x^2)^2}\,\mathrm{d}x - \int_{\gamma_\rho} \frac{\log z}{(1+z^2)^2}\,\mathrm{d}z + \int_{\rho}^{R} \frac{\ln x}{(1+x^2)^2}\,\mathrm{d}x + \int_{\gamma_R} \frac{\log z}{(1+z^2)^2}\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}(f,\mathrm{i})$$

For the second term, since $\displaystyle\lim_{z\to 0} z\cdot\frac{\log z}{(1+z^2)^2} = 0$, 0 is a simple pole of f. By Lemma 3.11, we have:

$$\lim_{\rho\to 0} \int_{\gamma_\rho} \frac{\log z}{(1+z^2)^2}\,\mathrm{d}z = 0$$

For the fourth term, $\displaystyle\lim_{z\to\infty} z\cdot\frac{\log z}{(1+z^2)^2} = 0$, we have:

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{\log z}{(1+z^2)^2}\,\mathrm{d}z = 0$$

Combining the preceding equations we have:

$$\lim_{\substack{\rho\to 0\\ R\to\infty}} \int_{-R}^{-\rho} \frac{\ln|x| + \mathrm{i}\pi}{(1+x^2)^2}\,\mathrm{d}x + \lim_{\substack{\rho\to 0\\ R\to\infty}} \int_{\rho}^{R} \frac{\ln x}{(1+x^2)^2}\,\mathrm{d}x = 2\pi\mathrm{i}\,\mathrm{Res}(f,\mathrm{i})$$

$$\Longrightarrow 2\int_0^{\infty} \frac{\ln x}{(1+x^2)^2}\,\mathrm{d}x + \mathrm{i}\pi\int_{-\infty}^{0} \frac{1}{(1+x^2)^2}\,\mathrm{d}x = 2\pi\mathrm{i}\left(\frac{\pi}{8} + \frac{\mathrm{i}}{4}\right)$$

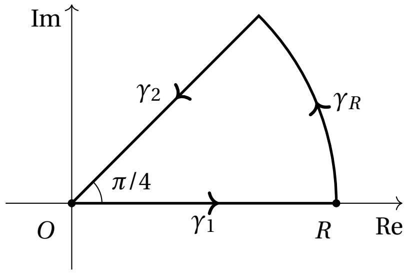

Figure 3.3: A sectorial contour

Taking the real part:

$$\int_0^\infty \frac{\ln x}{(1+x^2)^2}\,\mathrm{d}x = -\frac{\pi}{4}$$

By taking the imaginary part, we also obtain that:

$$\int_0^\infty \frac{1}{(1+x^2)^2}\,\mathrm{d}x = \frac{\pi}{4}$$

□

Remark. We will see later that the previous example can also be solved by a keyhole contour.

Example 3.14. Fresnel Integrals.

Evaluate $\displaystyle\int_0^\infty \sin x^2\,\mathrm{d}x$ and $\displaystyle\int_0^\infty \cos x^2\,\mathrm{d}x$.

Solution. We consider the function $f(z) = \mathrm{e}^{\mathrm{i}z^2}$, which is entire on $\mathbb{C}$. We wish to apply Cauchy's Theorem. A semicircular contour would not work, because $\mathrm{e}^{\mathrm{i}z^2}$ does not tends to zero as $R\to\infty$. Alternatively we can consider a contour that goes around a sector, as shown in Figure 3.3. Let the contour be $\Gamma := \gamma_1 \star \gamma_R \star \gamma_2$, where $\gamma_R(t) = R\mathrm{e}^{\mathrm{i}t},\ t\in[0,\pi/4]$ and $\gamma_2(t) = -t\,\mathrm{e}^{\pi\mathrm{i}/4},\ t\in[-R,0]$. By Cauchy's Theorem:

$$\int_0^R \mathrm{e}^{\mathrm{i}x^2}\,\mathrm{d}x + \int_{\gamma_R} \mathrm{e}^{\mathrm{i}z^2}\,\mathrm{d}z + \int_{\gamma_2} \mathrm{e}^{\mathrm{i}z^2}\,\mathrm{d}z = 0$$

Now the second term tends to zero as $R\to\infty$, following from the same argument in the proof of Jordan's Theorem (applying the Jensen's Inequality). For the third term,

$$\begin{aligned}\lim_{R\to\infty}\int_{\gamma_2} \mathrm{e}^{\mathrm{i}z^2}\,\mathrm{d}z &= -\int_0^\infty \mathrm{e}^{\mathrm{i}t^2\mathrm{e}^{\pi\mathrm{i}/2}}\,\mathrm{e}^{\pi\mathrm{i}/4}\,\mathrm{d}t = -\mathrm{e}^{\pi\mathrm{i}/4}\int_0^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t\\ &= -\mathrm{e}^{\pi\mathrm{i}/4}\frac{\sqrt{\pi}}{2} = -\left(\frac{\sqrt{2\pi}}{4} + \mathrm{i}\frac{\sqrt{2\pi}}{4}\right)\end{aligned}$$

Hence

$$\int_0^\infty \sin x^2\,\mathrm{d}x = \int_0^\infty \cos x^2\,\mathrm{d}x = \frac{\sqrt{2\pi}}{4}$$

□

Remark. In the previous example we have used the well-known result:

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$$

which is usually derived from a double integral in the polar coordinates. An alternative way which utilizes residue calculus is given in Example 3.23.

3.4 Keyhole Contour

We consider the integrals of the type $\displaystyle\int_0^\infty f(x)\,\mathrm{d}x$, where f depends explicitly or implicitly on logarithm. We need to select a holomorphic branch by drawing a cut line on the plane. For example we choose $\mathbb{R}_+$ as the cut line and consider the following contour Γ. Let $\gamma(0,R)$ and $\gamma(0,\rho)^-$ be two circular paths, where we make $R\to\infty$ and $\rho\to 0$. We join the two circles by two line segments with a narrow neck in between, as shown in Figure 3.4. Our choice will make $\log z$ holomorphic inside the contour. Moreover, we have $\log z = \ln|z|$ just above $\mathbb{R}_+$ and $\log z = \ln|z| + 2\pi\mathrm{i}$ just below $\mathbb{R}_+$.

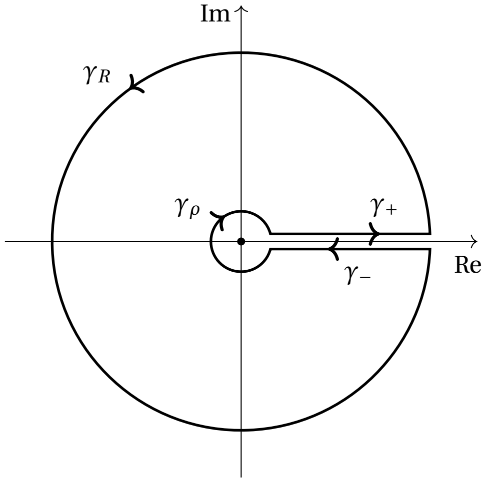

Figure 3.4: A keyhole contour.

Example 3.15

Evaluate $\displaystyle\int_0^\infty \frac{x^{p-1}}{(1+x)^m}\,\mathrm{d}x$, where $m \in \mathbb{Z}^+$ and $p \in (0,m) \setminus \mathbb{Z}$.

Solution. Let $f(z) = \dfrac{z^{p-1}}{(1+z)^m} = \dfrac{\mathrm{e}^{(p-1)\log z}}{(1+z)^m}$. Since $p \notin \mathbb{Z}$, f is multi-valued. We choose the positive real axis as the cut line and consider the keyhole contour $\Gamma = \gamma_R \star \gamma_- \star \gamma_\rho^- \star \gamma_+$ in Figure 3.4.

On γ_+, $z^{p-1} = \mathrm{e}^{(p-1)\ln x} = x^{p-1}$. On γ_-, $z^{p-1} = \mathrm{e}^{(p-1)\log z} = \mathrm{e}^{(p-1)(\ln x + 2\pi\mathrm{i})} = \mathrm{e}^{2p\pi\mathrm{i}}\,x^{p-1}$. We can see that f has a pole of order m at $z=-1$. By Residue Theorem, we have:

$$\int_\rho^R \frac{x^{p-1}}{(1+x)^m}\,\mathrm{d}x + \int_{\gamma_R} \frac{z^{p-1}}{(1+z)^m}\,\mathrm{d}z - \int_\rho^R \frac{\mathrm{e}^{2p\pi\mathrm{i}}\,x^{p-1}}{(1+x)^m}\,\mathrm{d}x - \int_{\gamma_\rho} \frac{z^{p-1}}{(1+z)^m}\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}(f,-1)$$

Since $\gamma_R(t) = R\mathrm{e}^{\mathrm{i}t},\ t \in [0,2\pi]$, we have:

$$\begin{aligned}\left|\int_{\gamma_R} \frac{z^{p-1}}{(1+z)^m}\,\mathrm{d}z\right| &= \left|\int_0^{2\pi} \frac{R^{p-1}\,\mathrm{e}^{(p-1)\mathrm{i}t}}{(1+R\mathrm{e}^{\mathrm{i}t})^m} R\mathrm{i}\mathrm{e}^{\mathrm{i}t}\,\mathrm{d}t\right| \\ &\leqslant 2\pi \cdot \sup_{z\in\gamma_R^*}\left|\frac{R^{p-1}\,\mathrm{e}^{(p-1)\mathrm{i}t}}{(1+R\mathrm{e}^{\mathrm{i}t})^m} R\mathrm{i}\mathrm{e}^{\mathrm{i}t}\right| \leqslant 2\pi\frac{R^p}{(R-1)^m} \to 0\end{aligned}$$

as $R \to \infty$, because $p < m$.

Similarly on γ_ρ, we have:

$$\left|\int_{\gamma_\rho} \frac{z^{p-1}}{(1+z)^m}\,\mathrm{d}z\right| \leqslant 2\pi\frac{\rho^p}{(\rho-1)^m} \to 0$$

as $\rho \to 0$.

Then by letting $R \to \infty$ and $\rho \to 0$ in the Residue Theorem, we obtain that

$$\left(1-\mathrm{e}^{2p\pi\mathrm{i}}\right)\int_0^\infty \frac{x^{p-1}}{(1+x)^m}\,\mathrm{d}x = 2\pi\mathrm{i}\,\mathrm{Res}(f,-1)$$

Now we compute the residue. If $m=1$, then

$$\mathrm{Res}(f,-1) = \lim_{z\to-1}(z+1)\frac{z^{p-1}}{(1+z)} = (-1)^{p-1} = -\mathrm{e}^{p\pi\mathrm{i}}$$

If $m > 1$, then

$$\begin{aligned}\mathrm{Res}(f,-1) &= \frac{1}{(m-1)!}\lim_{z\to-1}\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}\left((z+1)^m\frac{z^{p-1}}{(1+z)^m}\right) \\ &= \frac{1}{(m-1)!}\lim_{z\to-1}(p-1)(p-2)\cdots(p-m+1)z^{p-m}\end{aligned}$$

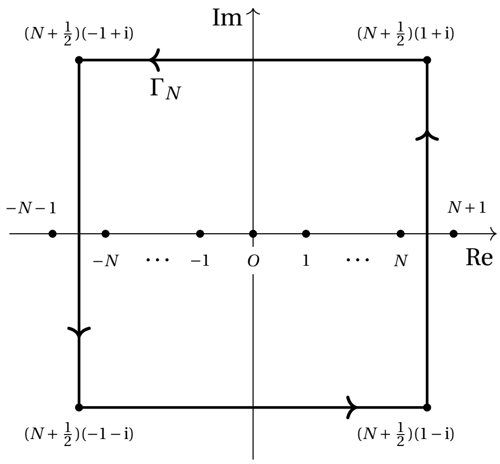

Figure 3.5: A square contour.

$$= -\frac{\mathrm{e}^{p\pi\mathrm{i}}(1-p)(2-p)\cdots(m-1-p)}{(m-1)!} = -\mathrm{e}^{p\pi\mathrm{i}}\prod_{j=1}^{m-1}\left(1-\frac{p}{j}\right)$$

Hence we have:

$$\int_0^\infty \frac{x^{p-1}}{1+x}\,\mathrm{d}x = 2\pi\mathrm{i}\frac{-\mathrm{e}^{p\pi\mathrm{i}}}{1-\mathrm{e}^{2p\pi\mathrm{i}}} = 2\pi\mathrm{i}\cdot\frac{1}{2\mathrm{i}\sin p\pi} = \frac{\pi}{\sin p\pi}$$

and for $m>1$:

$$\int_0^\infty \frac{x^{p-1}}{(1+x)^m}\,\mathrm{d}x = \frac{\pi}{\sin p\pi}\prod_{j=1}^{m-1}\left(1-\frac{p}{j}\right)$$

□

3.5 Infinite Series

Residue Theorem are also useful in calculating the sum of an infinite series. We will use the function $\cot\pi z$. Since $\sin z$ has simple zeros at $\pi\mathbb{Z}$, $\cot\pi z$ has simple poles at $\mathbb{Z}$. We have the following lemma:

Lemma 3.16

Suppose that φ is holomorphic near $n\in\mathbb{Z}$ with $\varphi(n)\neq 0$. Then $f(z):=\varphi(z)\cot\pi z$ has a simple pole at $z=n$ with residue $\operatorname{Res}(f,n)=\varphi(n)/\pi$.

Proof. It is easy to see that $z=n$ is a simple pole of f. So the residue is calculated by

$$\begin{aligned}\operatorname{Res}(f,n) &= \lim_{z\to n}(z-n)\varphi(z)\cot\pi z = \varphi(n)\lim_{z\to n}\frac{(z-n)\cos\pi z}{\sin\pi z}\\ &= \varphi(n)\lim_{z\to n}(z-n)\frac{\cos\pi z\cos\pi n+\sin\pi z\sin\pi n}{\sin\pi z\cos\pi n-\cos\pi z\sin\pi n} \quad (\sin\pi n=0)\\ &= \varphi(n)\lim_{z\to n}\frac{(z-n)\cos\pi(z-n)}{\sin\pi(z-n)} = \frac{\varphi(n)}{\pi}\lim_{w\to 0}\frac{w(1+O(w^2))}{w+O(w^3)} \quad (w:=\pi(z-n))\\ &= \varphi(n)/\pi\end{aligned}$$

□

Now let us consider the positively oriented square contour Γ_N with vertices at $(N+\frac{1}{2})(\pm1\pm\mathrm{i})$. The next lemma shows that $\cot\pi z$ is uniformly bounded on all Γ_N^*.

Lemma 3.17

There exists a $C>0$ such that $\sup\left\{|\cot\pi z|:\ z\in\Gamma_N^*\right\}<C$ for all $N\in\mathbb{N}$.

Proof. We consider the horizontal and vertical sides of the square separately. On the horizontal sides, we have $z=x\pm(N+\frac{1}{2})\mathrm{i},\ x\in[-(N+\frac{1}{2}),N+\frac{1}{2}]$.

$$|\cot\pi z| = \left|\frac{\mathrm{e}^{\mathrm{i}\pi z}+\mathrm{e}^{-\mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z}-\mathrm{e}^{-\mathrm{i}\pi z}}\right| = \left|\frac{\mathrm{e}^{\mathrm{i}\pi(x\pm\mathrm{i}(N+\frac{1}{2}))}+\mathrm{e}^{-\mathrm{i}\pi(x\pm\mathrm{i}(N+\frac{1}{2}))}}{\mathrm{e}^{\mathrm{i}\pi\mathrm{i}(x\pm(N+\frac{1}{2}))}-\mathrm{e}^{-\mathrm{i}\pi(x\pm\mathrm{i}(N+\frac{1}{2}))}}\right|$$

$$\begin{aligned}
&= \left| \frac{e^{i\pi x \mp \pi(N+\frac{1}{2})} + e^{-i\pi x \pm \pi(N+\frac{1}{2})}}{e^{i\pi x \mp \pi(N+\frac{1}{2})} - e^{-i\pi x \pm \pi(N+\frac{1}{2})}} \right| \\
&\leqslant \left| \frac{e^{\pi(N+\frac{1}{2})} + e^{-\pi(N+\frac{1}{2})}}{e^{\pi(N+\frac{1}{2})} - e^{-\pi(N+\frac{1}{2})}} \right| = |\coth \pi(N+1/2))| \\
&\leqslant \left| \coth \frac{\pi}{2} \right| \qquad \text{(since } \coth x \text{ is decreasing when } x > 0)
\end{aligned}$$

On the vertical sides, we have $z = \pm(N+\frac{1}{2}) + yi$, $y \in \left[-(N+\frac{1}{2}), N+\frac{1}{2}\right]$. From elementary trigonometric equalities, we know that $\cot(z+\pi/2) = -\tan z$ and that $\tan(z+N\pi) = \tan z$ for $N \in \mathbb{Z}$. We have:

$$|\cot \pi z| = |\cot(\pm(\pi/2 + N\pi) + i\pi y)| = |\tan i\pi y| = |\tanh \pi y| < 1$$

Hence we have $\sup_{z \in \Gamma_N^*} |\cot \pi z| < \min\left(1, \left|\coth \frac{\pi}{2}\right|\right)$ for all $N \in \mathbb{N}$. □

Remark. Given the two preceding lemmata, we are now able to sum some infinite series.

Example 3.18

Evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Solution. By Cauchy's integral test the series converges. Let $f(z) = \cot \pi z / z^2$. We consider the square contour Γ_N defined above. By Lemma 3.16, we know that f has simple poles at $z = n \in \mathbb{Z} \setminus \{0\}$ with residues $\operatorname{Res}(f, n) = \frac{1}{\pi n^2}$. In addition, $z = 0$ is a triple pole of f. The residue is given by

$$\begin{aligned}
\operatorname{Res}(f, 0) &= \frac{1}{2} \lim_{z \to 0} \frac{d^2}{dz^2} z \cot \pi z = \frac{1}{2} \lim_{z \to 0} \left(-2 \frac{\pi}{\sin^2 \pi z} + 2 \frac{\pi^2 z \cos \pi z}{\sin^3 \pi z} \right) \\
&= \pi \lim_{z \to 0} \frac{\pi z \cos \pi z - \sin \pi z}{\sin^3 \pi z} \\
&= \pi \lim_{z \to 0} \frac{\pi z(1 - \pi^2 z^2/2 + O(z^4)) - (\pi z - \pi^3 z^3/6 + O(z^5))}{(\pi z + O(z^3))^3} \\
&= \pi \frac{-\pi^3/2 + \pi^3/6}{\pi^3} = -\frac{\pi}{3}
\end{aligned}$$

Apply the Residue Theorem to f along Γ_N:

$$\oint_{\Gamma_N} f(z)\, dz = \sum_{n=-N}^{N} \operatorname{Res}(f, n) = \frac{\pi}{3} + 2 \sum_{n=1}^{N} \frac{1}{\pi n^2}$$

By Lemma 3.17, we have

$$\left| \oint_{\Gamma_N} f(z)\, dz \right| = \left| \oint_{\Gamma_N} \frac{\cot \pi z}{z^2}\, dz \right| \leqslant 4 \cdot (2N+1) \cdot \frac{C}{(N+1/2)^2} \to 0$$

as $N \to \infty$.

Therefore:

$$0 = -\frac{\pi}{3} + 2 \sum_{n=1}^{\infty} \frac{1}{\pi n^2} \implies \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

□

Remark. In the previous example, the residue of $\cot \pi z / z^2$ at $z = 0$ can also be calculated by directly expanding $\sin z$ and $\cos z$ at $z = 0$:

$$\begin{aligned}
\cot z = \frac{\cos z}{\sin z} &= \left(1 - \frac{z^2}{2} + O(z^4)\right)\left(z - \frac{z^3}{6} + O(z^5)\right)^{-1} \\
&= \left(1 - \frac{z^2}{2} + O(z^4)\right) \frac{1}{z} \left(1 - z^2\left(\frac{1}{6} + O(z^2)\right)\right)^{-1} \\
&= \frac{1}{z}\left(1 - \frac{z^2}{2} + O(z^4)\right)\left(1 + z^2\left(\frac{1}{6} + O(z^2)\right) + O(z^4)\right)
\end{aligned}$$

$$= \frac{1}{z} - \frac{z}{3} + O(z^3)$$

Therefore we obtain the Laurent expansion of f at $z = 0$:

$$f(z) = \frac{\cot \pi z}{z^2} = \frac{1}{\pi z^3} - \frac{\pi}{3z} + O(z)$$

Hence we have $\operatorname{Res}(f, 0) = -\pi/3$, which is the same as in the example.

Remark. We will give another example, which sums an alternating series. Instead of $\cot \pi z$ we use $\csc \pi z$, for which we have the similar lemmata:

Lemma 3.19

Suppose φ is holomorphic near $n \in \mathbb{Z}$ with $\varphi(n) \neq 0$. Then $f(z) := \varphi(z)/\sin \pi z$ has a simple pole at $z = n$ with residue $\operatorname{Res}(f, n) = (-1)^n \varphi(n)/\pi$.

Lemma 3.20

There exists a $C > 0$ such that $\sup\left\{|\csc \pi z| : z \in \Gamma_N^*\right\} < C$ for all $N \in \mathbb{N}$.

The proofs are very much similar (and in fact easier).

Example 3.21

Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$.

Solution. By Leibniz's alternating series test the series converges. Let $f(z) = \dfrac{1}{(z^2+1)\sin \pi z}$. By Lemma 3.19, f has simple poles at every integer $n \in \mathbb{Z}$ with residues $\operatorname{Res}(f, n) = \dfrac{(-1)^n}{\pi(1+n^2)}$. In addition, f also has simple poles at $z = \pm \mathrm{i}$, where the residues are given by:

$$\operatorname{Res}(f, \mathrm{i}) = \lim_{z \to \mathrm{i}} \frac{(z-\mathrm{i})}{(z^2+1)\sin \pi z} = \frac{1}{2\mathrm{i}\sin(\mathrm{i}\pi)} = -\frac{1}{2\sinh \pi}$$
$$\operatorname{Res}(f, -\mathrm{i}) = \lim_{z \to -\mathrm{i}} \frac{(z+\mathrm{i})}{(z^2+1)\sin \pi z} = -\frac{1}{2\mathrm{i}\sin(-\mathrm{i}\pi)} = -\frac{1}{2\sinh \pi}$$

The integration of f along Γ_N:

$$\left|\oint_{\Gamma_N} f(z)\,\mathrm{d}z\right| = \left|\oint_{\Gamma_N} \frac{(-1)^n}{z^2+1}\,\mathrm{d}z\right| \leqslant 4 \cdot (2N+1) \cdot \frac{C}{(N+1/2)^2+1} \to 0$$

as $N \to \infty$.

Therefore by Residue Theorem:

$$\oint_{\Gamma_N} f(z)\,\mathrm{d}z = \operatorname{Res}(f, \mathrm{i}) + \operatorname{Res}(f, -\mathrm{i}) + \sum_{n=-N}^{N} \operatorname{Res}(f, n)$$
$$\xrightarrow{N\to\infty} \quad 0 = 2 \cdot -\frac{1}{2\sinh \pi} + \frac{1}{\pi} + 2\sum_{n=1}^{\infty} \frac{1}{\pi(n^2+1)} \quad \Longrightarrow \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{\pi}{2\sinh \pi} - \frac{1}{2}$$

□

3.6 Some More Examples

Example 3.22

Evaluate $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{\sinh x}\,\mathrm{d}x$

Solution. Let $f(z) = z/\sinh z$. This is a function with infinitely many poles, so we will not be able to choose a contour that encloses all the poles. However, we can use the periodicity of the function. At $z = 0$, observe that

$$\lim_{z\to 0}\frac{z}{\sinh z} = \lim_{z\to 0}\frac{z}{\sum_{n=0}^{\infty} z^{2n+1}/(2n+1)!} = 1$$

Then $z = 0$ is a removable singularity of f. Notice the periodicity of $\sinh z$: $\sinh(x) = \sinh(x + \mathrm{i}\pi)$ for $x \in \mathbb{R}$. $\sinh z$ has simple zeros at $\mathrm{i}\pi\mathbb{Z}$. f has simple poles at $\mathrm{i}\pi\mathbb{Z}\setminus\{0\}$ with residues:

$$\operatorname{Res}(f, n\pi\mathrm{i}) = \lim_{z\to n\pi\mathrm{i}}\frac{z(z-n\pi\mathrm{i})}{\sinh z} = n\pi\mathrm{i}\lim_{z\to 0}\frac{z}{\sinh z} = n\pi\mathrm{i}$$

We consider the rectangular contour with indentation at $z = \mathrm{i}\pi$, as shown in Figure 3.6.

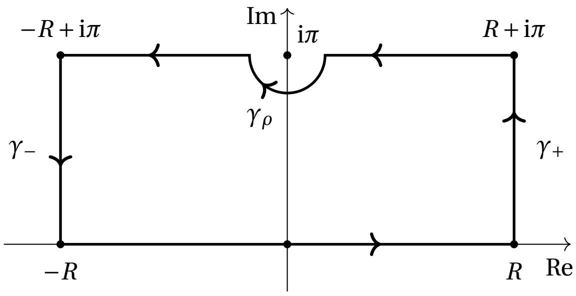

Figure 3.6: A rectangular contour with indentation at $z = \mathrm{i}\pi$.

We observe that f is holomorphic inside and along the above contour. Apply Cauchy's Theorem:

$$\int_{-R}^{R}\frac{x}{\sinh x}\,\mathrm{d}x + \int_{\gamma_+}\frac{z}{\sinh z}\,\mathrm{d}z + \int_{R}^{\rho}\frac{x+\mathrm{i}\pi}{\sinh(x+\mathrm{i}\pi)}\,\mathrm{d}x$$
$$+\int_{\gamma_\rho}\frac{z}{\sinh z}\,\mathrm{d}z + \int_{-\rho}^{-R}\frac{x+\mathrm{i}\pi}{\sinh(x+\mathrm{i}\pi)}\,\mathrm{d}x + \int_{\gamma_-}\frac{z}{\sinh z}\,\mathrm{d}z = 0$$

First we inspect the vertical sides of the contour:

$$\left|\int_{\gamma_+}\frac{z}{\sinh z}\,\mathrm{d}z\right| = \left|\int_0^{\pi}\frac{R+\mathrm{i}t}{\sinh(R+\mathrm{i}t)}\,\mathrm{i}\mathrm{d}t\right| = \left|\int_0^{\pi}\frac{R+\mathrm{i}t}{\sinh R\cos t + \mathrm{i}\cosh R\sin t}\,\mathrm{i}\mathrm{d}t\right|$$
$$\leqslant \pi\cdot\frac{R+\pi}{\sinh R}\to 0$$

as $R\to 0$. And it is similar for γ_-.

For the indentation near $z = \mathrm{i}\pi$, by Lemma 3.11, we have:

$$\lim_{\rho\to 0}\int_{\gamma_\rho}\frac{z}{\sinh z}\,\mathrm{d}z = \mathrm{i}\pi\operatorname{Res}(f,\mathrm{i}\pi) = -\pi^2$$

Let $R\to\infty$ and $\rho\to 0$ in Cauchy's Theorem. Substitute the preceding equations into the equation:

$$\int_{-\infty}^{+\infty}\frac{x}{\sinh x}\,\mathrm{d}x - \pi^2 - \int_{-\infty}^{+\infty}\frac{x+\mathrm{i}\pi}{-\sinh x}\,\mathrm{d}x = 0$$

Taking the real part, we have:

$$\int_{-\infty}^{+\infty}\frac{x}{\sinh x}\,\mathrm{d}x = \frac{\pi^2}{2}$$

□

Example 3.23. Gaussian Integral.

Evaluate $\displaystyle\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x$.

Solution. Normally we may consider the function e^{-z^2}. But this is entire and is not well-behaved as $z\to\infty$. Here we define $f(z) = \mathrm{e}^{\mathrm{i}\pi z^2}/\sin\pi z$ and consider its integral on a parallelogram contour, as shown in Figure 3.7.

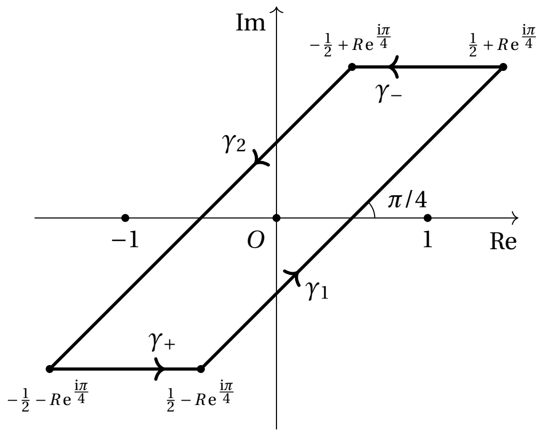

Figure 3.7: Contour for the Gaussian Integral.

The nominator of f is entire. By Lemma 3.19, we know that f has a simple pole at $z=0$ with residue $\mathrm{Res}(f,0)=1/\pi$. By Residue Theorem, we have

$$\int_{\gamma_1} f(z)\,\mathrm{d}z+\int_{\gamma_-} f(z)\,\mathrm{d}z+\int_{\gamma_2} f(z)\,\mathrm{d}z+\int_{\gamma_+} f(z)\,\mathrm{d}z=2\pi\mathrm{i}\,\mathrm{Res}(f,0)=2\mathrm{i}$$

On γ_1, we have $\gamma_1(t)=\frac{1}{2}+t\,\mathrm{e}^{\mathrm{i}\pi/4},\ t\in[-R,R]$. The integral:

$$\begin{aligned}\int_{\gamma_1}\frac{\mathrm{e}^{\mathrm{i}\pi z^2}}{\sin\pi z}\,\mathrm{d}z&=\int_{-R}^{R}\frac{\exp(\mathrm{i}\pi(\frac{1}{2}+t\,\mathrm{e}^{\mathrm{i}\pi/4})^2)}{\sin(\pi(\frac{1}{2}+t\,\mathrm{e}^{\mathrm{i}\pi/4}))}\cdot\mathrm{e}^{\mathrm{i}\pi/4}\,\mathrm{d}t\\&=\int_{-R}^{R}\frac{\exp(\mathrm{i}\pi(\frac{1}{2}+\mathrm{i}t^2+t\,\mathrm{e}^{\mathrm{i}\pi/4}))}{\cos(\pi t\,\mathrm{e}^{\mathrm{i}\pi/4})}\,\mathrm{d}t\\&=\mathrm{i}\int_{-R}^{R}\mathrm{e}^{-\pi t^2}\,\frac{\exp(\mathrm{i}\pi t\,\mathrm{e}^{\mathrm{i}\pi/4})}{\cos(\pi t\,\mathrm{e}^{\mathrm{i}\pi/4})}\,\mathrm{d}t\end{aligned}$$

Similarly on γ_2, we have

$$\int_{\gamma_2}\frac{\mathrm{e}^{\mathrm{i}\pi z^2}}{\sin\pi z}\,\mathrm{d}z=\mathrm{i}\int_{-R}^{R}\mathrm{e}^{-\pi t^2}\,\frac{\exp(-\mathrm{i}\pi t\,\mathrm{e}^{\mathrm{i}\pi/4})}{\cos(\pi t\,\mathrm{e}^{\mathrm{i}\pi/4})}\,\mathrm{d}t$$

Adding up:

$$\begin{aligned}\int_{\gamma_1}\frac{\mathrm{e}^{\mathrm{i}\pi z^2}}{\sin\pi z}\,\mathrm{d}z+\int_{\gamma_2}\frac{\mathrm{e}^{\mathrm{i}\pi z^2}}{\sin\pi z}\,\mathrm{d}z&=\mathrm{i}\int_{-R}^{R}\mathrm{e}^{-\pi t^2}\,\frac{\exp(\mathrm{i}\pi t\,\mathrm{e}^{\mathrm{i}\pi/4})+\exp(-\mathrm{i}\pi t\,\mathrm{e}^{\mathrm{i}\pi/4})}{\cos(\pi t\,\mathrm{e}^{\mathrm{i}\pi/4})}\,\mathrm{d}t\\&=\mathrm{i}\int_{-R}^{R}\mathrm{e}^{-\pi t^2}\,\frac{2\cos(\pi t\,\mathrm{e}^{\mathrm{i}\pi/4})}{\cos(\pi t\,\mathrm{e}^{\mathrm{i}\pi/4})}\,\mathrm{d}t=2\mathrm{i}\int_{-R}^{R}\mathrm{e}^{-\pi t^2}\,\mathrm{d}t\end{aligned}$$

On γ_-, the integral:

$$\begin{aligned}\left|\int_{\gamma_-}\frac{\mathrm{e}^{\mathrm{i}\pi z^2}}{\sin\pi z}\,\mathrm{d}z\right|&=\left|\int_{1/2}^{-1/2}\frac{\exp(\mathrm{i}\pi(x+R\,\mathrm{e}^{\mathrm{i}\pi/4})^2)}{\sin(\pi(x+R\,\mathrm{e}^{\mathrm{i}\pi/4}))}\,\mathrm{d}x\right|\\&=2\left|\int_{-1/2}^{1/2}\frac{\mathrm{e}^{\mathrm{i}\pi x(x+\sqrt{2}R)}\,\mathrm{e}^{-\pi R(R+\sqrt{2}x)}}{\mathrm{e}^{\mathrm{i}\pi(x+R/\sqrt{2})}\,\mathrm{e}^{-\pi R/\sqrt{2}}+\mathrm{e}^{-\mathrm{i}\pi(x+R/\sqrt{2})}\,\mathrm{e}^{\pi R/\sqrt{2}}}\,\mathrm{d}x\right|\\&\leqslant 2\sup_{x\in[-\frac{1}{2},\frac{1}{2}]}\left|\frac{\mathrm{e}^{-\pi R(R+\sqrt{2}x)}}{\mathrm{e}^{\pi R/\sqrt{2}}-\mathrm{e}^{-\pi R/\sqrt{2}}}\right|\to 0\end{aligned}$$

as $R\to\infty$. Similarly, the integral along γ_+ also vanishes as $R\to\infty$.

By letting $R\to\infty$ in the Residue Theorem, we have:

$$2\mathrm{i}\int_{-\infty}^{+\infty}\mathrm{e}^{-\pi x^2}\,\mathrm{d}x=2\mathrm{i}\Longrightarrow\int_{-\infty}^{+\infty}\mathrm{e}^{-x^2}\,\mathrm{d}x=\sqrt{\pi}$$

□

Example 3.24

Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{3+\cos\theta+2\sin\theta}$.

Remark. This is an example of integral with finite interval. More generally, consider integrals of the form $\displaystyle\int_0^{2\pi} R(\sin\theta,\cos\theta)\, d\theta$, where the integrand is a rational function of $\sin\theta$ and $\cos\theta$. We use the substitution $z = e^{i\theta}$. Then the integral can be regarded as a contour integral along the unit disk. Moreover, we have:

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) = \frac{1}{2i}(z - z^{-1})$$
$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + z^{-1})$$
$$d\theta = (i e^{i\theta})^{-1} d(e^{i\theta}) = -i z^{-1} dz$$

So we can use the Residue Theorem to compute the integral.

Solution. Let $\gamma(0,1)$ be the positively oriented contour along the unit disk. We have:

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{3+\cos\theta+2\sin\theta} &= \oint_{\gamma(0,1)} \frac{1}{3+\frac{1}{2}(z+z^{-1})+\frac{1}{i}(z-z^{-1})} \frac{1}{iz} dz \\ &= \frac{10}{i+2}\oint_{\gamma(0,1)} \frac{dz}{(5z+1+2i)(z+1+2i)}\end{aligned}$$

Let f be the integrand above. Then f has two simple poles: $z = -\frac{1}{5}(1+2i)$ and $z = -(1+2i)$. By Residue Theorem, we have:

$$\begin{aligned}\oint_{\gamma(0,1)} \frac{dz}{(5z+1+2i)(z+1+2i)} &= 2\pi i \operatorname{Res}(f, -\frac{1}{5}(1+2i)) \\ &= 2\pi i \lim_{z\to -\frac{1}{5}(1+2i)} \frac{1}{5(z+1+2i)} = \frac{\pi i}{2(1+2i)}\end{aligned}$$

Hence

$$\int_0^{2\pi} \frac{d\theta}{3+\cos\theta+2\sin\theta} = \frac{10}{i+2}\cdot\frac{\pi i}{2(1+2i)} = \pi$$

□

Example 3.25

Evaluate $\displaystyle\int_{-1}^{1} \frac{dx}{(1+x)^{2/3}(1-x)^{1/3}}$.

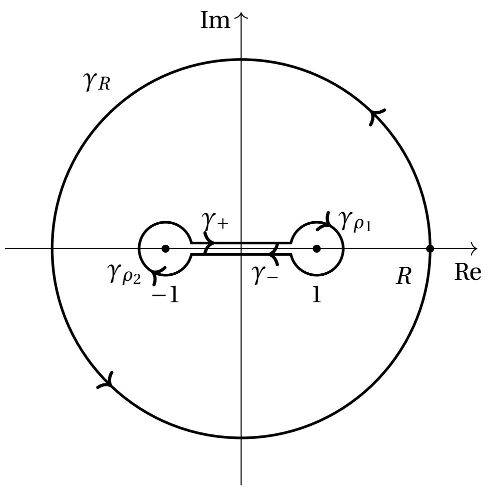

Figure 3.8: A dumbbell contour.

Solution. Let $f(z) = (1-z)^{-1/3}(1+z)^{-2/3}$. This is a multifunction with branch points at $z=1$ and $z=-1$. We shift to polar coordinates (see Example 0.23 for detail) and write $z = 1 + r\,\mathrm{e}^{\mathrm{i}\theta} = -1 + s\,\mathrm{e}^{\mathrm{i}\varphi}$. Then $f(z) = r^{-1/3}s^{-2/3}\,\mathrm{e}^{-\mathrm{i}(\theta+2\varphi)/3}$. The admissible contours are those which winds around both or none of the two branch points, so we can obtain a holomorphic branch of f by doing the branch cut at $[-1,1]$. We consider a "dumbbell-shaped" contour $\gamma_d := \gamma_{\rho_1} \star \gamma_- \star \gamma_{\rho_2} \star \gamma_+$ that encloses both poles, as shown in Figure 3.8. Here $\gamma_{\rho_1} := \gamma(1,\rho)$ and $\gamma_{\rho_2} := \gamma(-1,\rho)$. Let $\gamma_R := \gamma(0,R)$ be the positively oriented circular contour. Then f is holomorphic in the interior of the cycle $\Gamma := \gamma_R + \gamma_d$. By Cauchy's Theorem we have:

$$\oint_{\gamma_R} f(z)\,\mathrm{d}z + \int_{\gamma_{\rho_1}} f(z)\,\mathrm{d}z + \int_{\gamma_{\rho_2}} f(z)\,\mathrm{d}z + \int_{\gamma_+} f(z)\,\mathrm{d}z + \int_{\gamma_-} f(z)\,\mathrm{d}z = 0$$

Notice that

$$\begin{aligned}
\lim_{z\to\infty} z f(z) &= \mathrm{e}^{\mathrm{i}\pi/3} \lim_{z\to\infty} \frac{z}{(z-1)^{1/3}(1+z)^{2/3}} \\
&= \mathrm{e}^{\mathrm{i}\pi/3} \lim_{z\to\infty} \left|\frac{z}{z-1}\right|^{1/3} \left|\frac{z}{1+z}\right|^{2/3} \exp\left(-\frac{\mathrm{i}}{3}\arg(z-1) - \frac{2\mathrm{i}}{3}\arg(1+z) + \mathrm{i}\arg(z)\right) \\
&= \mathrm{e}^{\mathrm{i}\pi/3} \lim_{z\to\infty} \exp\left(-\frac{\mathrm{i}}{3}\arg(z) - \frac{2\mathrm{i}}{3}\arg(z) + \mathrm{i}\arg(z)\right) = \mathrm{e}^{\mathrm{i}\pi/3}
\end{aligned}$$

On the circular contour γ_R, the calculation is very similar to the proof of the Indentation Lemma. Let $g(z) = zf(z) - \mathrm{e}^{-\mathrm{i}\pi/3}$. Then $\lim_{z\to\infty} g(z) = 0$. Hence:

$$\begin{aligned}
\lim_{R\to\infty} \oint_{\gamma_R} f(z)\,\mathrm{d}z &= \lim_{R\to\infty} \oint_{\gamma_R} \frac{g(z) + \mathrm{e}^{-\mathrm{i}\pi/3}}{z}\,\mathrm{d}z \\
&= \lim_{R\to\infty} \oint_{\gamma_R} \frac{g(z)}{z}\,\mathrm{d}z + \mathrm{e}^{-\mathrm{i}\pi/3} \lim_{R\to\infty} \oint_{\gamma_R} \frac{1}{z}\,\mathrm{d}z \\
&= 2\pi\mathrm{i}\,\mathrm{e}^{\mathrm{i}\pi/3}
\end{aligned}$$

On the indented circular arcs γ_{ρ_1} and γ_{ρ_2}, since

$$\lim_{z\to1}(z-1)f(z) = \mathrm{e}^{-\mathrm{i}\pi/3} \lim_{z\to1} \left(\frac{z-1}{z+1}\right)^{2/3} = 0$$
$$\lim_{z\to-1}(z+1)f(z) = \lim_{z\to-1} \left(\frac{z+1}{1-z}\right)^{1/3} = 0$$

By the Indentation Lemma, we have:

$$\lim_{\rho\to0} \int_{\gamma_{\rho_1}} f(z)\,\mathrm{d}z = 0, \quad \lim_{\rho\to0} \int_{\gamma_{\rho_2}} f(z)\,\mathrm{d}z = 0$$

On the line segment γ_+, $\arg(1-z) = 0$, $\arg(z+1) = 0$, therefore

$$f(z) = (1-x)^{-1/3}(1+x)^{-2/3}$$

On the line segment γ_-, $\arg(1-z) = -2\pi$, $\arg(z+1) = 0$, therefore

$$f(z) = (\mathrm{e}^{-2\pi\mathrm{i}}(1-x))^{-1/3}(1+x))^{-2/3} = \mathrm{e}^{2\pi\mathrm{i}/3}(1-x)^{-1/3}(1+x)^{-2/3}$$

Letting $R\to\infty$ and $\rho\to0$ in Cauchy's Theorem, we have:

$$\begin{aligned}
&\int_{-1}^{1} \frac{\mathrm{d}x}{(1-x)^{1/3}(1+x)^{2/3}} + \int_{1}^{-1} \frac{\mathrm{e}^{2\pi\mathrm{i}/3}\,\mathrm{d}x}{(1-x)^{1/3}(1+x)^{2/3}} + 2\pi\mathrm{i}\,\mathrm{e}^{\mathrm{i}\pi/3} = 0 \\
\implies &\int_{-1}^{1} \frac{\mathrm{d}x}{(1-x)^{1/3}(1+x)^{2/3}} = \frac{-2\pi\mathrm{i}\,\mathrm{e}^{\mathrm{i}\pi/3}}{1-\mathrm{e}^{2\pi\mathrm{i}/3}} = \frac{\pi}{\sin(\pi/3)} = \frac{2\pi}{\sqrt{3}}
\end{aligned}$$

□

Chapter 4

Conformal Mappings

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4.1 Extended Complex Plane

In the classification of singularities, we are motivated to process functions in the extended plane $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ rather than in $\mathbb{C}$. This is an example of one-point compactification of $\mathbb{C}$. To fully make sense of continuity and holomorphicity in $\mathbb{C}_\infty$, we shall use two approaches:

1. **Riemann Sphere** $\mathbb{S}^2$. We will identify points on $\mathbb{S}^2 := \{\boldsymbol{x} \in \mathbb{R}^3 : \|\boldsymbol{x}\| = 1\}$ with points on $\mathbb{C}$ through the stereograhic projection. The north pole on $\mathbb{S}^2$ will be identified as ∞.

2. **Projective Line** $\mathbb{CP}^1$. We denote the set of all one-dimensional subspaces in $\mathbb{C}^2$ by $\mathbb{CP}^1$. We shall indentify points $z \in \mathbb{C}$ with the subspace $\langle(z,1)\rangle \subseteq \mathbb{C}^2$. Then the point of infinity is identified as the subspace $\langle(1,0)\rangle$.

4.1.1 Riemann Sphere.

We begin our discussion with the stereograhic projection.

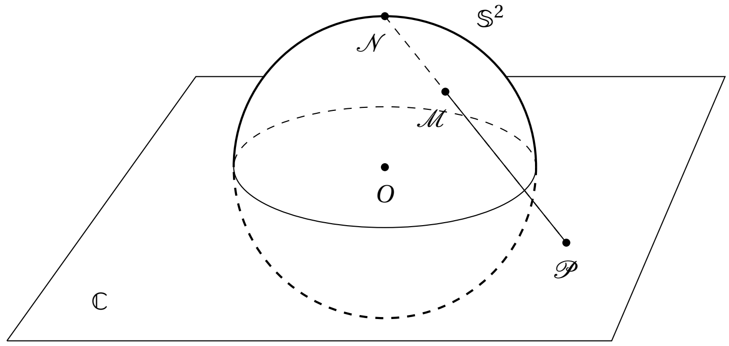

Figure 4.1: Stereographic Projection.

Definition 4.1. Stereographic Projection.

Consider the unit sphere $\mathbb{S}^2 := \{\boldsymbol{x} \in \mathbb{R}^3 : \|\boldsymbol{x}\| = 1\}$ in $\mathbb{R}^3$. Let $\mathcal{N}(0,0,1)$ be the north pole of S^2. We view the complex plane $\mathbb{C}$ as a copy of $\mathbb{R}^2$ in $\mathbb{R}^3$: $\{(x,y,0) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$. Given a point $M \in S^2 \setminus \{\mathcal{N}\}$, the line connecting $\mathcal{N}$ and $\mathcal{M}$ has a unique intersection with $\mathbb{C}$ at $\mathcal{P}$. The mapping $\mathcal{M} \mapsto \mathcal{P}$ is a bijection between $\mathbb{S}^2 \setminus \{\mathcal{N}\}$ and $\mathbb{C}$, and is called the stereographic projection.

Remark. By some simple calculations the stereographic projection is given explicitly by:

$$(X,Y,Z) \mapsto \frac{X+Y\mathrm{i}}{1-Z}$$

The inverse mapping is given by:

$$x+\mathrm{i}y \mapsto \frac{1}{x^2+y^2+1}\left(2x, 2y, x^2+y^2-1\right) = \frac{1}{|z|^2+1}(2\operatorname{Re} z, 2\operatorname{Im} z, |z|^2-1)$$

Definition 4.2. Riemann Sphere.

If we identify $\mathcal{N}$ with ∞ on the plane, then the stereographic projection becomes a bijection between $\mathbb{S}^2$ and $\mathbb{C}_\infty$. The sphere $\mathbb{S}^2$ is called the Riemann sphere.

Lemma 4.3

As a subset $\mathbb{R}^3$, the Riemann sphere $\mathbb{S}^2$ is naturally a metric space. It induces a metric on $\mathbb{C}_\infty$:

$$d(z,w) = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}, \quad d(z,\infty) = \frac{2}{\sqrt{1+|z|^2}}$$

for any $z, w \in \mathbb{C}$.

Proof. For convenience, we denote the image of $z \in \mathbb{C}$ under the inverse mapping of stereographic projection by $S(z)$. Then from the previous remark we have

$$S(z) = \frac{1}{|z|^2+1}(2\mathrm{Re}(z), 2\mathrm{Im}(z), |z|^2-1)$$

For $z, w \in \mathbb{C}$,

$$\begin{aligned}
d(z,w) &:= \|S(z)-S(w)\| = \sqrt{2-2S(z)\cdot S(w)} \\
&= \sqrt{2-\frac{2}{(|z|^2+1)(|w|^2+1)}(2\mathrm{Re}(z), 2\mathrm{Im}(z), |z|^2-1)\cdot(2\mathrm{Re}(w), 2\mathrm{Im}(w), |w|^2-1)} \\
&= \sqrt{2-\frac{8\mathrm{Re}(zw)+8\mathrm{Im}(zw)+2(|z|^2-1)(|w|^2-1)}{(|z|^2+1)(|w|^2+1)}} \\
&= \sqrt{\frac{4|z|^2+4|w|^2-4(z\overline{w}+\overline{z}w)}{(|z|^2+1)(|w|^2+1)}} = \frac{2|z-w|}{\sqrt{1+|z|^2}\sqrt{1+|w|^2}}
\end{aligned}$$

$$d(z,\infty) := \|S(z)-(0,0,1)\| = \sqrt{2-2S(z)\cdot(0,0,1)}$$
$$= \sqrt{2-2\frac{|z|^2-1}{1+|z|^2}} = \frac{2}{\sqrt{1+|z|^2}}$$

□

Remark. We have made $\mathbb{C}_\infty$ into a metric space, so that we can define continuity of functions on $\tilde{\mathbb{C}}$. We can also see that $S(z) \to \mathcal{N}(0,0,1)$ as $|z| \to \infty$, so it is legitimate to identify $\mathcal{N}$ with ∞. In particular, a meromorphic function $f : U \to \mathbb{C}$ naturally extends to a continuous function from U to $\mathbb{C}_\infty$.

Remark. The geometry of Riemann sphere nicely unites lines and circles on the complex plane, as shown in the following proposition:

Proposition 4.4

The mapping $S : \mathbb{C} \to \mathbb{S}^2$ induces the following bijective correspondence:

(i) Lines in $\mathbb{C} \Leftrightarrow$ circles in $\mathbb{S}^2$ containing $\mathcal{N}$;

(ii) Circles in $\mathbb{C} \Leftrightarrow$ circles in $\mathbb{S}^2$ not containing $\mathcal{N}$.

Proof. This is purely calculation in analytic geometry. A circle on $\mathbb{S}^2$ is obtained by intersecting the sphere with a plane $\{(X,Y,Z) \in \mathbb{R}^3 : aX + bY + cZ = d\}$ where $a^2 + b^2 + c^2 > d^2$. The plane contains $\mathcal{N}(0,0,1)$ if and only if $c = d$.

For $z = x + \mathrm{i}y \in \mathbb{C}$, $S(z)$ lies in the above plane, if and only if:

$$2ax + 2by + c(x^2 + y^2 - 1) = d(x^2 + y^2 + 1)$$
$$\Leftrightarrow (c-d)(x^2+y^2) + 2ax + 2by - (c+d) = 0$$

We can see that this is the equation of a line for $c = d$. If $c \neq d$, the equation simplifies to

$$\left(x + \frac{a}{c-d}\right)^2 + \left(y + \frac{b}{c-d}\right)^2 = \frac{a^2+b^2+c^2-d^2}{(c-d)^2} \tag{4.1}$$

The RHS>0. It is indeed an equation of a circle. Hence circles on $\mathbb{S}^2$ corresponds to circles and lines in $\mathbb{C}$.

Conversely, for a line in $\mathbb{C}$, its equation can certainly be expressed as $2ax + 2by = c + d$, which corresponds to a circle on $\mathbb{S}^2$ containing $\mathcal{N}$. For a circle in $\mathbb{C}$ of the form $(x+A)^2 + (y+B)^2 = C^2$, we put $c - d = 1$ and $a = A, b = B, c = (C^2 - A^2 - B^2 + 1)/2$. Then the equation of the circle becomes Equation (4.1). So it corresponds to a circle on $\mathbb{S}^2$ not containing $\mathcal{N}$. □

Proposition 4.5

Suppose $U \in \mathbb{C}$ is a domain. Then U is simply-connected if and only if $\mathbb{C}_\infty \setminus U$ is connected.

Proof. "$\Longleftarrow$": Suppose $\mathbb{C}_\infty \setminus U$ is connected. Let γ be a closed path in U and V be the exterior of γ. Then V is connected and has ∞ as its limit point. Since $\mathbb{C}_\infty \setminus U \subseteq \mathbb{C}_\infty \setminus \gamma^*$ is connected and contains ∞, we have $\mathbb{C}_\infty \setminus U \subseteq V \cup \{\infty\}$. Therefore, we have $I(\gamma, z) = 0$ for all $z \in \mathbb{C} \setminus U$. Hence the interior of γ is contained in U. Since γ is arbitrary, by Proposition 1.38, U is simply-connected.

"$\Longrightarrow$": See the Appendix B.1 of **[Stein]**. □

4.1.2 Projective Line.

Definition 4.6. Projective Line.

The complex projective line $\mathbb{CP}^1$ is defined by the set of all one-dimensional subspaces of $\mathbb{C}^2$. The subspace generated by $(z,w) \in \mathbb{C}^2 \setminus \{0\}$ would be denoted by $[z:w]$. The coordinates z, w are called **homogeneous coordinates**, which are determined up to simultaneous rescaling.

The mapping that identifies $z \in \mathbb{C}$ with $[z:1] \in \mathbb{CP}^1$ is a bijection between $\mathbb{C}$ and $\mathbb{CP}^1 \setminus \{[1:0]\}$. If we identify ∞ with $[1:0]$, then the mapping becomes a bijection between $\mathbb{C}_\infty$ and $\mathbb{CP}^1$.

Lemma 4.7

$\tilde{\mathbb{C}}$ (equipped with the mertic induced by $\mathbb{S}^2$) induces a metric on $\mathbb{CP}^1$:

$$d(L_1, L_2) = 2\sqrt{1 - \frac{|\langle u, v\rangle|^2}{\|u\|^2\|v\|^2}}$$

for any $u \in L_1 \setminus \{0\}$ and $v \in L_2 \setminus \{0\}$.

Proof. Just some simple calculations. □

Definition 4.8. Differentiability on $\mathbb{CP}^1$.

Now we wish to introduce a differential structure on $\mathbb{CP}^1$.

Let $U_0 := \mathbb{CP}^1 \setminus \{[1:0]\}$ and $U_\infty := \mathbb{CP}^1 \setminus \{[0:1]\}$ be two subsets of $\mathbb{CP}^1$. Then the mapping $\iota_0(z) := [z:1]$ and $\iota_\infty(z) := [1:z]$ are two embeddings from $\mathbb{C}$ into $\mathbb{CP}^1$, whose images are exactly U_0 and U_∞. We also have $\mathbb{CP}^1 = U_0 \cup U_\infty$. We say that $V \in \mathbb{CP}^1$ is an open set, if both $\iota_0^{-1}(V)$ and $\iota_\infty^{-1}(V)$ are open sets in $\mathbb{C}$. This makes $\mathbb{CP}^1$ a one-dimensional **smooth complex manifold** or a **Riemann surface**. More explicitly, (U_0, ι_0^{-1}) and $(U_\infty, \iota_\infty^{-1})$ are two **coordinate charts**, and $\{(U_0, \iota_0^{-1}), (U_\infty, \iota_\infty^{-1})\}$ is an **altas** of $\mathbb{CP}^1$ (check that it meets all the definitions).

Suppose $V \subseteq \mathbb{CP}^1$ is open. Suppose $f : V \to \mathbb{CP}^1$ is continuous. For $L \in V$ and $f(L) \in \mathbb{CP}^1$, we have $L \in U_\alpha$ and $f(L) \in U_\beta$ for some $\alpha, \beta \in \{0, \infty\}$. Now f induces a mapping $\tilde{f} : \iota_\alpha^{-1}(V) \to \mathbb{C}$ via the following diagram:

$$\begin{array}{ccc} V \cap U_\alpha & \xrightarrow{\iota_\alpha^{-1}} & \iota_\alpha^{-1}(V) \\ \big\downarrow f & & \big\downarrow \tilde{f} \\ U_\beta & \xleftarrow[\iota_\beta]{} & \mathbb{C} \end{array}$$

We say that f is differentiable at $L \in \mathbb{CP}^1$, if $\tilde{f} := \iota_\beta^{-1} \circ f \circ \iota_\alpha : \iota_\alpha^{-1}(V) \to \mathbb{C}$ is differentiable at $\iota_\alpha^{-1}(L) \in \mathbb{C}$. (One should check that this is well-defined and is independent of the choice of U_α and U_β.)

If we identify $\mathbb{C}_\infty$ with $\mathbb{CP}^1$, then the above discussion defines holomorphic functions on $\mathbb{C}_\infty$.

Remark. The above definition uses the language of differential geometry. It is essentially saying that, a function $f : \mathbb{C}_\infty \to \mathbb{C}$ is differentiable at $z_0 \in \mathbb{C}_\infty$, if:

1. $z_0, f(z_0) \neq \infty$: f is differentiable under the usual definition;
2. $z_0 = \infty, f(z_0) \neq \infty$: $g(z) := f(1/z)$ is differentiable at 0;
3. $z_0 \neq \infty, f(z_0) = \infty$: $g(z) := 1/f(z)$ is differentiable at z_0;
4. $z_0 = f(z_0) = \infty$: $g(z) := 1/f(1/z)$ is differentiable at 0.

Proposition 4.9

(i) $f : \mathbb{C}_\infty \to \mathbb{C}$ is holomorphic (in the sense of Definition 4.8) if and only if $f \circ \iota_0$ is constant;

(ii) Suppose $f : \mathbb{C} \to \mathbb{C}_\infty$ is non-constant. Then f is holomorphic (in the sense of Definition 4.8) if and only if $\iota_0^{-1} \circ f$ is meromorphic;

(iii) Suppose $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is non-constant. Then f is holomorphic (in the sense of Definition 4.8) if and only if $\iota_0^{-1} \circ f \circ \iota_0$ is a rational function.

Proof. (i): The backward argument is trivial. For the forward argument, since $\mathbb{C}_\infty$ is compact and f is continuous, then $f(\mathbb{C}_\infty)$ is bounded. By Liouville's Theorem, f is constant.

(ii): The backward argument is trivial. For the forward argument, it suffices to show that if $f(z_0) = \infty = [1:0]$ then z_0 is a pole of $\iota_0^{-1} \circ f$. By continuity of f at z_0, there exists $r > 0$ such that $f(z) \neq 0 = [0:1]$ for $z \in B(z_0, r)$. Hence

$f(B(z_0,r)\setminus\{0\}) \subseteq U_\infty \cap U_0$. Let $f_\infty := \iota_\infty^{-1} \circ f$ and $f_0 := \iota_0^{-1} \circ f$. Then $f_\infty(z) = 1/f_0(z)$ for $z \in B(z_0,r)\setminus\{z_0\}$ and $f_\infty(z) = 0$. Hence z_0 is a pole of f_0.

(iii): The backward argument is trivial. For the forward argument, since f is holomorphic, then $\bar{f} := \iota_0^{-1} \circ f \circ \iota_0$ is a meromorphic function with ∞ as a pole or removable singularity. Since $\mathbb{C}_\infty$ is compact and the singularities of $\bar{f}$ are isolated, $\bar{f}$ can only have finitely many poles. By Mittag-Leffler Theorem, there exists a meromorphic function g with the same poles and orders. Hence $\bar{f} - g$ is holomorphic and bounded on $\mathbb{C}$. By Liouville's Theorem, $\bar{f}(z) - g(z) = c$ is constant. Moreover, g is given by a finite sum of some rational functions. Hence $\bar{f}(z) = g(z) + c$ is also a rational function. □

Remark. A meromorphic function is said to be **transcendental**, if it is not a rational function. As a corollary of the previous proposition, a transcendental meromorphic function either has ∞ as an essential singularity, or as a limit of poles.

4.2 Conformal Equivalence and Möbius Transformations

In this chapter we will shift our focus from analysis to geometry. The main problem is that, given two open sets U and V in $\mathbb{C}$, if there exists a holomorphic bijection between these sets. We begin with the definition of conformal mappings, which are mappings that preserve angles. Informally stated, suppose there are two rays γ_1 and γ_2 starting at z_0. We can define the angle between the two rays by the difference of their arguments at z_0. We say that f is angle-preserving at z_0, if $f\circ\gamma_1$ and $f\circ\gamma_2$ make the same angle at z_0 as γ_1 and γ_2 do. The formal definition is as follows:

4.2.1 Conformal Equivalence.

Definition 4.10. Angle Preservation.

Suppose that $U \subseteq \mathbb{C}$ is open and $f : U \to \mathbb{C}$ is a function. For $z_0 \in U$, suppose z_0 has a deleted neighbourhood $B(z_0,r)\setminus\{z_0\}$ such that $f(z_0) \notin f(B(z_0,r)\setminus\{z_0\})$ (i.e. f is locally injective at z_0). We say that f preserves angle at z_0, if the limit

$$\lim_{r\to 0} e^{-i\theta} \frac{f(z_0 + r\,e^{i\theta}) - f(z_0)}{\left|f(z_0 + r\,e^{i\theta}) - f(z_0)\right|}$$

exists and is independent of θ.

Proposition 4.11

Suppose $f : U \subseteq \mathbb{C}$ is a function. For $z_0 \in U$,

(i) If f is complex differentiable at z_0 and $f'(z_0) \neq 0$, then f preserves angle at z_0;

(ii) If f is real differentiable (as a mapping $U \to \mathbb{R}^2$) at z_0 with $\mathrm{d}f_{z_0} \neq 0$ and preserves angle at z_0, then f is complex differentiable at z_0 and $f'(z_0) \neq 0$.

Proof. (i): Suppose $f'(z_0) = a$. Then the limit

$$\lim_{r\to 0} \frac{f(z_0 + r\,e^{i\theta}) - f(z_0)}{r\,e^{i\theta}} = a$$

holds for any $\theta \in \mathbb{R}$.

Then the limit is given by

$$\lim_{r\to 0} e^{-i\theta} \frac{f(z_0 + r\,e^{i\theta}) - f(z_0)}{\left|f(z_0 + r\,e^{i\theta}) - f(z_0)\right|} = a \lim_{r\to 0} \frac{r}{\left|f(z_0 + r\,e^{i\theta}) - f(z_0)\right|} = \frac{a}{|a|}$$

Hence f preserves angle at z_0.

(ii): Since f is real differentiable at z_0, we can write

$$f(z) = f(z_0) + \lambda(z - z_0) + \eta\left(\overline{z - z_0}\right) + |z| \cdot O(z) \qquad \lambda, \eta \in \mathbb{C}$$

One can check that this is equivalent to real differentiability at z_0.

Angle preservation at z_0 implies that:

$$\alpha = \lim_{r\to 0} \mathrm{e}^{-\mathrm{i}\theta} \frac{f(z_0 + r\,\mathrm{e}^{\mathrm{i}\theta}) - f(z_0)}{|f(z_0 + r\,\mathrm{e}^{\mathrm{i}\theta}) - f(z_0)|} = \lim_{r\to 0} \mathrm{e}^{-\mathrm{i}\theta} \frac{\lambda r\,\mathrm{e}^{\mathrm{i}\theta} + \eta r\,\mathrm{e}^{-\mathrm{i}\theta}}{|\lambda r\,\mathrm{e}^{\mathrm{i}\theta} + \eta r\,\mathrm{e}^{-\mathrm{i}\theta}|} = \frac{\lambda + \eta\,\mathrm{e}^{-2\mathrm{i}\theta}}{|\lambda + \eta\,\mathrm{e}^{-2\mathrm{i}\theta}|}$$

The value of α is independent of θ. We can infer that $\eta = 0$. Since $\mathrm{d}f_0 \neq 0$, we must have $\lambda \neq 0$. Hence

$$f(z) = f(z_0) + \lambda(z - z_0) + |z| \cdot O(z)$$

We conclude that f is complex differentiable at z_0 with $f'(z_0) = \lambda \neq 0$. □

Remark. Under our definition, there exists angle-preserving mappings which are not holomorphic. For example, $f(z) = z|z|$ preserves angle at $z = 0$ and the differential at that point is a zero map, but clearly f is not holomorphic at $z = 0$.

Definition 4.12. Conformal Mapping.

We say that f is a conformal mapping, if f is holomorphic with non-vanishing derivatives.

Corollary 4.13

A conformal mapping preserves angle at every point in its domain.

Remark. If $f : U \to \mathbb{C}$ is holomorphic and injective, then we must have $f'(z) \neq 0$ for all $z \in U$. Hence biholomorphisms (and their inverses) are conformal mappings.

Definition 4.14. Conformal Equivalence.

Suppose $U, V \in \mathbb{C}$ are two open sets. We say that U and V are conformally equivalent, if there exists a bijective conformal mapping $f : U \to V$.

4.2.2 Möbius Transformations.

Definition 4.15. Projective General Linear Group $\mathrm{PGL}(2,\mathbb{C})$**.**

The **General Linear Group** $\mathrm{GL}(2,\mathbb{C})$ is the group of all invertible linear operators in $\mathbb{C}^2$ with compositions. That is,

$$\mathrm{GL}(2,\mathbb{C}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C};\ ad - bc \neq 0 \right\}$$

$\mathrm{GL}(2,\mathbb{C})$ acts naturally on $\mathbb{C}^2$ and hence on $\mathbb{CP}^1$. The operators that fix all elements of $\mathbb{CP}^1$ are exactly all the scalar multiplications, $\mathrm{Z}(2,\mathbb{C})$, which is exactly the center of $\mathrm{GL}(2,\mathbb{C})$. The quotient group, $\mathrm{GL}(2,\mathbb{C})/Z(2,\mathbb{C})$, which is the group of all invertiable projective transformations in $\mathbb{CP}^1$, is called the projective general linear group and is denoted by $\mathrm{PGL}(2,\mathbb{C})$.

Remark. Now let us calculate the action of $\mathrm{PGL}(2,\mathbb{C})$ on $\mathbb{CP}^1$. For

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C})$$

it induces $\bar{f} \in \mathrm{PGL}(2,\mathbb{C})$. For $[z,1] \in \mathbb{CP}^1$:

$$\bar{f}([z,1]) = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} \right] = [az + b : cz + d] = \left[\frac{az+b}{cz+d} : 1 \right] \quad (z \neq -d/c)$$

(If $z = -d/c$, then $\bar{f}([z:1]) = [1:0]$.)

For $[1,0] \in \mathbb{CP}^1$,

$$\bar{f}([1,0]) = [a:c] = [a/c:1]$$

If we identify $\mathbb{CP}^1$ with $\mathbb{C}_\infty$, then $\mathrm{PGL}(2,\mathbb{C})$ is isomorphic to a group of automorphisms on $\mathbb{C}_\infty$, namely the Möbius transformations.

Definition 4.16. Möbius Transformations.

For $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$, the following mapping on $\mathbb{C}_\infty$ are called the fractral linear transformations or the Möbius transformations:

$$z \mapsto \frac{az+b}{cz+d}, \quad \infty \mapsto \frac{a}{c}$$

All Möbius transformations forms a group under mapping compositions, which is denoted by Mob. From the previous remark we have $\text{Mob} \cong \text{PGL}(2, \mathbb{C})$.

Definition 4.17. Dilations, Translations, and Inversion

There are three special types of Möbius transformations:

- For $a \in \mathbb{C} \setminus \{0\}$, $z \mapsto az$ is called a dilation.
- For $b \in \mathbb{C}$, $z \mapsto z + b$ is called a translation.
- $z \mapsto z^{-1}$ is called inversion.

Lemma 4.18

Any Möbius transformation is a composition of dilations, translations, and an inversion.

Proof. Suppose $f(z) = \dfrac{az+b}{cz+d}$ is a Möbius transformation. If $c \neq 0$, we observe that

$$f(z) = \frac{az+b}{cz+d} = \frac{a}{c} + \left(b - \frac{ad}{c}\right)\frac{1}{cz+d} = t_1 \circ d_1 \circ i \circ t_2 \circ d_2(z)$$

where

$$d_2(z) = cz, \quad t_2(z) = z + d, \quad i(z) = z^{-1}, \quad d_1(z) = (b - ad/c)z, \quad t_1(z) = z + a/c$$

If $c = 0$, we trivially have $f(z) = a/d + (b/d)z$. □

Remark. The subgroup of Mob generated by translations and dilations is the group of $\mathbb{C}$-affine transformations $\text{Aff}(\mathbb{C}) := \{f(z) = az + b : a \neq 0\}$ of the complex plane. It is the automorphism group of $\mathbb{C}$ (see 4.44) and is the stablizer of ∞ in Mob.

Proposition 4.19

Möbius transformation $f(z) = \dfrac{az+b}{cz+d}$ is a biholomorphism on $\mathbb{C}_\infty$.

Proof. Trivial. □

Now we shall explore some geometric properties of the Möbius transformations.

Proposition 4.20

Möbius transformations preserve circles in $\mathbb{S}^2$ (which are circles or lines on $\mathbb{C}$ by Proposition 4.4).

Proof. It suffices to prove that circles in $\mathbb{S}^2$ are preserved under dilations, translations, and inversion. The first two cases are trivial. For the inversion mapping $f(z) = 1/z = \bar{z}/|z|^2$, we apply the inverse stereographic mapping:

$$z \mapsto \frac{1}{|z|^2+1}(2\operatorname{Re} z, 2\operatorname{Im} z, |z|^2 - 1);$$

$$\frac{\bar{z}}{|z|^2} \mapsto \frac{1}{\frac{1}{|z|^2}+1}\left(2\frac{1}{|z|^2}\operatorname{Re} z, -2\frac{1}{|z|^2}\operatorname{Im} z, \frac{1}{|z|^2} - 1\right) = \frac{1}{|z|^2+1}(2\operatorname{Re} z, -2\operatorname{Im} z, 1 - |z|^2)$$

Hence f induces a transformation $(x, y, z) \mapsto (x, -y, -z)$ on $\mathbb{S}^2$, which is a rotation about the x-axis by π and certainly preserves circles in $\mathbb{S}^2$. □

Proposition 4.21

Given two triples of distinct points z_1, z_2, z_3 and w_1, w_2, w_3 in $\mathbb{C}_\infty$, there exists a unique Möbius transformation f such that $f(z_i) = w_i$ for $i = 1, 2, 3$.

Proof. Consider the mapping $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ defined by

$$f(z) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

It is a Möbius transformation and sends z_1, z_2, z_3 to $0, 1, \infty$. Similarly, consider $g : \mathbb{C}_\infty \to \mathbb{C}_\infty$ defined by

$$g(z) = \frac{(z - w_1)(w_2 - w_3)}{(z - w_3)(w_2 - w_1)}$$

which sends w_1, w_2, w_3 to $0, 1, \infty$.

Hence $g^{-1} \circ f$ is a Möbius transformation that sends z_1, z_2, z_3 to w_1, w_2, w_3.

For the uniqueness part, suppose h is another such mapping. Then $g \circ h \circ f^{-1}$ is a Möbius transformation that sends $0, 1, \infty$ to $0, 1, \infty$. Suppose

$$g \circ h \circ f^{-1}(z) = \frac{az + b}{cz + d}$$

We observe that

$$g \circ h \circ f^{-1}(0) = 0 \Longrightarrow b = 0$$
$$g \circ h \circ f^{-1}(1) = 1 \Longrightarrow a = (c + d)$$
$$g \circ h \circ f^{-1}(\infty) = \infty \Longrightarrow c = 0$$

Therefore we have $a = d$ and $g \circ h \circ f^{-1} = \mathrm{id}_{\mathbb{C}_\infty} \Longrightarrow h = g^{-1} \circ f$. □

Remark. The Möbius transformation f given in the previous proposition is very crucial. It motivates us to give the following concept:

Definition 4.22. Cross Ratio.

For a quadruple of points $z, z_1, z_2, z_3 \in \mathbb{C}_\infty$, we define the cross ratio of it to be

$$(z, z_1, z_2, z_3) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

Proposition 4.23. Möbius transformation preserves cross ratio.

Suppose f is a Möbius transformation that maps $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ to $w_1, w_2, w_3, w_4 \in \mathbb{C}_\infty$, then the cross ratio:

$$(z_1, z_2, z_3, z_4) = (w_1, w_2, w_3, w_4)$$

Proof. The only non-trivial case is that z_2, z_3, z_4 are distinct. By Proposition 4.21, the unique mapping $g^{-1} \circ f$ maps z_2, z_3, z_4 to w_2, w_3, w_4. Since $f(z) = (z, z_2, z_3, z_4)$, $g(z) = (z, w_2, w_3, w_4)$, and $g(w_1) = f(z_1)$, then we have

$$(z_1, z_2, z_3, z_4) = (w_1, w_2, w_3, w_4)$$

as claimed. □

Definition 4.24. Symmetric Points.

Suppose that $C \subseteq \mathbb{C}$ is a circle centered at a with radius r. We say that $z, w \in \mathbb{C}_\infty$ is a pair of symmetric points with respect to C, if they lies on the same ray starting from a and satisfies

$$|z - a| \cdot |w - a| = r^2$$

Suppose that $L \subseteq \mathbb{C}$ is a line. r. We say that $z, w \in \mathbb{C}_\infty$ is a pair of symmetric points with respect to L if L is the perpendicular bisector of z and w.

Lemma 4.25

z and w are symmetric with respect to a circle $C \in \mathbb{C}_\infty$ (which is a circle or a line in $\mathbb{C}$) if and only if the cross ratio

$$(w, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$$

for any $z_1, z_2, z_3 \in C$.

Proof. We have to do some explicit calculations.

First, suppose that C is a line in $\mathbb{C}$ that passes through a and makes angle θ with the real axis. We have $|z-a| = |w-a|$ and $\arg(z-a) + \arg(w-a) = 2\theta$ (draw a diagram and do some middle school geometry). Hence

$$w - a = |w-a|\,\mathrm{e}^{\mathrm{i}\arg(w-a)} = |z-a|\,\mathrm{e}^{\mathrm{i}(2\theta - \arg(z-a))} = \overline{(z-a)}\,\mathrm{e}^{2\mathrm{i}\theta}$$
$$\implies w = a + (\bar{z} - \bar{a})\,\mathrm{e}^{2\mathrm{i}\theta}$$

Compute the cross ratio:

$$(w, z_1, z_2, z_3) = \frac{(w-z_1)(z_2-z_3)}{(w-z_3)(z_2-z_1)} = \frac{(a + (\bar{z}-\bar{a})\,\mathrm{e}^{2\mathrm{i}\theta} - z_1)(z_2-z_3)}{(a + (\bar{z}-\bar{a})\,\mathrm{e}^{2\mathrm{i}\theta} - z_3)(z_2-z_1)}$$

Since $z_1, z_2, z_3 \in C$, we have $z_j = a + (\bar{z}_j - \bar{a})\,\mathrm{e}^{2\mathrm{i}\theta}$ for $j = 1,2,3$. Then

$$\begin{aligned}(w, z_1, z_2, z_3) &= \frac{(a + (\bar{z}-\bar{a})\,\mathrm{e}^{2\mathrm{i}\theta} - (a + (\bar{z}_1-\bar{a})\,\mathrm{e}^{2\mathrm{i}\theta}))(a + (\bar{z}_2-\bar{a})\,\mathrm{e}^{2\mathrm{i}\theta} - (a + (\bar{z}_3-\bar{a})\,\mathrm{e}^{2\mathrm{i}\theta}))}{(a + (\bar{z}-\bar{a})\,\mathrm{e}^{2\mathrm{i}\theta} - (a + (\bar{z}_3-\bar{a})\,\mathrm{e}^{2\mathrm{i}\theta}))(a + (\bar{z}_2-\bar{a})\,\mathrm{e}^{2\mathrm{i}\theta} - (a + (\bar{z}_1-\bar{a})\,\mathrm{e}^{2\mathrm{i}\theta}))} \\ &= \frac{(\bar{z}-\bar{z}_1)(\bar{z}_2-\bar{z}_3)}{(\bar{z}-\bar{z}_3)(\bar{z}_2-\bar{z}_1)} = \overline{(z, z_1, z_2, z_3)}\end{aligned}$$

Second, suppose that C is a circle in $\mathbb{C}$ centered at a with radius r. We have $|z-a|\cdot|w-a| = r^2$ and $\arg(z-a) = \arg(w-a)$. Hence

$$w - a = |w-a|\,\mathrm{e}^{\mathrm{i}\arg(w-a)} = \frac{r^2}{|z-a|}\,\mathrm{e}^{\mathrm{i}\arg(z-a)} = \frac{r^2}{\overline{(z-a)}}$$
$$\implies w = a + \frac{r^2}{\bar{z}-\bar{a}}$$

Compute the cross ratio:

$$\begin{aligned}(w, z_1, z_2, z_3) &= \frac{(w-z_1)(z_2-z_3)}{(w-z_3)(z_2-z_1)} = \frac{(a + \frac{r^2}{\bar{z}-\bar{a}} - z_1)(z_2-z_3)}{(a + \frac{r^2}{\bar{z}-\bar{a}} - z_3)(z_2-z_1)} \\ &= \frac{\left(\frac{1}{\bar{z}-\bar{a}} - \frac{1}{\bar{z}_1-\bar{a}}\right)\left(\frac{1}{\bar{z}_2-\bar{a}} - \frac{1}{\bar{z}_3-\bar{a}}\right)}{\left(\frac{1}{\bar{z}-\bar{a}} - \frac{1}{\bar{z}_3-\bar{a}}\right)\left(\frac{1}{\bar{z}_2-\bar{a}} - \frac{1}{\bar{z}_1-\bar{a}}\right)} = \frac{(\bar{z}-\bar{z}_1)(\bar{z}_2-\bar{z}_3)}{(\bar{z}-\bar{z}_3)(\bar{z}_2-\bar{z}_1)} \\ &= \overline{(z, z_1, z_2, z_3)}\end{aligned}$$

□

Proposition 4.26. Möbius transformation preserves symmetric points.

Suppose f is a Möbius transformation. Suppose z_1 and z_2 are symmetric with respect to a circle $C \in \mathbb{C}_\infty$. Then $f(z_1)$ and $f(z_2)$ are symmetric with respect to $f(C)$.

Proof. This is immediate by Lemma 4.25 and Proposition 4.23. □

In next section, we shall see how we utilize Möbius transformations and other elementary functions to build up common conformal mappings.

4.3 Examples of Conformal Mappings

Definition 4.27. Upper Half Plane; Unit Disk.

We will frequently encounter these special subsets of $\mathbb{C}$ in this chapter. For notational convenience, we write $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and $\mathbb{D} := B(0,1) = \{z \in \mathbb{C} : |z| < 1\}$.

4.3.1 Using Möbius Transformations.

Example 4.28

Find a conformal mapping from $\mathbb{H}$ onto $\mathbb{D}$.

Solution. We express $\mathbb{H}$ as follows:

$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\} = \{z \in \mathbb{C} : |z - \mathrm{i}| < |z + \mathrm{i}|\} = \left\{z \in \mathbb{C} : \left|\frac{z - \mathrm{i}}{z + \mathrm{i}}\right| < 1\right\}$$

We immediately observe that the Möbius transformation

$$f(z) = \frac{z - \mathrm{i}}{z + \mathrm{i}}$$

is a conformal mapping from $\mathbb{H}$ onto $\mathbb{D}$. The inverse mapping from $\mathbb{D}$ onto $\mathbb{H}$ is given by

$$f^{-1}(z) = \mathrm{i}\,\frac{z - 1}{z + 1}$$ □

Remark. Some conformal mappings from other half planes to $\mathbb{D}$ are given by:

- From lower half plane to $\mathbb{D}$: $f(z) = \dfrac{z - \mathrm{i}}{z + \mathrm{i}}$;
- From left half plane to $\mathbb{D}$: $f(z) = \dfrac{z + 1}{z - 1}$;
- From right half plane to $\mathbb{D}$: $f(z) = \dfrac{z - 1}{z + 1}$ (notice that this mapping is self-inverse).

Example 4.29

Find a conformal mapping from $\mathbb{H}$ onto $\mathbb{D}$ that maps $a \in \mathbb{H}$ to 0.

Solution. We can make use of Proposition 4.26 to find a Möbius transformation satisfying the conditions. Suppose f is such a mapping. For $a \in \mathbb{H}$, the symmetric point of a is $\bar{a}$. As f should maps the real axis to the unit circle, $f(a)$ and $f(\bar{a})$ are symmetric with respect to the unit circle. $f(a) = 0$ implies $f(\bar{a}) = \infty$. Hence we can choose

$$f(z) = \frac{z - a}{z - \bar{a}}$$

It is easy to observe that $|f(z)| \leqslant 1$ and that $f'(z) \neq 0$. Then f is a conformal mapping. Moreover, f is unique up to composition of dilations (this is obvious after we prove Riemann Mapping Theorem). □

Example 4.30

For $|a| < 1$, find *the* conformal mapping from $\mathbb{D}$ onto $\mathbb{D}$ that swaps a with 0.

Solution. We again use Proposition 4.26 to construct a Möbius transformation. Suppose φ_a is such a mapping. For $a \in \mathbb{D}$, the symmetric point of a is $1/\bar{a}$. φ_a should fix the unit circle and maps a to 0. Hence $\varphi_a(1/\bar{a}) = \infty$. We write

$$\varphi_a(z) = \lambda \frac{a - z}{1 - \bar{a}z}$$

where $\lambda \in \mathbb{C}$. Moreover, we have $\varphi_a(0) = a$. Hence $a = \lambda a \Longrightarrow \lambda = 1$. The mapping with desired properties is given by

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}$$ □

Remark. The mapping φ_a constructed in the previous example is an automorphism of the unit disk. In next section we shall prove that any automorphism of the disk is a composition of such mappings and rotations. Moreover, φ_a is self-inverse, which means that $\varphi_a^2 = \mathrm{id}_{\mathbb{D}}$ or $\varphi_a = \varphi_a^{-1}$.

Example 4.31

Find a conformal mapping from a cresent $U := \{z \in \mathbb{C} : |z| < 2,\ |z - 1| > 1\}$ onto a strip $V := \{w \in \mathbb{C} : -1 < \mathrm{Re}\, w < 0\}$.

Solution. We want a Möbius transformation that maps the circle $|z-1|=1$ to the imaginary axis and maps circle $|z|=2$ to the line $\operatorname{Re} w=-2$. By geometric intuitions, this could be done by mapping $0,2,-2$ to $0,\infty,-2$, as shown in Figure 4.2.

Let $f(z)=\lambda\dfrac{z}{z-2}$ for some $\lambda\in\mathbb{C}$. Since $f(-2)=-2$, we have $\lambda=4$. The desired mapping is given by $f(z)=\dfrac{4z}{z-2}$. □

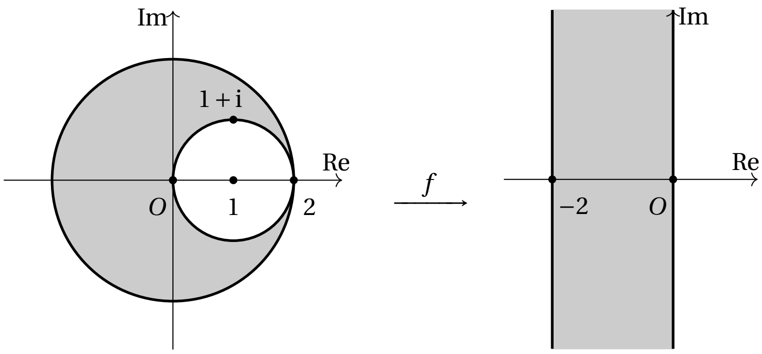

Figure 4.2: Conformal mapping from a cresent area to a strip area.

Example 4.32

Find a conformal mapping from a lozenge-shaped area $\{z\in\mathbb{C}:|z-\mathrm{i}|<\sqrt{2},|z+\mathrm{i}|<\sqrt{2}\}$ onto the first quadrant $\{z\in\mathbb{C}:\operatorname{Re} z>0,\operatorname{Im} z>0\}$.

Solution. The lozenge area is bound by two arcs which intersects at -1 and 1. If we apply a Möbius transformation that maps -1 to 0 and 1 to ∞, then the two arcs will be mapped to two rays starting from 0, and the image of the lozenge would be a sector. Let $f(z)=\lambda\dfrac{z+1}{z-1}$ for some $\lambda\in\mathbb{C}$. Since Möbius transformation preserves angle at -1, we know that the angle of the sector is exactly $\pi/2$. In order to map the lozenge onto the first quadrant, we may need a rotation, which is determined by $\lambda\in\partial B(0,1)$. f maps $\mathrm{i}-\sqrt{2}$ to a point on the real axis. Hence

$$\operatorname{Im} f(\mathrm{i}-\sqrt{2})=0\Longrightarrow \operatorname{Im}\left(\lambda\frac{\mathrm{i}-\sqrt{2}+1}{\mathrm{i}-\sqrt{2}-1}\right)=0\Longrightarrow \operatorname{Im}(\lambda(\mathrm{i}-1))=0$$
$$\Longrightarrow \lambda=(\mathrm{i}+1)\eta \qquad \text{for some } \eta\in\mathbb{R}$$

We choose $\eta=-1$ (the sign is to ensure that $f(0)$ lies in the first quadrant). Then the desired mapping is given by $f(z)=(\mathrm{i}+1)\dfrac{1+z}{1-z}$. □

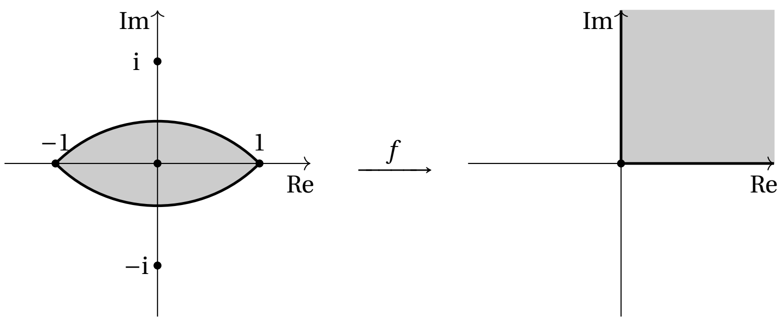

Figure 4.3: Conformal mapping from a lozenge to a sector.

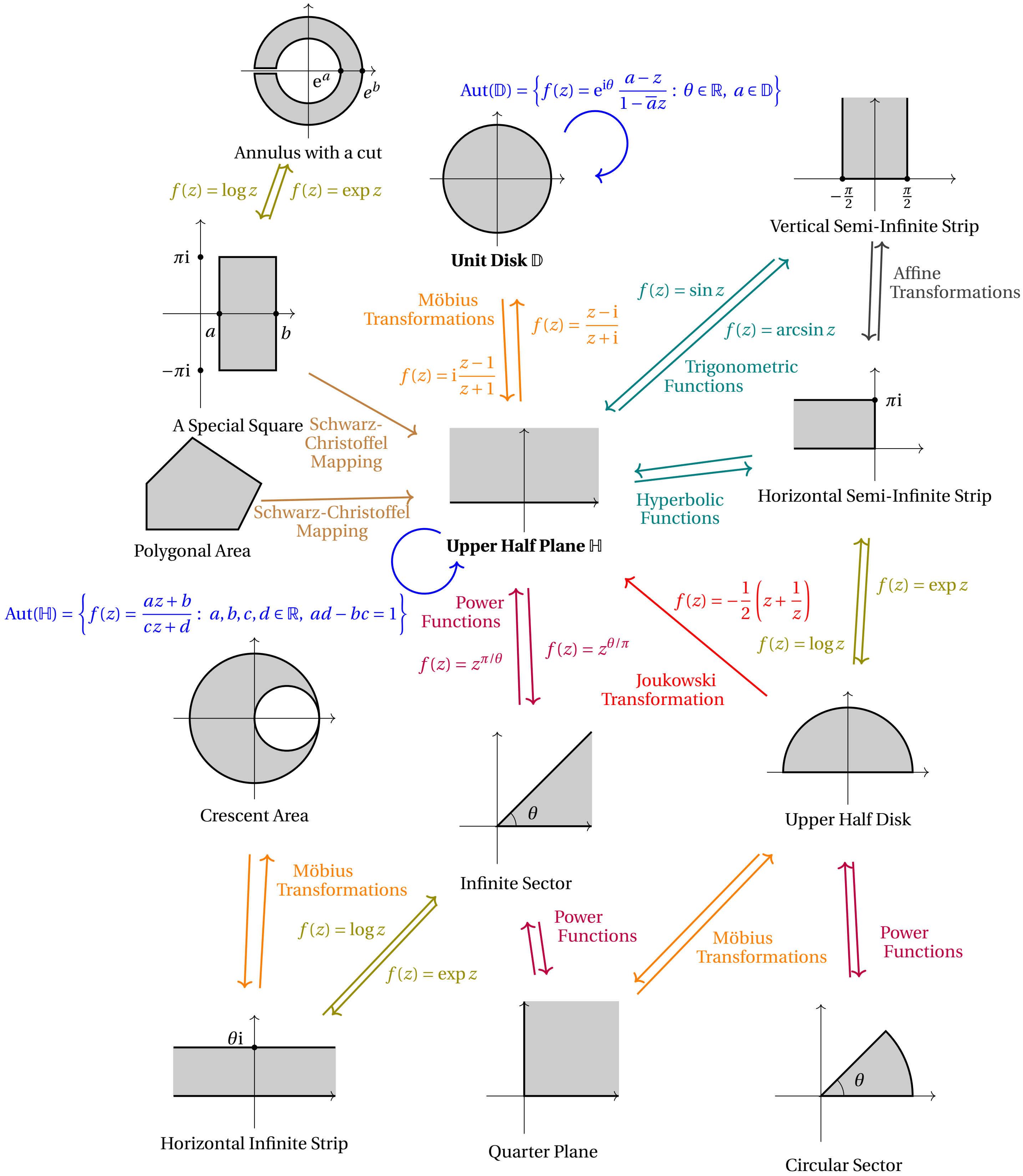

Figure 4.4: Summary of Bihomomorphisms between Common Regions

4.3.2 Using other Elementary Functions.

Example 4.33. Power Functions.

Consider the power function $f(z) = z^n$ for $n \in \mathbb{Z}_+$. f preserves angle everywhere except at $z = 0$, where angles are magnified by a factor of n.

f maps the sector $\{z \in \mathbb{C} : \arg z \in (0, \pi/n)\}$ to $\mathbb{H}$, as shown in Figure 4.5. The inverse function $f^{-1}(z) = z^{1/n}$ maps $\mathbb{H}$ to $\{z \in \mathbb{C} : \arg z \in (0, \pi/n)\}$ (we need to take a branch cut in this case).

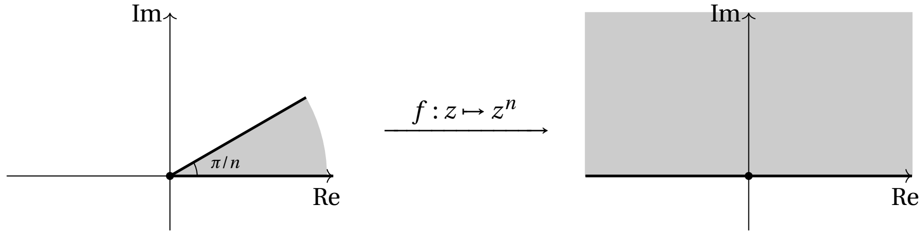

Figure 4.5: Conformal mapping from a sector to $\mathbb{H}$.

Example 4.34. Exponential Function.

The exponential function $\exp z$ is conformal everywhere. It maps the vertical line $x = a$ to the circle $|z| = e^a$, and maps the horizontal line $y = b$ to the ray $\arg z = c$.

Therefore $\exp z$ maps the vertical strip $\{z \in \mathbb{C} : \operatorname{Re} z \in (a, b)\}$ to the annulus $A(0, e^a, e^b)$ (*this mapping is not injective!*), and maps the horizontal strip $\{z \in \mathbb{C} : \operatorname{Im} z \in (c, d)\}$ to the sector $\{z \in \mathbb{C} : \arg z \in (c, d)\}$, as shown in Figure 4.6.

In reverse $\log z$ can maps a sector to a vertical strip, but again we should choose the branch cut carefully.

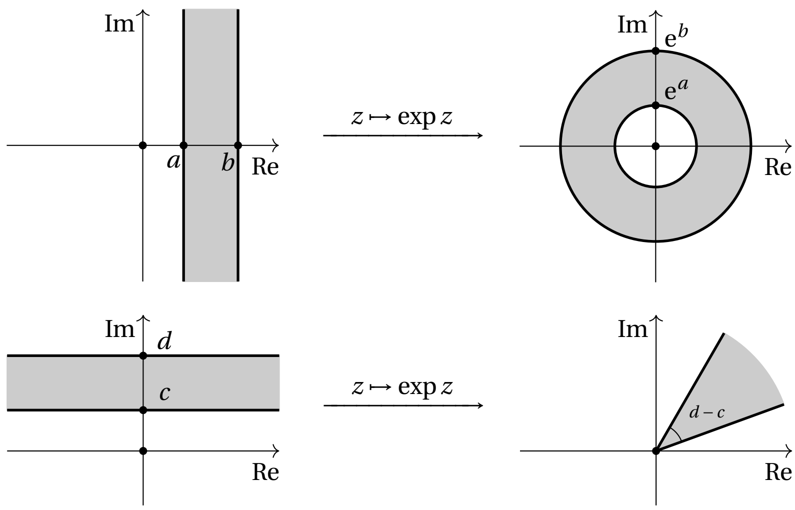

Figure 4.6: Conformal mapping of strips by exponential function.

Example 4.35. Trigonometric Functions.

The cosine function $\cos z = (e^{iz} + e^{-iz})/2$ is conformal everywhere. Consider the strip area $U := \{z \in \mathbb{C} : \operatorname{Re} z \in (0, \pi), \operatorname{Im} z > 0\}$. We would like to investigate how $\cos z$ maps the boundary lines. It is obvious that $\cos z$ maps $(0, \pi)$ to $(-1, 1)$ with orientation reversed. For positive imaginary axis, notice that

$$\cos(iy) = \cosh y \in (1, \infty) \quad \text{for } y > 0$$

Hence $\cos z$ maps the positive imaginary axis to $(1,\infty)$. For the half-line $\{z \in \mathbb{C} : \operatorname{Re} z = \pi, \operatorname{Im} z > 0\}$, notice that

$$\cos(\pi + \mathrm{i}y) = -\cosh y \in (-\infty, -1) \quad \text{for } y > 0$$

Hence $\cos z$ maps the half-line to $(-1,\infty)$. In conclusion, $\cos z$ maps ∂U to the entire real axis. However, it is still unclear whether $\cos z$ maps U to the upper half plane or the lower half plane. So consider $z = \pi/2 + \mathrm{i}y$ for some $y > 0$. Then $\cos(\pi/2 + \mathrm{i}y) = -\sin(\mathrm{i}y) = -\mathrm{i}\sinh y$ has a negative imaginary part. Hence we conclude that $\cos z$ maps U to the lower half plane, as shown in Figure 4.7.

Similarly, the hyperbolic function $\cosh z$ maps a horizontal semi-infinite strip to a half plane.

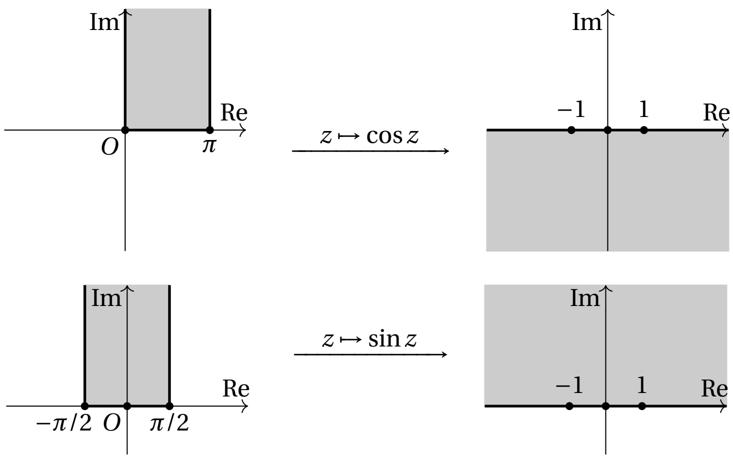

Figure 4.7: Conformal mapping by trigonometric functions.

Example 4.36. Joukowsky Transformation.

The simplest form of **Joukowsky Transformation** is given by:

$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

The derivative is given by $f'(z) = \frac{1}{2}\left(1 - \frac{1}{z^2}\right)$. Hence the mapping is holomorphic on $\mathbb{C} \setminus \{0\}$ and preserves angle everywhere except at $z = \pm 1$. If we put $w = f(z)$, then

$$2wz = z^2 + 1 \quad \text{or} \quad \frac{w+1}{w-1} = \left(\frac{z+1}{z-1}\right)^2$$

We can regard the Joukowsky transformation as the composition $g^{-1} \circ h \circ g$, where $g(z) := \frac{z+1}{z-1}$ maps the left half plane to the unit disk and $h(z) := z^2$ doubles the angles subtended at 0.

Suppose that $z = r\,\mathrm{e}^{\mathrm{i}\theta}$ is mapped to $w = u + \mathrm{i}v$. Then

$$u = \frac{1}{2}(r + r^{-1})\cos\theta \qquad v = \frac{1}{2}(r - r^{-1})\sin\theta$$

The circles $\{z \in \mathbb{C} : |z| = \rho\}$ $(\rho > 0)$ are mapped to

$$\frac{u^2}{\frac{1}{4}(\rho + \rho^{-1})^2} + \frac{v^2}{\frac{1}{4}(\rho + \rho^{-1})^2} = 1$$

which are ellipses on the plane for $\rho \neq 1$. When $\rho = 1$, the Joukowsky transformation maps the unit circle to $(-1,1)$, as shown in Figure 4.8. In particular, Joukowsky transformation maps the unit disk $\mathbb{D}$ to the "cut plane" $\mathbb{C} \setminus [-1,1]$, and maps the upper unit disk $\{z \in \mathbb{C} : |z| < 1, \operatorname{Im} z > 0\}$ to the *lower half plane* $\{z \in \mathbb{C} : \operatorname{Im} z < 0\}$.

The Joukowsky transformation is important in fluid dynamics as it maps certain circles to aerofoil shapes, as shown in Figure 4.8.

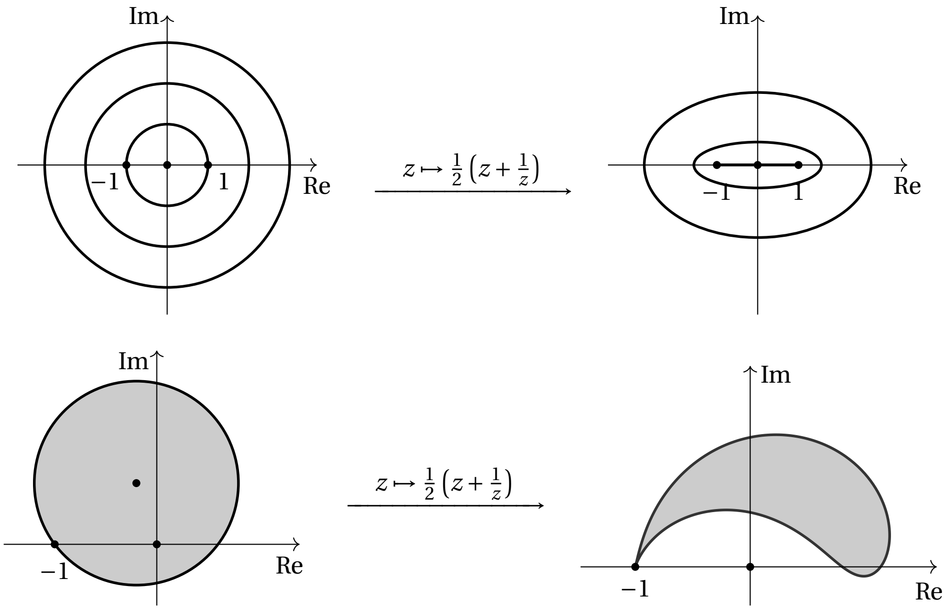

Figure 4.8: Joukowsky Transformation.

4.4 Automorphism Groups*

Definition 4.37. Automorphisms.

A conformal mapping from an open set U onto itself is called an automorphism on U. The set of all automorphisms on U forms a group $\mathrm{Aut}(U)$.

4.4.1 Automorphisms of the Unit Disk.

We begin with discussion of the automorphisms on the unit disk $\mathbb{D}$.

Lemma 4.38. Schwarz's Lemma.

Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic with $f(0) = 0$. Then we have:

(i) $|f(z)| \leqslant |z|$ for all $z \in \mathbb{D}$;

(ii) $|f'(0)| \leqslant 1$;

(iii) if $\exists\, z_0 \in \mathbb{D} \setminus \{0\}\ \big(|f(z_0)| = |z_0|\big)$ or $|f'(0)| = 1$, then f is a rotation.
That is, $f(z) = e^{i\theta} z$ for some $\theta \in [0, 2\pi]$.

Proof. (i). Let $g : \mathbb{D} \setminus \{0\} \to \mathbb{D}$ defined by $g(z) = f(z)/z$. Notice that $z = 0$ is a zero of f of multiplicity at least 1. So $z = 0$ is a removable singularity of g. If we put $g(0) = f'(0)$, then g is holomorphic on $\mathbb{D}$. Fix $r \in (0, 1)$, then by Corollary 2.11:

$$\sup_{z \in \overline{B}(0,r)} |g(z)| = \sup_{z \in \partial B(0,r)} |g(z)| = \sup_{|z|=r} \frac{|f(z)|}{|z|} \leqslant \frac{1}{r}$$

We let $r \to 1$ and obtain $\sup\limits_{z \in \mathbb{D}} |g(z)| \leqslant 1$. Hence $|f(z)| \leqslant |z|$ for all $z \in \mathbb{D}$.

(ii). This is immediate as $|f'(0)| = |g(0)| \leqslant 1$.

(iii). First we assume that $\exists\, z_0 \in \mathbb{D} \setminus \{0\}\ \big(|f(z_0)| = |z_0|\big)$. Then g attains maximum modulus 1 at some $z_0 \in \mathbb{D}$. By Maximum Modulus Principle, g is constant. Hence $f(z) = cz$ for some $c \in \mathbb{C}$. Since $|g(z_0)| = |c| = 1$, we can write $c = e^{i\theta}$ and conclude that f is a rotation.

Second, we assume that $|f'(0)| = 1$. This implies that $g(0) = 1$ and g attains maximum modulus 1 at $z = 0$. By the similar argument, f is a rotation. □

Remark. We have shown that the Möbius transformation

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}$$

is an automorphism of $\mathbb{D}$ that swaps $a \in \mathbb{D}$ with 0. Schwarz's Lemma can be generalized as follows:

Corollary 4.39. Schwarz-Pick Lemma.

Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic with $f(a) = b$ for some $a, b \in \mathbb{D}$, then:

(i) $|\varphi_b \circ f(z)| \leqslant |\varphi_a(z)|$ for all $z \in \mathbb{D}$;

(ii) $|f'(a)| \leqslant \dfrac{1-|b|^2}{1-|a|^2}$;

(iii) if $\exists\, z_0 \in \mathbb{D} \setminus \{a\}$ $(|\varphi_b \circ f(z_0)| = |\varphi_a(z_0)|)$ or $|f'(a)| = \dfrac{1-|b|^2}{1-|a|^2}$, then $f \in \operatorname{Aut}(\mathbb{D})$.

Proof. Let $g := \varphi_b \circ f \circ \varphi_a$. Then g is an automorphism of $\mathbb{D}$ and $g(0) = \varphi_b \circ f \circ \varphi_a(0) = \varphi_b \circ f(a) = \varphi_b(b) = 0$. We can apply Schwarz's Lemma to g.

(i). By Schwarz's Lemma (i), $|g(z)| \leqslant |z| \Longrightarrow |\varphi_b \circ f \circ \varphi_a(z)| \leqslant |z| \Longrightarrow |\varphi_b \circ f(z)| \leqslant |\varphi_a(z)|$.

(ii). By Schwarz's Lemma (ii), $|g'(0)| \leqslant 1$, where

$$\begin{aligned} g'(0) &= \varphi_b'(b) \cdot f'(a) \cdot \varphi_a'(0) \\ &= f'(a) \cdot \left(\frac{|b|^2 - 1}{(1-\bar{b}z)^2}\right)_{z=b} \cdot \left(\frac{|a|^2-1}{(1-\bar{a}z)^2}\right)_{z=0} \\ &= f'(a) \cdot \frac{1-|a|^2}{1-|b|^2} \end{aligned}$$

Hence $|f'(a)| \leqslant \dfrac{1-|b|^2}{1-|a|^2}$.

(iii). If $\exists\, z_0 \in \mathbb{D} \setminus \{a\}$ $(|\varphi_b \circ f(z_0)| = |\varphi_a(z_0)|)$, then $|g(z_0)| = |z_0|$. If $|f'(a)| = \dfrac{1-|b|^2}{1-|a|^2}$, then $|g'(0)| = 1$. In either case we deduce from Schwarz's Lemma that g is an rotation. Hence $f = \varphi_b \circ g \circ \varphi_a \in \operatorname{Aut}(\mathbb{D})$. □

Theorem 4.40. Automorphism Group of the Unit Disk.

$$\operatorname{Aut}(\mathbb{D}) = \left\{ f(z) = \mathrm{e}^{\mathrm{i}\theta}\, \frac{a-z}{1-\bar{a}z} : \theta \in \mathbb{R},\ a \in \mathbb{D} \right\}$$

Proof. Suppose that f is an automorphism of $\mathbb{D}$. Then there exists a unique $a \in \mathbb{D}$ such that $f(a) = 0$. Let $g := f \circ \varphi_a$. g is also an automorphism of $\mathbb{D}$. We have $g(0) = f \circ \varphi_a(0) = f(a) = 0$ and $g^{-1}(0) = 0$. Apply Schwarz's Lemma to g and g^{-1}:

$$\begin{aligned} &\forall\, z \in \mathbb{D} : \quad |g(z)| \leqslant |z|, \quad |g^{-1}(z)| \leqslant |z| \\ \Longrightarrow &\forall\, z \in \mathbb{D} : \quad |g(z)| \leqslant |z|, \quad |z| \leqslant |g(z)| \\ \Longrightarrow &\forall\, z \in \mathbb{D} : \quad |g(z)| = |z| \end{aligned}$$

Hence by Schwarz's Lemma (iii), g is a rotation. There exists $\theta \in [0, 2\pi]$ such that $g(z) = \mathrm{e}^{\mathrm{i}\theta} z$. Hence $f(z) = g \circ \varphi_a^{-1}(z) = \mathrm{e}^{\mathrm{i}\theta}\, \dfrac{a-z}{1-\bar{a}z}$ as claimed. □

Corollary 4.41

The only mappings in $\operatorname{Aut}(\mathbb{D})$ that fix the origin are the rotations.

Remark. We can see that $\operatorname{Aut}(\mathbb{D})$ acts transitively on $\mathbb{D}$ in the sense that for any $a, b \in \mathbb{D}$, we have $\varphi_a, \varphi_b \in \operatorname{Aut}(\mathbb{D})$ and $\varphi_b \circ \varphi_a : a \mapsto b$.

4.4.2 Automorphisms of the Upper Half Plane.

We have proven in Example 4.28 and 4.29 that $\mathbb{H}$ and $\mathbb{D}$ are conformally equivalent. This leads to the group isomorphism $\mathrm{Aut}(\mathbb{D}) \cong \mathrm{Aut}(\mathbb{H})$ via $\Gamma : \mathrm{Aut}(\mathbb{D}) \to \mathrm{Aut}(\mathbb{H})$ defined by conjugation $\Gamma(\psi) = f^{-1} \circ \psi \circ f$, where f is any conformal mapping from $\mathbb{H}$ onto $\mathbb{D}$. In the next theorem, we shall give the elements of the group $\mathrm{Aut}(\mathbb{H})$ and show that it is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$.

Definition 4.42. Projective Special Linear Group $\mathrm{PSL}(2,\mathbb{R})$**.**

The **Special Linear Group** $\mathrm{SL}(2,\mathbb{R})$ is the group of all linear operators in $\mathbb{R}^2$ with determinant 1. That is,

$$\mathrm{SL}(2,\mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in \mathbb{R};\ ad - bc = 1 \right\}$$

Just like $\mathrm{PGL}(2,\mathbb{C})$, $\mathrm{PSL}(2,\mathbb{R})$ is the group of all special projective transformations in $\mathbb{RP}^1$:

$$\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{I, -I\}$$

Remark. We recall from Definition 4.16 that $\mathrm{Mob} \cong \mathrm{PGL}(2,\mathbb{C})$. We should warn readers that, while $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{PGL}(2,\mathbb{C})$, in fact $\mathrm{PSL}(2,\mathbb{R}) < \mathrm{PGL}(2,\mathbb{R})$. This corresponds to $\mathbb{RP}^1$ being orientable, and $\mathrm{PSL}(2,\mathbb{R})$ only being the orientation-preserving transformations.

Theorem 4.43. Automorphism Group of the Upper Half Plane.

$$\mathrm{Aut}(\mathbb{H}) = \left\{ f(z) = \frac{az+b}{cz+d} :\ a,b,c,d \in \mathbb{R},\ ad - bc = 1 \right\} \cong \mathrm{PSL}(2,\mathbb{R})$$

Proof. Let $\mathrm{SL}(2,\mathbb{R})$ acts on $\mathbb{H}$ by

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto f_M(z) = \frac{az+b}{cz+d}$$

It is easy to check that $f_M(\mathbb{H}) \subseteq \mathbb{H}$ and that $f_M \circ f_N = f_{MN},\ f_I = \mathrm{id}_{\mathbb{H}}$ so that this is indeed a group action. In particular, $f_M \in \mathrm{Aut}(\mathbb{H})$ for every $M \in \mathrm{SL}(2,\mathbb{R})$ as it has a holomorphic inverse $f_{M^{-1}}$.

First we shall prove that the group action is transitive. It suffices to prove that $\forall\, \alpha \in \mathbb{H}\ \exists\, M \in \mathrm{SL}(2,\mathbb{R}) :\ f_M(\alpha) = \mathrm{i}$. Notice that

$$M = \begin{pmatrix} \mathrm{Re}\,\alpha & -|\alpha|^2 \\ \mathrm{Im}\,\alpha & 0 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}), \quad f_M(\alpha) = \frac{\alpha\,\mathrm{Re}\,\alpha - |\alpha|^2}{\alpha\,\mathrm{Im}\,\alpha} = \mathrm{i}$$

Next, for $\theta \in \mathbb{R}$, let $M_\theta \in \mathrm{SL}(2,\mathbb{R})$ defined by the rotation

$$M_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

And let $F := \dfrac{z-\mathrm{i}}{z+\mathrm{i}}$ be a conformal mapping from $\mathbb{H}$ onto $\mathbb{D}$. Then $F \circ f_{M_\theta} \circ F^{-1}(z) = \mathrm{e}^{-2\mathrm{i}\theta} z$ is a rotation on $\mathbb{D}$.

Finally, for any automorphism $f \in \mathrm{Aut}(\mathbb{H})$, suppose that $f(\mathrm{i}) = \alpha$. There exists $N \in \mathrm{SL}(2,\mathbb{R})$ such that $f_N(\alpha) = \mathrm{i}$. Hence $f_N \circ f(\mathrm{i}) = \mathrm{i}$. Since F maps i to 0, $F \circ f_N \circ f \circ F^{-1} \in \mathrm{Aut}(\mathbb{D})$ fixes the origin. By Corollary 4.41, this is a rotation. Hence there exists $\theta \in \mathbb{R}$ such that $F \circ f_N \circ f \circ F^{-1} = F \circ f_{M_\theta} \circ F^{-1}$. Hence $f = f_N^{-1} \circ f_{M_\theta} = f_{N^{-1}M_\theta}$. There exists $a,b,c,d \in \mathbb{R}$ with $ad - bc = 1$ such that $f(z) = \dfrac{az+b}{cz+d}$. □

4.4.3 Automorphisms of the Complex Plane.

Theorem 4.44. Automorphism Group of the Complex Plane.

$$\mathrm{Aut}(\mathbb{C}) = \{ f(z) = az + b :\ a,b \in \mathbb{C},\ a \neq 0 \} = \mathrm{Aff}(\mathbb{C})$$

Proof. Suppose that $f \in \mathrm{Aut}(\mathbb{C})$. We first show that ∞ cannot be an essential singularity of f. Suppose for contradiction that ∞ is an essential singularity. Let $g(z) := f(1/z)$. Then g has an essential singularity at $z = 0$. By Casorati-Weierstrass Theorem, $g(\mathbb{D} \setminus \{0\})$ is dense in $\mathbb{C}$. Notice that the inversion $z \mapsto 1/z$ maps $\mathbb{C} \setminus \overline{B}(0,1)$ to $B(0,1) \setminus \{0\} = \mathbb{D} \setminus \{0\}$. Therefore

$f(\mathbb{C}\setminus\overline{B}(0,1))$ is dense in $\mathbb{C}$. Since f is continuous and bijective, it maps the closure of $\mathbb{C}\setminus\overline{B}(0,1)$ to the closure of $f(\mathbb{C}\setminus\overline{B}(0,1))$ [1], which implies that $f(\mathbb{C}\setminus B(0,1)) = \mathbb{C}$. Hence f cannot be injective, contradicting that f is an automorphism.

We know that ∞ is a removable singularity or a pole of f. By Proposition 4.9 (iii), f is a rational function. Moreover f must be a polynomial since it is holomorphic on $\mathbb{C}$. By injectivity of f let $f^{-1}(0) = b \in \mathbb{C}$. Then $f(z) = a(z-b)^n$ where $n = \deg f$ and $a \in \mathbb{C}$. If $n \geqslant 2$, then f cannot be injective, as $f(b+1) = f(b+\mathrm{e}^{2\pi\mathrm{i}/n}) = a$. Hence $f = az+b$ for some $a, b \in \mathbb{C}$. □

Remark. Notice that we define Aut($\mathbb{C}$) to be the group of all *biholomorphisms* on $\mathbb{C}$, It should not be confused with the group of all (continuous) *field automorphisms* of $\mathbb{C}$, which appears in field theory and sometimes shares the same notation.

4.4.4 Automorphisms of the Extended Complex Plane.

Theorem 4.45. Automorphism Group of the Extended Complex Plane.

$$\mathrm{Aut}(\mathbb{C}_\infty) = \{f(z) = \frac{az+b}{cz+d} :\ a,b,c,d \in \mathbb{C},\ ad-bc \neq 0\} = \mathrm{Mob}$$

Proof. Suppose that $f \in \mathrm{Aut}(\mathbb{C}_\infty)$ and that $f(\infty) = \infty$. Since f is bijective we have $f|_{\mathbb{C}} \in \mathrm{Aut}(\mathbb{C})$ (more rigorously $f \circ \iota_0 \in \mathrm{Aut}(\mathbb{C})$). By Theorem 4.44 we have $f(z) = az+b$ for some $a, b \in \mathbb{C}$.

Suppose that $f \in \mathrm{Aut}(\mathbb{C}_\infty)$ and that $f(\infty) = z_0 \in \mathbb{C}$. Then $g(z) := 1/(f(z)-z_0)$ satisfies $g(\infty) = \infty$. Hence $g(z) = cz+d$ for some $c, d \in \mathbb{C}$.

$$cz+d = \frac{1}{f(z)-z_0} \implies f(z) = z_0 + \frac{1}{cz+d} = \frac{az+b}{cz+d}$$

where $a := cz_0$ and $b := dz_0 + 1$. □

Remark. In this section we present four examples of automorphism groups: Aut($\mathbb{D}$), Aut($\mathbb{H}$), Aut($\mathbb{C}$) and Aut($\mathbb{C}_\infty$). We have Aut($\mathbb{D}$) ≅ Aut($\mathbb{H}$), but they are not isomorphic to Aut($\mathbb{C}$) or Aut($\mathbb{C}_\infty$). They are all subgroups of Mob, the group of all Möbius Transformations. The Uniformisation Theorem 4.55 tell us that these groups can completely discribe any automorphism group of simply-connected domain in $\mathbb{C}_\infty$ up to isomorphism.

4.4.5 Automorphisms of an Annulus.

Theorem 4.46. Automorphism Group of Annulus.

Suppose that $A = A(0, r, R)$ is an annulus, where $0 < r < R < \infty$. Then we have

$$\mathrm{Aut}(A) = \left\{ f(z) = \mathrm{e}^{\mathrm{i}\theta} z \text{ or } \mathrm{e}^{\mathrm{i}\theta}\frac{Rr}{z} :\ \theta \in \mathbb{R} \right\}$$

We will present the proof in next section, after we prove the generalized Schwarz Reflection Principle.

4.5 Schwarz Reflection Principle*

In this section we will investigate holomorphic extension of functions and its application in constructing conformal mappings. The key theorem is Schwarz Reflection Theorem and its generalisation.

Lemma 4.47. Painlevé's Theorem.

Suppose that $U \subseteq \mathbb{C}$ is a domain and $\gamma : [0,1] \to \mathbb{C}$ is a piecewise smooth path. Suppose that $f : U \to \mathbb{C}$ is continuous on U and holomorphic on $U \setminus \gamma^*$. Then f is holomorphic in the whole U.

[1] We know that if $f : X \to Y$ is a continuous mapping between topological spaces, and $Z \subseteq X$, then $f(\overline{Z}) \subseteq \overline{f(Z)}$. Moreover, if f is bijective, then

$$Z = f^{-1}(f(Z)) \implies f^{-1}(\overline{f(Z)}) \subseteq \overline{Z} \implies \overline{f(Z)} \subseteq f(\overline{Z})$$

Hence we have $f(\overline{Z}) = \overline{f(Z)}$.

Sketch of Proof. It suffices to prove that f is holomorphic in any open disk $B(z_0, r) \subseteq U$. By Morera's Theorem, it suffices to prove that Cauchy's Theorem holds for any triangle $T \subseteq B(z_0, r)$:

$$\oint_T f(z)\,\mathrm{d}z = 0$$

γ^* divides T into finitely many parts, each part is enclosed by a closed curve with a part of it on γ^*. For each closed curve we choose the positive orientation to integrate so that the integral on γ^* cancels out. Apply Cauchy's Theorem to each closed path and we obtain that $\oint_T f(z)\,\mathrm{d}z = 0$ as claimed. □

Theorem 4.48. Schwarz Reflection Principle.

Suppose that $U \subseteq \mathbb{H}$ is a domain with $I = (a, b) \in \mathbb{R} \cap \partial U$. Suppose that $f : U \cup I \to \mathbb{C}$ satisfies

1. f is continuous on $U \cup I$;
2. f is holomorphic in U;
3. f is real-valued on I.

Then f can be holomorphically extended to $U' \subseteq \mathbb{C}$, a domain symmetric to U with respect to the real axis, such that $f(z) = \overline{f(\bar{z})}$ on U'.

Proof. We extend f to U' by defining $f(z) = \overline{f(\bar{z})}$ on U'. For $x_0 \in I$, the limit from the lower part:

$$\lim_{U' \ni z \to x_0} f(z) = \lim_{U \ni z \to x_0} \overline{f(\bar{z})} = \overline{f(x_0)} = f(x_0)$$

Hence f is continuous on $U \cup I \cup U'$.

To prove that f is holomorphic in U', for $z, z_0 \in U'$, we have $\bar{z}, \bar{z}_0 \in U$. The Taylor expansion of f near $\bar{z}_0$ is given by $f(\bar{z}) = \sum a_n (\bar{z} - \bar{z}_0)^n$, which implies that $f(z) = \sum \bar{a}_n (z - z_0)^n$. Hence f is analytic in $U \cup U'$.

Now by Painlevé's Theorem, f is holomorphic on $U \cup I \cup U'$. □

We can replace $\mathbb{R}$ by any circle in $\mathbb{C}_\infty$. Recall that symmetric in Definition 4.24 we give the definition of a pair of symmetric points with repsect to a circle or line. We say that U and V are symmetric with respect to the circle or line, if V contains all the symmetric points of points in U and vise versa.

Theorem 4.49. Schwarz Reflection Principle for a Circle.

Suppose that $\gamma(z_0, r)$ is a path whose image $\gamma^* = \partial B(z_0, r)$ is a circle in $\mathbb{C}$. We denote the two connected components of $\mathbb{C} \setminus \gamma^*$ by Ω_+ and Ω_-. Suppose that $U \subseteq \Omega_+$ is a domain such that $I := \partial U \cap \gamma^*$ is non-empty. Suppose that $f : U \cup I \to \mathbb{C}$ satisfies

1. f is continuous on $U \cup I$;
2. f is holomorphic in U;
3. $f(I) \subseteq \Gamma^* := \partial B(w_0, \rho)$;
4. $w_0 \notin f(U)$.

Then f can be holomorphically extended to $U' \subseteq \Omega_-$, a domain symmetric to U with respect to γ^*, such that f maps a pair of symmetric points with respect to γ^* to a pair of symmetric points with respect to Γ^*.

Proof. By Lemma 4.25, we know that z is symmetric to $z_0 + \dfrac{r^2}{\bar{z} - \bar{z}_0}$ with respect to $B(z_0, r)$ and w is symmetric to $w_0 + \dfrac{\rho^2}{\bar{w} - \bar{w}_0}$ with respect to $B(w_0, \rho)$. We define f on U' by

$$f(z) = w_0 + \frac{\rho^2}{\overline{f\left(z_0 + \dfrac{r^2}{\bar{z} - \bar{z}_0}\right)} - \bar{w}_0}$$

f is well defined on U' as $w_0 \notin f(U)$, which suggests that the denominator is never zero. Let $w = f(z)$. Then we have

$$w_0 + \frac{\rho^2}{\bar{w} - \bar{w}_0} = f\left(z_0 + \frac{r^2}{\bar{z} - \bar{z}_0}\right)$$

Let $\phi_\rho(w) := w_0 + \dfrac{\rho^2}{\bar{w} - \bar{w}_0}$ and $\phi_r(z) := z_0 + \dfrac{r^2}{\bar{z} - \bar{z}_0}$. Then $\phi_\rho \circ f(z) = \phi_\rho(w) = f \circ \phi_r(z)$. Notice that ϕ_r and ϕ_ρ are self-inverse. For each open disk $B(z_1, \delta) \subseteq U'$, we can expand $f(z)$ into power series near $\phi_r(z_1) \in U$:

$$f \circ \phi_r(z) = \sum_{n=0}^{\infty} a_n(\phi_r(z) - \phi_r(z_1))^n = \sum_{n=0}^{\infty} a_n \left(\frac{r^2}{\bar{z} - \bar{z}_0} + \frac{r^2}{\bar{z}_1 - \bar{z}_0}\right)^n = \sum_{n=0}^{\infty} a_n r^{2n} \left(\frac{\bar{z}_1 - \bar{z}}{(\bar{z} - \bar{z}_0)(\bar{z}_1 - \bar{z}_0)}\right)^n$$

Hence

$$\begin{aligned} f(z) = \phi_\rho \circ f \circ \phi_r(z) &= \phi_\rho\left(\sum_{n=0}^{\infty} a_n r^{2n} \left(\frac{\bar{z}_1 - \bar{z}}{(\bar{z} - \bar{z}_0)(\bar{z}_1 - \bar{z}_0)}\right)^n\right) \\ &= w_0 + \frac{\rho^2}{\sum_{n=0}^{\infty} \bar{a}_n r^{2n} \dfrac{(z_1 - z)^n}{(z - z_0)^n (z_1 - z_0)^n} - \bar{w}_0} \end{aligned}$$

Then f is holomorphic in U'.

Next we prove that f is continuous on I. Fix any $\zeta \in I$. For $z \in U'$, $\phi_r(z) \in U$. As $z \to \zeta$, we also have $\phi_r(z) \to \zeta$. Since f is continuous on $U \cup I$, we have $f \circ \phi_r(z) \to f(\zeta) \in \Gamma^*$. Then $f(z) = \phi_\rho \circ f \circ \phi_r(z) \to f(\zeta)$. Hence f is continuous at $\zeta \in I$.

By Painlevé's Theorem, f is holomorphic on $U \cup I \cup U'$. □

Remark. The theorem still holds if we replace γ^* and Γ^* by circles in $\mathbb{C}_\infty$ (circles or lines in $\mathbb{C}$). The proof is very similar and we are not going to do it here.

Theorem 4.50. Conformal equivalence classes of annuli.

Let $A(0, r, R) = \{z : r < |z| < R\}$ be an annulus with the smaller radius r and the larger radius R. In the case $0 < r_i$ and $R_i < \infty$, $A(0, r_1, R_1)$ and $A(0, r_2, R_2)$ are conformally equivalent if and only if $R_1/r_1 = R_2/r_2$.

For the degenerate cases, the annulus $A(0, 0, \infty) = \mathbb{C} \setminus \{0\}$ is not conformally equivalent to any other annulus. The annuli $A(0, 0, R)$ and $A(0, r, \infty)$ with $r > 0$ and $R < \infty$ are conformally equivalent to each other and not equivalent to any other annuli.

Proof. We first consider the non-degenerate case. Suppose that $R_1/r_1 = R_2/r_2$. Then $f(z) = \dfrac{R_2}{R_1} z$ maps $A(0, r_1, R_1)$ bijectively to $A(0, r_2, R_2)$. Conversely, suppose that $f : A(0, r_1, R_1) \to A(0, r_2, R_2)$ is a biholomorphism. We claim that f extends to a homeomorphism from $\overline{A(0, r_1, R_1)}$ to $\overline{A(0, r_2, R_2)}$. For the boundary behavior of biholomorphisms, we need to use the tools discussed in Section 4.7. Since the boundary of $A(0, r_1, R_1)$ is the union of two disjoint simple closed paths, with some minor adaptation of Proposition 4.62 and Proposition 4.63 we can prove that the claim is true. In particular, f maps $\partial A(0, r_1, R_1)$ continuously and bijectively to $\partial A(0, r_2, R_2)$.

Suppose that f maps $\partial B(0, r_1)$ to $\partial B(0, r_2)$ and maps $\partial B(0, R_1)$ to $\partial B(0, R_2)$. By Schwarz Reflection Theorem for a circle, we can "reflect" $A(0, r_1, R_1)$ across $\partial B(0, r_1)$. Since R_1 is symmetric to r_1^2/R_1 with respect to $\partial B(0, r_1)$, we extend f holomorphically to $A\left(0, \dfrac{r_1^2}{R_1}, R_1\right)$. Now f is a biholomorphism from $A\left(0, \dfrac{r_1^2}{R_1}, R_1\right)$ to $A\left(0, \dfrac{r_2^2}{R_2}, R_2\right)$. We repeat the reflection. After n times, f is a biholomorphism from $A\left(0, R_1\left(\dfrac{r_1}{R_1}\right)^{2n}, R_1\right)$ to $A\left(0, R_1\left(\dfrac{r_2}{R_2}\right)^{2n}, R_2\right)$. Let $n \to \infty$. Then f is biholomorphism from $A(0, 0, R_1)$ to $A(0, 0, R_2)$. In particular f maps 0 to 0. Then f is a biholomorphism from $B(0, R_1)$ to $B(0, R_2)$. Let $\phi_1(z) := z/R_1$ and $\phi_2(z) := R_2 z$. Then $\phi_2 \circ f \circ \phi_1 \in \mathrm{Aut}(\mathbb{D})$. By Theorem 4.40, there exists $\theta \in \mathbb{R}$ such that $f(z) = \mathrm{e}^{\mathrm{i}\theta} \dfrac{R_2}{R_1} z$. Hence we must have $R_1/r_1 = R_2/r_2$.

Suppose that f maps $\partial B(0, r_2)$ to $\partial B(0, r_1)$ and maps $\partial B(0, R_2)$ to $\partial B(0, R_1)$. Then $g(z) := R_2 r_2 / f(z)$ is a biholomorphism that maps $\partial B(0, r_1)$ to $\partial B(0, r_2)$. We return to the previous case and conclude that $R_1/r_1 = R_2/r_2$.

Now we consider the degenerate case. $f(z) = \dfrac{r_2}{r_1} z$ is a biholomorphism from $A(0, r_1, \infty)$ to $A(0, r_2, \infty)$; $f(z) = \dfrac{R_2}{R_1} z$ is a biholomorphism from $A(0, 0, R_1)$ to $A(0, 0, R_2)$; $f(z) = \dfrac{Rr}{z}$ is a biholomorphism from $A(0, 0, R)$ to $A(0, r, \infty)$. The remaining parts are trivial if we again invoke the boundary correspondence. Suppose that $A(0, 0, R_1)$ are conformally equivalent to $A(0, r_2, R_2)$ via biholomorphism f, where $r_2, R_1, R_2 \in (0, \infty)$. Then f is a homeomorphism from $\partial A(0, 0, R_1) = \{0\} \cup \partial B(0, R_1)$ tp $\partial A(0, r_2, R_2) = \partial B(0, r_2) \cup \partial B(0, R_2)$. Such f cannot be bijective, contradiction. Similarly we can prove that $A(0, 0, \infty)$ is not conformally equivalent to any other annuli. □

Proof of Theorem 4.46. Suppose that $f \in \mathrm{Aut}(A)$. In the proof of Theorem 4.46, we have already shown that there are only two cases about how f maps the boundary of A. They corresponds to $f(z) = e^{i\theta} z$ and $f(z) = e^{i\theta}\dfrac{Rr}{z}$ respectively. □

4.6 Riemann Mapping Theorem*

Theorem 4.51. Riemann Mapping Theorem.

Suppose that $U \subsetneq \mathbb{C}$ is a simply-connected domain and $z_0 \in U$. Then there exists a unique bijective holomorphic function f which maps U onto the unit disk $\mathbb{D}$ such that $f(z_0) = 0$ and $f'(z_0) > 0$.

Remark. Riemann Mapping Theorem in fact implies that **any two proper simply-connected domains in $\mathbb{C}$ are conformally equivalent**. This is a very profound result which establishes the connection between topological properties and holomorphic properties. We regard this theorem as the cornerstone of the theory of simple-variable complex analysis.

4.6.1 Normal Families

Before proving Riemann Mapping Theorem we shall first introduce normal families and Montel's Theorem.

Definition 4.52. Normal Families.

Suppose $U \subseteq \mathbb{C}$ is an open set. A set $\mathscr{F}$ of functions $U \to \mathbb{C}$ is called a normal family, if any sequence in $\mathscr{F}$ has a subsequence that converges uniformly on every compact subset of U (the limit is not necessarily in $\mathscr{F}$). The convergence is called normal convergence.

Theorem 4.53. Montel's Theorem.

Suppose that $\mathscr{F}$ is a set of holomorphic functions on U such that $\mathscr{F}$ is uniformly bounded on any compact subset of U, then $\mathscr{F}$ is a normal family.

Proof. Since $\mathscr{F}$ is uniformly bounded, ther exists $M > 0$ such that $|f(z)| \leqslant M$ for all $z \in U$ and $f \in \mathscr{F}$. Now we define a sequence of compact subsets of U:

$$K_n := \left\{ z \in U : \operatorname{dist}(z, \mathbb{C} \setminus U) \geqslant \frac{1}{n} \right\} = \left\{ z \in U : \inf_{w \in \mathbb{C} \setminus U} |z - w| \geqslant \frac{1}{n} \right\}$$

It follows easily from definition that $K_n \subseteq K_{n+1}$ for $n \in \mathbb{Z}_+$ and that $\bigcup_{n=1}^{\infty} K_n = U$. We say that $\{K_n\}$ is an **exhaustion** of U.

First we shall prove that $\mathscr{F}$ is equicontinuous. For each compact subset $K_n \in U$, let $r := 3 \cdot \operatorname{dist}(K_{n-1}, K_n) > 0$. For $z, w \in K_n$ such that $|z - w| < r$, we consider the circular path $\gamma(z, 2r)$ which lies entirely in K_{n+1} and apply Cauchy's Intgeral Formula to any given $f \in \mathscr{F}$:

$$f(z) - f(w) = \oint_{\gamma(z,2r)} f(\zeta)\left(\frac{1}{\zeta - z} - \frac{1}{\zeta - w}\right) d\zeta$$

Hence

$$\begin{aligned} |f(z) - f(w)| &\leqslant 2\pi r \sup_{\zeta \in \partial B(z,2r)} \left| f(\zeta)\left(\frac{1}{\zeta - z} - \frac{1}{\zeta - w}\right)\right| \\ &\leqslant 2\pi r M \sup_{\zeta \in \partial B(z,2r)} \frac{|z - w|}{|\zeta - z||\zeta - w|} \\ &\leqslant 2\pi r M \frac{|z - w|}{2r \cdot r} = 2\pi M r^{-1} |z - w| \end{aligned}$$

Let $L := 2\pi M r^{-1}$ be a constant which only depends on K. Then $|f(z) - f(w)| \leqslant L|z - w|$ for all $f \in \mathscr{F}$ and all $z, w \in K_n$ that is sufficiently closed. This is a uniform Lipschitz property of $\mathscr{F}$ and it implies that $\mathscr{F}$ is equicontinuous on K_n.

Next, suppose $\{f_n\} \subseteq \mathscr{F}$ is a sequence. By Arzelà-Ascoli Theorem 0.75, $\{f_n\}$ has a subsequence $\{f_{s(1,n)}\}$ that converges uniformly on K_1 (without loss of generality we may assume that $K_1 \neq \varnothing$), where $n \mapsto s(1, n)$ is an injective increasing function on $\mathbb{Z}_+$.

By the same argument $\{f_{s(1,n)}\}$ has a subsequence $\{f_{s(2,n)}\}$ that converges uniformly on K_2. Inductively we obtain a sequence of subsequences $\{f_{s(j,n)}\}$. By the diagonal argument, $\{f_{s(n,n)}\}$ is a subsequence of $\{f_n\}$ which converges uniformly on every K_n. We further notice that any compact subset of U lies entirely in some K_n. Hence we conclude that $\{f_n\}$ converges normally. $\mathscr{F}$ is a normal family. □

Remark. Montel's Theorem guarantees the existence of the limit function, but it says nothing about the behavior of the limit function other than it is holomorphic. The following theorem gives a particular case which we can have some dichotomy information about the limit function.

Theorem 4.54. Hurwitz's Theorem.

Suppose that $\{f_n\}$ is a sequence of injective holomorphic functions on U. If f_n converges normally to f on U, then f is either injective or constant.

Proof. Suppose that f is non-constant. We argue for contradiction and suppose that f is not injective. There exists $z_1, z_2 \in U$ such that $f(z_1) = f(z_2)$. Let $g_n(z) := f_n(z) - f_n(z_2)$ and $g(z) = f(z) - f(z_2)$. Since g is holomorphic, by identity theorem the roots of g are isolated. There exists $r > 0$ such that g is non-zero in the deleted closed disk $\overline{B}(z_1, r) \setminus \{z_1\}$. We know that $f_n \to f$ normally on U. Therefore $g_n \to g$ uniformly on $\overline{B}(z_1, r)$. Let

$$\varepsilon = \inf_{z \in \partial B(z_1, r)} |g(z)| > 0$$

By uniformly convergence, $\sup\limits_{z \in \partial B(z_1, r)} |g_n(z) - g(z)| < \varepsilon$ for sufficiently large n.
It implies that $|g(z)| > |g_n(z) - g(z)|$ for all $z \in \partial B(z_1, r)$. By Rouché's Theorem, $g(z)$ and $g_n(z)$ has the same number of zeros in $B(z_1, r)$. However, f_n is injective and g_n has no zeros in $B(z_1, r)$, whereas g has one zero $g(z_1) = 0$ in $B(z_1, r)$. This is a contradiction. Hence f is injective. □

4.6.2 Proof and Consequences of RMT

Now we can proceed to the proof of Riemann Mapping Theorem.

Proof of Riemann Mapping Theorem. .

Step 1: U is conformally equivalent to a bounded simply-connected domain.

Since $U \neq \mathbb{C}$, we pick $\alpha \in \mathbb{C} \setminus U$. Since U is simply-connected, by Proposition 1.38 (vii) we can define a holomorphic function $f(z) = \log(z - \alpha)$ on U, which has all the desired properties of logarithm. f is injective as $\exp \circ f(z) = z - \alpha$. Fix any $w \in U$. We claim that $f(z) \neq f(w) + 2\pi\mathrm{i}$ for all $z \in U$. If this is not the case, then $f(z) = f(w)$ for some $z \in U$. Then

$$z - \alpha = \exp \circ f(z) = \exp(f(w) + 2\pi\mathrm{i}) = w - \alpha \Longrightarrow z = w \Longrightarrow f(z) = f(w)$$

which is a contradiction.

In fact, $f(w) + 2\pi\mathrm{i} \notin \overline{f(U)}$. Suppose that there exists $\{z_n\} \subseteq U$ such that $\lim\limits_{n\to\infty} f(z_n) = f(w) + 2\pi\mathrm{i}$, Then $\lim\limits_{n\to\infty} z_n = w$ as the exponential function is continuous. But this implies that $\lim\limits_{n\to\infty} f(z_n) = f(w)$, which is a contradiction. Hence we can define $F : U \to \mathbb{C}$ by

$$F(z) = \frac{1}{f(z) - (f(w) + 2\pi\mathrm{i})}$$

The above discussion suggests that $F(U)$ is bounded. Moreover, F is injective holomorphic. Hence U and $F(U)$ are conformally equivalent. We have hence proven that any simply-connected domain is conformally equivalent to some bounded simply-connected domain. So from now on we may assume that U is bounded.

Step 2: We consider the following set of holomorphic function:

$$\mathscr{F} = \left\{ f : U \to \mathbb{D} \,\middle|\, f \text{ is injective holomorphic, } f(z_0) = 0 \right\}$$

We claim that $\mathscr{F}$ is non-empty.

Since U is bounded, there exists $R > 0$ such that $|g(z)| \leq R$. We consider the mapping $g(z) = (z - z_0)/2R$. Clearly f is injective holomorphic; $g(z_0) = 0$; and $|g(z)| = |z - z_0|/2R \leq 1$. Hence $g \in \mathscr{F}$.

Step 3: $\mathscr{F}$ is a normal family, because it is uniformly bounded by 1 and we can use Montel's Theorem.

Step 4: The maximiser of the functional $f \mapsto |f'(z_0)|$ lies in $\mathscr{F}$.

Since U is open and $z_0 \in U$, there exists $r > 0$ such that $B(z_0, r) \subseteq U$. For any $f \in \mathscr{F}$, by Cauchy's Inequality 1.22 we have

$$|f'(z_0)| \leqslant \frac{1}{r} \sup_{z\in\partial B(z_0,r)} |f(z)| \leqslant \frac{1}{r}$$

Hence $M := \sup_{f\in\mathscr{F}} |f'(z_0)| < +\infty$. We can choose a sequence $\{f_n\} \subseteq \mathscr{F}$ such that $\lim_{n\to\infty} |f'(z_0)| = M$. Since $\mathscr{F}$ is a normal family, $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ that converges uniformly to f_0 on every compact subset of U. Then $|f_0'(z_0)| = M$. By Hurwitz's Theorem, f_0 is either injective or constant. If f_0 is constant, then $|f_0'(z_0)| = M = 0$. However, $g(z) = (z - z_0)/2R \in \mathscr{F}$ and $|g'(z_0)| = 1/2R > 0$, which implies that $M \geqslant 1/2R > 0$. This is a contradiction. Hence f_0 is injective holomorphic. We have $f_0 \in \mathscr{F}$.

Step 5: The maximiser f_0 is surjective.

Suppose for contradiction that $f_0(U) \neq \mathbb{D}$. We shall construct explicitly $h \in \mathscr{F}$ such that $|h'(z_0)| > |f'(z_0)|$. We pick $u \in \mathbb{D} \setminus f_0(U)$. The following Möbius transformation maps u to 0:

$$\varphi_u(z) = \frac{z-u}{1-\bar{u}z}$$

Notice that $\varphi_u \circ f_0(U)$ is simply-connected and does not contain 0, we can define a holomorphic branch of the square root function $\eta(z) = z^{1/2}$ on $\varphi_u \circ f_0(U)$.

Let $v = \eta \circ \varphi_u \circ f_0(z_0)$ and $h = \varphi_v \circ \eta \circ \varphi_u \circ f_0$. We observe that h is a compositon of injective holomorphic functions and $|h(z)| \leqslant 1$. Moreover, $h(z_0) = \varphi_v(v) = 0$. Hence $h \in \mathscr{F}$.

We write $f_0 = \varphi_u^{-1} \circ \lambda \circ \varphi_v^{-1} \circ h$ where $\lambda(z) := z^2$. Let $\Phi = \varphi_u^{-1} \circ \lambda \circ \varphi_v^{-1}$. Notice that Φ maps $\mathbb{D}$ into $\mathbb{D}$ and that $\Phi(v) = 0$. By Schwarz's Lemma, $|\Phi'(0)| < 1$ because Φ is not a rotation. Hence

$$|f_0'(z_0)| = |\Phi'(0) \cdot h'(z_0)| < |h'(z_0)|$$

which contradicts that $|f_0'(z_0)| = \sup_{f\in\mathscr{F}} |f'(z_0)|$. Hence f_0 is surjective.

We have obtained a bijective holomorphic function $f_0 : U \to \mathbb{D}$. To adjust the derivative at z_0, we put

$$\tilde{f}_0(z) = \frac{|f_0'(z_0)|}{f_0'(z_0)} f_0(z)$$

so that $\tilde{f}_0'(z_0) = |f_0'(z_0)| > 0$. $\tilde{f}_0$ is a conformal mapping $U \to \mathbb{D}$ such that $\tilde{f}_0(z_0) = 0$ and $\tilde{f}_0'(z_0) > 0$.

Step 6: $\tilde{f}_0$ is unique.

Suppose that $\tilde{f}_1$ is another function that satisfies the desired properties. Then $\psi := \tilde{f}_0 \circ \tilde{f}_1^{-1}$ is an automorphism of $\mathbb{D}$. By Theorem 4.40, there exists $a \in \mathbb{D}$ and $\theta \in [0, 2\pi]$ such that

$$\psi(z) = \mathrm{e}^{\mathrm{i}\theta} \frac{z-a}{1-\bar{a}z}$$

Since $\psi(0) = 0$ and $\psi'(0) > 0$, we must have $\theta = 0$ and $a = 0$. Hence $\psi = \mathrm{id}$ and $\tilde{f}_0 = \tilde{f}_1$ as claimed. □

Remark. The only property of simply-connected domain we use is the existence of holomorphic logarithm (existence of holomorphic square root follows directly). Thus we have finished the proof of Proposition 1.38 (vii)$\Longrightarrow$(i) by proving that the domains satisfying (vii) are conformally equivalent (which is much stronger than homeomorphic) to the unit disk $\mathbb{D}$.

In the language of extended complex plane (more rigorously Rimeann surfaces, or one-dimensional complex manifolds), Riemann Mapping Theorem can be slightly strengthened as follows:

Corollary 4.55. Poincaré-Koebe Uniformisation Theorem.

Suppose $U \subseteq \mathbb{C}_\infty$ is a simply-connected domain. Then U is conformally equivalent to one of $\mathbb{C}_\infty$, $\mathbb{C}$ and $\mathbb{D}$. More specifically, if $\mathbb{C}_\infty \setminus U$ contains more than two points, then U is conformally equivalent to $\mathbb{D}$; if $\mathbb{C}_\infty \setminus U$ contains exactly one point, then U is conformally equivalent to $\mathbb{C}$; if $\mathbb{C}_\infty \setminus U$ is empty, then U is conformally equivalent to $\mathbb{C}_\infty$. Furthermore, $\mathbb{C}_\infty$, $\mathbb{C}$ and $\mathbb{D}$ are not conformally equivalent.

Proof. $\mathbb{C}_\infty$ is not conformally equivalent to $\mathbb{C}$ or $\mathbb{D}$, as they are not even homeomorphic. $\mathbb{C}$ is not conformally equivalent to $\mathbb{D}$, as any holomorphic function from $\mathbb{C}$ to $\mathbb{D}$ is constant by Liouville's Theorem.

For $U = \mathbb{C}_\infty$, there is really nothing to prove. For $U = \mathbb{C}_\infty \setminus \{w\}$ and $w \in \mathbb{C}$, the Möbius transformation $f(z) = 1/(z-w)$ maps U conformally onto $\mathbb{C}$. For the case that $\mathbb{C}_\infty \setminus U$ contains more than two points, if $\infty \in \mathbb{C}_\infty \setminus U$, then this is just Riemann Mapping Theorem; if $\infty \notin \mathbb{C}_\infty \setminus U$, we can always use a Möbius transformation to map some $w \in \mathbb{C}_\infty \setminus U$ to ∞, which changes the problem to the previous case. □

Remark. The classification of the conformal equivalence classes of multiply-connected domains are much more complicated. The conformal equivalence of annuli has been given in Theorem 4.50. The following theorem completely describes the conformal equivalence class of doubly-connected domains. We are not going to give the proof here.

Theorem 4.56. Conformal equivalence of doubly-connected domains.

Suppose that $U \subseteq \mathbb{C}$ is a doubly-connected domain. Then U is conformally equivalent to an annulus with outer radius 1. Moreover, the conformal mapping is unique up to translations, rotations, and inversion.

Proof. See **[Belyaev]** Theorem 2.6.3. □

4.7 Boundary Correspondence*

In the previous section, we have proven that any two proper simply-connected domains are conformally equivalent. Suppose that $f : U \to V$ is a biholomorphism between two simply-connected domains. It is generally unclear if f can maps ∂U bijectively onto ∂V. The main result we are about to prove is as follows:

Theorem 4.57. Carathéodory Extension Theorem.

Suppose that U is a domain and f is a biholomorphism from U onto $\mathbb{D}$. Then f can be continuous extended to a homeomorphism $\overline{U} \to \overline{B}(0,1)$ if and only if ∂U is the image of a simple closed path.

The result has the following equivalent statement:

Theorem 4.58. Boundary Correspondence Theorem.

Suppose that U is a domain and f is a biholomorphism from U onto $\mathbb{D}$. Then f maps ∂U bijectively to $\partial \mathbb{D}$ with orientation preserved if and only if ∂U is the image of a simple closed path.

Definition 4.59. Accessible Points.

Suppose that $U \subseteq \mathbb{C}$ is open and $\alpha \in \partial U$. We say that ζ is an accessible point of U, if for any sequence $\{z_n\} \subseteq U$ with $\lim\limits_{n\to\infty} z_n = \zeta$ there exists a path $\gamma : [0,1] \to \mathbb{C}$ such that:

$$\gamma(1) = \zeta; \quad \gamma([0,1)) \subseteq U;$$

and an increasing sequence $\{t_n\} \subseteq [0,1)$ such that $\gamma(t_n) = z_n$.

In other words, the path γ passes through all points in $\{z_n\}$ and lies entirely in U except for one of the end points.

Example 4.60. Inaccessible Points.

Let $U := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1,\ 0 < \operatorname{Im} z < 1\}$ and $I_n := \{x + 2^{-n}\mathrm{i} \in \mathbb{C} : 0 \leqslant x \leqslant 1/2\}$. Let $V := U \setminus \left(\bigcup_{n=1}^{\infty} I_n\right)$. Then $[0,1/2)$ are inaccessible points of V, as shown in Figure 4.9. In other words, there are no paths in V that can approach points on $[0,1/2)$.

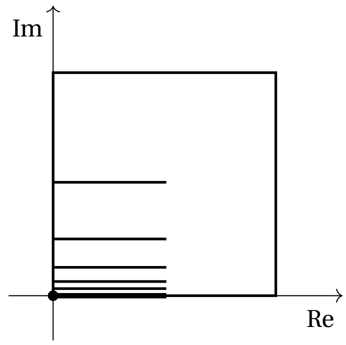

Figure 4.9: A domain with inaccessible boundary points.

The key result is that for accessible points, the limit of the mapping on the boundary exists. First we present a lemma due to Lindelöf and Koebe.

Lemma 4.61. Koebe's Lemma

Let $\{z_n\}, \{z_n'\} \subseteq \mathbb{D}$ be two sequences such that $z_n \to \zeta$ and $z_n' \to \zeta'$, where $\zeta, \zeta' \in \partial\mathbb{D}$ and $\zeta \neq \zeta'$. Let γ_n be a simple path joining z_n and z_n' which lies in the annulus $A(0, 1-\varepsilon_n, 1)$ where $\varepsilon_n \to 0$ as $n \to \infty$.

Suppose that $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and bounded. If $\sup_{z\in\gamma_n^*} |f(z)| \to 0$ as $n \to \infty$, then f is identically zero on $\mathbb{D}$.

Proof. Suppose that f is not identically zero. Without loss of generality we may assume that $f(0) = 0$. If f has a zero of multiplicity n at 0, then we can replace f by $f(z)/z^n$ which satisfies all conditions of the lemma. By applying a rotation, we may further assume that $\overline{\zeta} = \zeta'$. That is, ζ and ζ' are symmetric about the real axis.

We can find a angle $\pi/m < \arg\zeta$ and a sector $S := \{z \in \mathbb{C} : \arg z \in (-\pi/m, \pi/m)\}$ such that there are infinitely many n with $\gamma_n^* \cap S \neq \varnothing$ and $z_n, z_n' \notin S$, as shown in Figure 4.10.

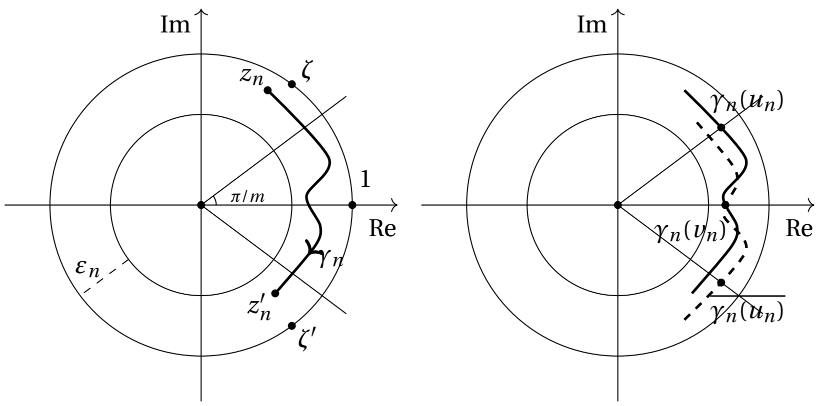

Figure 4.10: The dashed line on the second diagram is the reflection of γ_n about the real axis.

Let $u_n \in [0, 1]$ be the largest number such that $\arg\gamma_n(u_n) = \pi/m$ and let $v_n > u_n$ be the smallest number such that $\arg\gamma_n(v_n) = 0$. Then γ_n restricted to $[u_n, v_n]$ is a path in the sector which joins $\gamma_n(u_n)$ and $\gamma_n(v_n)$. Let $\overline{\gamma}$ be the reflection of γ about the real axis. Then we obtain a path joining $\gamma_n(v_n)$ and $\overline{\gamma_n(u_n)}$, as shown in Figure 4.10. Let σ_n be the concatenation of the two paths such that σ_n is a path joining $\gamma_n(u_n)$ and $\overline{\gamma_n(u_n)}$ and is symmetric about the real axis.

Let $T(z) := e^{\pi i/m} z$ be the rotation by $2\pi/m$. We consider the concatenation of successive rotations of σ_n:

$$\eta_n := \sigma_n \star (T \circ \sigma_n) \star \cdots \star \left(T^{m-1} \circ \sigma_n\right)$$

Hence we obtain a simple closed curve η_n which lies entirely in the annulus $A(0, 1-\varepsilon_n, 1)$.

Finally, define $F(z) := f(z)\overline{f(\bar{z})}$ and

$$G(z) := F(z) \cdot F \circ T(z) \cdot \ldots \cdot F \circ T^{m-1}(z)$$

Suppose that $r_n := \sup_{z\in\gamma_n^*}|f(z)|$ and $M := \sup_{z\in\mathbb{D}}|f(z)|$. By Schwarz Reflection Principle, $\overline{f(\bar{z})}$ is also holomorphic in $\mathbb{D}$ and bounded by M. In particular F is bounded by r_nM on σ_n. G is holomorphic in $\mathbb{D}$. For any $z\in\eta_n^*$, one of the factors of G is bounded by r_nM and the rest are bounded by M^2. Hence G is bounded by r_nM^{2m-1} on η_n^*. Suppose that f is non-constant. By Maximum Modulus Principle, we have

$$|f(0)|^{2m} = |G(0)| < \sup_{z\in\eta_n^*}|G(z)| \leqslant r_nM^{2m-1}$$

Let $n\to\infty$. Since $r_n\to 0$, we have $f(0)=0$, which is a contradiction. Hence f is identically zero. □

Proposition 4.62. Existence of Continuous Extension.

Suppose that $f: U\to\mathbb{D}$ is a biholomorphism and $\zeta\in\partial U$ is accessible. If there exists $r>0$ such that $B(\zeta,r)\cap U$ is connected, then $\lim_{z\to\zeta} f(z)$ exists and has modulus 1.

Proof. First, for any $\{z_n\}\subseteq U$ with $\lim_{n\to\infty} z_n = \zeta$, let γ be a path that satisfies the conditions in Definition 4.59. $\gamma(t_n) = z_n$ for each $n\in\mathbb{Z}_+$. We claim that $\lim_{t\to 1}|f\circ\gamma(t)| = 1$. Suppose for contradiction that there exists $\varepsilon>0$ and a subsequence $\{s_n\}$ of $\{t_n\}$ such that $|f\circ\gamma(s_n)|\leqslant 1-\varepsilon$ for all $n\in\mathbb{Z}_+$. Let $\{u_n\}$ be a subsequence of $\{s_n\}$ such that $w := \lim_{n\to\infty} f\circ\gamma(u_n)$ exists. Then $|w|\leqslant 1-\varepsilon$ or $w\in\mathbb{D}$. The inverse function $f^{-1}:\mathbb{D}\to U$ maps

$$f^{-1}(w) = \lim_{n\to\infty} f^{-1}\circ f\circ\gamma(u_n) = \lim_{n\to\infty}\gamma(u_n) = \gamma(1)$$

which suggests that $\zeta=\gamma(1)$ does not lie on the boundary of U, contradiction.

Next, supppose for contradiction that $\lim_{z\to\zeta} f(z)$ does not exists. Since $\overline{B}(0,1)$ is compact, there exists $\{a_n\},\{b_n\}\subseteq U$ such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \zeta$ but $\lim_{n\to\infty} f(a_n) =: a \neq b := \lim_{n\to\infty} f(b_n)$. By connectivity, we can find a simple path $\gamma:[0,1]\to U\cup\{\zeta\}$ joining $a_1,b_1,a_2,b_2,\cdots$ and $\gamma(1)=\zeta$. For each $n\in\mathbb{Z}_+$, let γ_n be the restriction of γ on $[\gamma^{-1}(a_n),\gamma^{-1}(b_n)]$. Then $f\circ\gamma_n$ is a simple path joining $f(a_n)$ and $f(b_n)$.

Let $g(z) := f^{-1}(z)-\zeta$ defined on $\mathbb{D}$. We apply Koebe's Lemma to g. By the previous part, we know that $\lim_{t\to 1}|f\circ\gamma(t)| = 1$, which implies that $\varepsilon_n := \inf_{z\in\gamma_n^*}|f(z)|\to 0$ as $n\to\infty$. In other words, $f\circ\gamma_n$ tends to the unit circle uniformly. Moreover, we have

$$\sup_{w\in(f\circ\gamma_n)^*}|g(w)| = \sup_{z\in\gamma_n^*}|z-\zeta|\to 0 \qquad \text{as } n\to\infty$$

since γ is uniformly continuous and $\gamma(1)=\zeta$. By Koebe's Lemma, g is identically zero, which is obviously impossible. In conclusion, $\lim_{z\to\zeta} f(z)$ exists and is unique. We can continuous extend f to $\zeta\in\partial U$. □

Remark. For the case that $\zeta\in\partial U$ has no connected neighbourhood in U, the limit does not exists generally. However, if we extend the definition of accessible boundary points, and associate each connected component of $B(\zeta,r)\cap U$ with one "accessible point", then the proposition still holds.

We should also point out that there are cases when $B(\zeta,r)\cap U$ has no connected components. The formal definition of an accessible point is an equivalence class of paths whose end point is ζ. This defintion works even if $B(\zeta,r)\cap U$ has no connected components, where ζ corresponds to infinitely many accessible points.

Proposition 4.63. Injectivity of Continuous Extension.

Suppose that $f: U\to\mathbb{D}$ is a biholomorphism and $\zeta_1,\zeta_2\in\partial U$ are two distinct accessible points. Suppose that f extends to ζ_1,ζ_2 by continuity (existence proven in the previous proposition), then $f(\zeta_1)\neq f(\zeta_2)$.

Proof. Suppose for contradiction that $f(\zeta_1) = f(\zeta_2) = w_0\in\partial\mathbb{D}$. By applying a rotation we may assume that $w_0=-1$. Let $\gamma_1:[0,1]\to\mathbb{C}$ and $\gamma_2:[0,1]\to\mathbb{C}$ be two paths such that $\gamma_1(1)=\zeta_1$, $\gamma_2(1)=\zeta_2$, and $\gamma_1([0,1)),\gamma_2([0,1))\subseteq U$. Since $\zeta_1\neq\zeta_2$, there exists $t_0\in(0,1)$ such that $|\gamma_1(t)-\gamma_2(t)|>|\zeta_1-\zeta_2|/2$ for all $t\in(t_0,1)$. There exists $\delta>0$ such that $B(-1,\delta)$ does not intersect with $\gamma_1([0,t_0])$ or $\gamma_2([0,t_0])$.

Let $A := \mathbb{D}\cap B(-1,\delta)$. In the polar coordinates centered at -1, there exists a suitable function $\varphi(r)$ such that $A = \{-1+r\mathrm{e}^{\mathrm{i}\theta} : 0\leqslant r\leqslant\delta,\ -\varphi(r)\leqslant\theta\leqslant\varphi(r)\}$. For each $r\in(0,\delta)$, let $w_1\in(f\circ\gamma_1)^*\cap\partial B(-1,r)$ and $w_2\in(f\circ\gamma_2)^*\cap\partial B(-1,r)$. Let $g := f^{-1}$ be the inverse function. Then we have $|g(w_1)-g(w_2)|>|\zeta_1-\zeta_2|/2$.

Let η be the circular arc joining w_1 and w_2. Then we have:

$$\begin{aligned}
\frac{1}{2}|\zeta_1-\zeta_2| < |g(w_1)-g(w_2)| = \left|\int_\eta g'(z)\,\mathrm{d}z\right| &\leqslant \int_\eta |g'(z)|\,\mathrm{d}z \leqslant \int_{-\varphi(r)}^{\varphi(r)} |g'(-1+r\,\mathrm{e}^{\mathrm{i}\theta})|r\,\mathrm{d}\theta \\
\implies \frac{1}{4}|\zeta_1-\zeta_2|^2 &< \left(\int_{-\varphi(r)}^{\varphi(r)} |g'(-1+r\,\mathrm{e}^{\mathrm{i}\theta})|r\,\mathrm{d}\theta\right)^2 \\
&\leqslant \left(\int_{-\varphi(r)}^{\varphi(r)} |g'(-1+r\,\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\theta\right)^2 \left(\int_{-\varphi(r)}^{\varphi(r)} r\,\mathrm{d}\theta\right)^2 \qquad \text{(Cauchy-Schwarz Inequality)} \\
&\leqslant \pi^2 r^2 \cdot \int_{-\varphi(r)}^{\varphi(r)} |g'(-1+r\,\mathrm{e}^{\mathrm{i}\theta})|^2\,\mathrm{d}\theta
\end{aligned}$$

Integrate with respect to r:

$$\int_0^\delta \frac{|\zeta_1-\zeta_2|^2}{4\pi^2 r}\,\mathrm{d}r \leqslant \int_0^\delta \int_{-\varphi(r)}^{\varphi(r)} |g'(-1+r\,\mathrm{e}^{\mathrm{i}\theta})|^2 r\,\mathrm{d}\theta\,\mathrm{d}r = \iint_A |g'|^2$$

The left hand side diverges unless $\zeta_1=\zeta_2$, whereas the right hand side is equal to the area of $g(A)$ and is finite. Contradiction. Hence we must have $f(\zeta_1)\neq f(\zeta_2)$. □

Remark. By cosine theorem, $\varphi(r)$ is given by:

$$\varphi(r) = \arccos\left(\frac{r^2+1-\delta^2}{2r}\right)$$

We do not need this form in the proof of the proposition.

Proof of Theorem 4.57. "$\Longrightarrow$": If $f: U\to\mathbb{D}$ can be extended to a homeomorphism $f:\overline{U}\to\overline{B}(0,1)$, then $f|_{\partial U}$ is a homeomorphism from ∂U to the unit circle $\partial\mathbb{D}$. Therefore ∂U is a simple closed path.

"$\Longleftarrow$": Suppose that ∂U is a simple closed path. In particular, every point on ∂U is an accessible point of U. By Proposition 4.62 and Proposition 4.63, there exists a continuous injective extension of f to ∂U which is in fact bijective, as f is invertible. The continuity of the extension follows trivially. □

The following converse of Boundary Correspondence Theorem is very useful:

Theorem 4.64. Converse of Boundary Correspondence Theorem.

Suppose that $U\to\mathbb{C}$ is a simply-connected domain whose boundary ∂U is the image of a simple closed path. $f:\overline{U}\to\mathbb{C}$ is holomorphic in U and continuous on $\overline{U}$. If f maps ∂U bijectively to $\partial\mathbb{D}$, then f is a biholomorphism from U to $\mathbb{D}$.

Proof. Suppose that $w_0\in\mathbb{D}$. There exists a neighbourhood $V\subseteq U$ of ∂U such that $f(z)\neq w_0$ in V. For any simple closed curve $\gamma:[0,1]\to U$, we consider the increment of the argument of $f(z)-w_0$ as z goes along γ:

$$\Delta_\gamma \arg(f(z)-w_0)$$

which is invariant under homotopy. Suppose that Γ is a simple closed path with $\Gamma^* = \partial U$. and $\Lambda := \gamma(0,1)$. By the statement of the theorem we know that

$$\Delta_\Gamma \arg(f(z)-w_0) = \Delta_\Lambda \arg(w-w_0) = 2\pi$$

The same relation also holds for a positively-oriented simple closed path $\gamma:[0,1]\to V$ that is homotopic to Γ in $\overline{V}$. By Argument Principle, $f(z)-w_0$ has exactly one zero in D, the interior of γ. But $f(z)\neq w_0$ for $z\in U\setminus D\subseteq V$. We conclude that there exists exactly one $z_0\in U$ such that $f(z_0)=w_0$.

Similarly, we can prove that no points in U is mapped to $\partial\mathbb{D}$ or $\mathbb{C}\setminus\overline{B}(0,1)$. Hence f is a bijection between U and $\mathbb{D}$. □

4.8 Schwarz-Christoffel Mappings*

In this section, we aim to construct the explicit formula for a conformal mapping from the upper half plane to a polygonal area. We say that P is a polygonal area, if ∂P is the image of a piecewise-linear simple closed path.

We denote the vertices of ∂P by $w_1, ..., w_n$. By Riemann Mapping Theorem, there exists a biholomorphism $f : \mathbb{D} \to P$. By Boundary Correspondence Theorem, f extends to a bijection between $\partial\mathbb{D}$ and ∂P. On the other hand, we know that $\phi(z) := (z-\mathrm{i})/(z+\mathrm{i})$ is a biholomorphism from $\mathbb{H}$ to $\mathbb{D}$, which maps $\partial\mathbb{H} = \mathbb{R}$ bijectively to $\partial\mathbb{D}\setminus\{1\}$. Hence $f\circ\phi$ is a biholomorphism from $\mathbb{H}$ to P which maps $\mathbb{R}$ bijectively to ∂P with one point removed.

We shall first give the Schwarz-Christoffel Intgeral and prove that it maps $\mathbb{R}$ to a polygon. Then we shall show that any biholomorphism between the upper half plane and a polygonal area can be written in the form of the Schwarz-Christoffel Intgeral. After that we will investigate the behavior if we include the point of infinity. We will conclude the section with some examples of the use of the Schwarz-Christoffel Mapping.

For $a_i \in \mathbb{R}$ and $\beta_i < 1$, the function $(z-a_i)^{-\beta_i}$ could be multi-valued. We can define a holomorphic branch of it by cutting the plane along the ray $\{a_i + \mathrm{i}y \in \mathbb{C} : y \leqslant 0\}$. For $x \in \mathbb{R}$, we define:

$$(z-a_i)^{-\beta_i} = \begin{cases} |x-a_i|^{-\beta_i} & x > a_i \\ |x-a_i|^{-\beta_i}\,\mathrm{e}^{-\mathrm{i}\pi\beta_i} & x < a_i \end{cases}$$

In particular, $s(z) := (z-a_1)^{-\beta_1}\cdots(z-a_n)^{-\beta_n}$ has a holomorphic branch in the cut plane $\mathbb{C}\setminus\bigcup_{i=1}^n\{a_i + \mathrm{i}y \in \mathbb{C} : y \leqslant 0\}$. The cut plane is simply-connected, so $s(z)$ has a primitive, namely the Schwarz-Christoffel Integral:

Definition 4.65. Schwarz-Christoffel Integral.

Suppose that $-\infty < a_1 < \cdots < a_n < +\infty$ and $\beta_1, \cdots \beta_n \in (-\infty, 1)$. On the cut plane $\mathbb{C}\setminus\bigcup_{i=1}^n\{a_i + \mathrm{i}y \in \mathbb{C} : y \leqslant 0\}$, the function defined by the integral

$$S(z) = \int_{z_0}^{z} (\zeta - a_1)^{-\beta_1}\cdots(\zeta - a_n)^{-\beta_n}\,\mathrm{d}\zeta$$

is called the Schwarz-Christoffel Integral, where z_0 is a fixed point and the intgeral is taken along any piecewise smooth path from z_0 to z on the cut plane.

The condition $\beta_i < 1$ implies that $(z-a_i)^{-\beta_i}$ is integrable near the singularity a_i. Hence $S(z)$ can be continuously extended on the real line $\mathbb{R}$. In addition, if $\sum\limits_{i=1}^{n}\beta_i > 1$, then

$$(|z|-a_1)^{-\beta_1}\cdots(|z|-a_n)^{-\beta_n} \leqslant c|z|^{-\sum_{i=1}^n\beta_i}$$

for sufficiently large $|z|$. It is not difficult to show that the integral $S(z)$ converges as $|z| \to \infty$. We denote the limit by $w_\infty := \lim\limits_{z\to\infty} S(z)$.

Proposition 4.66

Let $S(z)$ be the Schwarz-Christoffel Integral defined in Definition 4.65. Let $w_i := S(a_i)$ for $i = 1, ..., n$.

(i) If $\sum\limits_{i=1}^{n}\beta_i = 2$, then S maps $\mathbb{R}$ to a $\partial P\setminus\{w_\infty\}$, where ∂P is a n-sided polygon whose vertices are given in order by $w_1, ..., w_n$. The point w_∞ lies on the line segment between w_n and w_1. Moreover, the interior angle of ∂P at the vertex w_i is $\pi(1-\beta_i)$.

(ii) If $1 < \sum\limits_{i=1}^{n}\beta_i < 2$, then S maps $\mathbb{R}$ to a $\partial P\setminus\{w_\infty\}$, where ∂P is a $(n+1)$-sided polygon whose vertices are given in order by $w_1, ..., w_n, w_\infty$. Moreover, the interior angle of ∂P at the vertex w_i is $\pi\beta_i$, and interior angle at the vertex w_∞ is $\pi\left(\sum\limits_{i=1}^{n}\beta_i - 1\right)$.

Proof. We can see that (i) is in fact a special case of (ii), where the interior angle at w_∞ is π. So we only need to prove (ii).

For $i \in \{0, ..., n\}$ and $x \in (a_{i-1}, a_i)$, we have

$$S'(x) = \prod_{k=1}^{i-1}(x-a_k)^{-\beta_k}\prod_{k=i}^{n}(x-a_k)^{-\beta_k} = \prod_{k=1}^{n}|x-a_k|^{-\beta_k}\,\mathrm{e}^{-\mathrm{i}\pi\sum_{k=i}^n\beta_i}$$

Hence $\arg S'(x) = -\pi \sum_{k=i}^{n} \beta_k$ is constant for $x \in (a_{i-1}, a_i)$. As

$$S(x) = S(a_i) + \int_{a_i}^{x} S'(t)\,\mathrm{d}t = w_i + \arg S'(x) \int_{a_i}^{x} |S'(t)|\,\mathrm{d}t$$

It suggests that S maps (a_{i-1}, a_i) to a line segment (w_{i-1}, w_i) on the complex plane, which makes angle $-\pi \sum_{k=i}^{n} \beta_i$ with the real axis. For $x > a_n$, we have $\arg S'(x) = 0$. Hence S maps $(a_n, +\infty)$ to a line segment (w_n, w_∞) parallel to the real axis. For $x < a_1$, $\arg S'(x) = -\pi \sum_{k=1}^{n} \beta_i$. S maps $(-\infty, a_1)$ to a line segment (w_∞, w_1).

For $i \in \{1, ..., n\}$, the interior angle θ_i at the vertex w_i is given by:

$$\theta_i = \pi - \left(\lim_{x \to a_i^-} \arg S'(x) - \lim_{x \to a_i^+} \arg S'(x) \right) = \pi - \left(\pi \sum_{k=i}^{n} \beta_i - \pi \sum_{k=i+1}^{n} \beta_i \right) = \pi(1 - \beta_i)$$

The interior angle θ_∞ at the vertex w_∞ is given by:

$$\theta_\infty = (n+1)\pi - \sum_{k=1}^{n} \theta_k = \pi \left(\sum_{i=1}^{n} \beta_i - 1 \right)$$

□

Theorem 4.67. Schwarz-Christoffel Theorem.

Suppose that the open set $P \subseteq \mathbb{C}$ is a polygonal area whose boundary ∂P is a n-sided polygon with vertices (in order) $w_1, ..., w_n$. Suppose that the interior angle of ∂P at the vertex w_i is $\pi(1 - \beta_i)$ where $\beta_1 \in (-1, 1)$. If $f : \mathbb{H} \to P$ is a biholomorphism, then there exists $-\infty < a_1 < \cdots < a_n < \infty$ such that

$$f(z) = C_1 \int_{z_0}^{z} (\zeta - a_1)^{-\beta_1} \cdots (\zeta - a_n)^{-\beta_n}\,\mathrm{d}\zeta + C_2$$

where z_0, C_1 and C_2 are complex constants. Moreover, the extension of f to the homeomorphism from $\overline{\mathbb{H}}$ to $\overline{P}$ implies that $f(a_i) = w_i$ for $i \in \{1, ..., n\}$.

Proof. By Boundary Correspondence Theorem f extends to a homeomorphism $f : \overline{\mathbb{H}} \to \overline{P}$. Let $a_i := f^{-1}(w_i)$ for $i = 1, ..., n$. f maps the real line bijectively to ∂P.

For $i \in \{2, ..., n-1\}$, the interior angle at the vertex w_i is $\pi(1 - \beta_i)$. Hence we define $g_i : \{z \in \mathbb{H} : \operatorname{Re} z \in (a_{i-1}, a_{i+1})\} \to \mathbb{C}$ by

$$g_i(z) = (f(z) - w_i)^{1/(1-\beta_i)}$$

Then $g_i(a_i) = 0$. And g_i extends the angle subtended by two line segments near w_0 to π. As a result, g_i maps the infinite half-strip to an infinite half-strip. By Schwarz Reflection Principle, g_i can be holomorphically extended across the real axis and becomes a holomorphic function on the infinite strip $a_{i-1} < \operatorname{Re} z < a_{i+1}$. On the upper half strip, we have:

$$f(z) = w_i + g_i(z)^{1-\beta_i} \quad \Longrightarrow \quad \frac{f'(z)}{f(z) - w_i} = (1 - \beta_i) \frac{g_i'(z)}{g_i(z)}$$

Since f is bijective, we have $f'(z) \neq 0$. Hence $g_i'(z) \neq 0$. The Schwarz Reflection Principle suggests that we also have $g_i'(z) \neq 0$ in the lower half strip. By some continuity argument we must have that g_i is injective on the real interval (a_{i-1}, a_{i+1}), which implies that $g_i'(z) \neq 0$ on (a_{i-1}, a_{i+1}). In conclusion, $g_i'(z) \neq 0$ on the whole strip $a_{i-1} < \operatorname{Re} z < a_{i+1}$.

Next we shall prove that a_i is a simple pole of f''/f'. The derivatives of f:

$$f'(z) = (1 - \beta_i) g_i(z)^{-\beta_i} g_i'(z) \qquad f''(z) = -\beta_i (1 - \beta_i) g_i(z)^{-\beta_i - 1} \left(g_i'(z)\right)^2 + (1 - \beta_i) g_i(z)^{-\beta_i} g_i''(z)$$

Hence

$$\frac{f''(z)}{f'(z)} = -\beta_i \frac{g_i'(z)}{g_i(z)} + \frac{g_i''(z)}{g_i'(z)}$$

Notice that the power series expansion of g_i near a_i is given by $g_i(z) = g_i'(a_i)(z - a_i) + O((z - a_i)^2)$ and that g_i is non-zero. Then a_i is a simple pole of g_i'/g_i with residue $\operatorname{Res}(g_i'/g_i, a_i) = 1$. In addition, g_i''/g_i' is holomorphic in the strip. Therefore there exists a holomorphic function q_i on $a_{i-1} < \operatorname{Re} z < a_{i+1}$ such that

$$\frac{f''(z)}{f'(z)} = -\beta_i \frac{1}{z - a_i} + q_i(z)$$

Similarly, for the infinite strip $\{z \in \mathbb{C} : -\infty < \operatorname{Re} z < a_2\}$ and $\{z \in \mathbb{C} : a_{n-1} < \operatorname{Re} z < +\infty\}$, there exists holomorphic functions q_1 and q_n defined on the respective strips such that

$$\frac{f''(z)}{f'(z)} = -\beta_i \frac{1}{z-a_1} + q_1(z) \qquad \frac{f''(z)}{f'(z)} = -\beta_i \frac{1}{z-a_n} + q_n(z)$$

Next we investigate the behavior of f''/f' at the infinity. For sufficiently large $R > 0$, f maps $(-\infty, -R) \cup (R, +\infty)$ to a line segment of $w_n - - w_1$. By Schwarz Reflection Principle, f is holomorphically extended to $\mathbb{C} \setminus \overline{B}(0,R)$. Since f maps $\infty \in \mathbb{C}$ to a point on the line segment $w_n - - w_1$, f is bounded and hence holomorphic at the infinity. There exists $m \in \mathbb{N}$ and $c_m \neq 0$ such that

$$f(z) = f(\infty) + \frac{c_m}{z^m} + \frac{c_{m+1}}{z^{m+1}} + \cdots$$

The derivative of f:

$$f'(z) = -m\frac{c_m}{z^{m+1}} - (m+1)\frac{c_{m+1}}{z^{m+2}} + \cdots = -m\frac{c_m}{z^{m+1}} + p(z)$$

where p is holomorphic on $\mathbb{C} \setminus \overline{B}(0,R)$ and $p(\infty) \neq 0$. Hence we have

$$\frac{f''(z)}{f'(z)} = -\frac{m+1}{z} + \frac{p'(z)}{p(z)}$$

Hence $\lim_{z\to\infty} \frac{f''(z)}{f'(z)} = 0$. In particular f''/f' is bounded near the infinity.

Finally we let $h(z) := \frac{f''(z)}{f(z)} + \sum_{i=1}^{n} \frac{\beta_i}{z-a_i}$. Then h is a bounded entire function with $\lim_{z\to\infty} h(z) = 0$. By Liouville's Theorem $h(z) = 0$ on the whole plane. Hence for $z \in \overline{\mathbb{H}}$ we have

$$\frac{f''(z)}{f'(z)} = -\sum_{i=1}^{n} \frac{\beta_i}{z-a_i}$$

We integrate the equation along any piecewise smooth curve from z_0 to z on the upper half plane. Then

$$\log f'(z) = -\sum_{i=1}^{n} \beta_i \log(z-a_i) + \text{const}$$

Hence

$$f'(z) = C_1 (z-a_1)^{-\beta_1} \cdots (z-a_n)^{-\beta_n}$$

Integrate again along any piecewise smooth curve from z_0 to z on the upper half plane:

$$f(z) = C_1 \int_{z_0}^{z} (\zeta - a_1)^{-\beta_1} \cdots (\zeta - a_n)^{-\beta_n} \, d\zeta + C_2$$

which completes the proof. □

Remark. If one of the vertices of ∂P is at the infinity, the mapping formula still applies. Suppose that $w_k = \infty$, as shown in Figure 4.11. We pick w_k' on the line segment $w_{k-1} - - w_k$ and w_k'' on the line segment $w_k - - w_{k+1}$. We connect w_k' and w_k'' by a line segment and obtain a $(n+1)$-sided polygon $\partial P'$: $w_1 - - \cdots - - w_{k-1} - - w_k' - - w_k'' - - w_{k+1} - - \cdots - - w_n$.

By Schwarz-Christoffel Theorem, the mapping from $\mathbb{H}$ to P' is given by

$$f(z) = C_1 \int_{z_0}^{z} (\zeta - a_1)^{-\beta_1} \cdots (\zeta - a_k')^{-\beta_k'} (\zeta - a_k'')^{-\beta_k''} \cdots (\zeta - a_n)^{-\beta_n} \, d\zeta + C_2$$

where $a_k' := f^{-1}(w_k')$ and $a_k'' := f^{-1}(w_k'')$. The interior angle at the vertices w_k' and w_k'' are $\pi(1-\beta_k')$ and $\pi(1-\beta_k'')$ respectively.

As w_k', $w_k'' \to \infty$, we have $P' \to P$ and a_k', $a_k'' \to a_k$. The factor $(z-a_k')^{-\beta_k'}(z-a_k'')^{-\beta_k''} \to (z-a_k)^{-\beta_k'-\beta_k''}$. If $-\pi(1-\beta_k)$ is the angle of intersection of the line $w_{k-1} - - w_k'$ and $w_k'' - - w_{k+1}$, then we have $\pi(1-\beta_k) + \pi(1-\beta_k') - \pi(1-\beta_k'') = \pi$ or $-\beta_k' - \beta_k'' = -\beta_k$. In this case, the mapping formula is exactly same as the finite polygon case:

$$f(z) = C_1 \int_{z_0}^{z} (\zeta - a_1)^{-\beta_1} \cdots (\zeta - a_k)^{-\beta_k} \cdots (\zeta - a_n)^{-\beta_n} \, d\zeta + C_2$$

But notice that we give a slightly different definition of β_k.

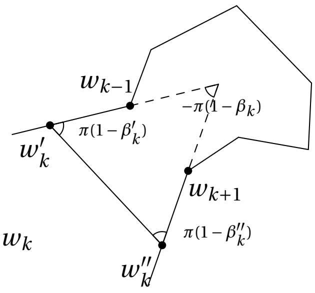

Figure 4.11: A polygonal area with one vertex at the infinity.

Schwarz-Christoffel Theorem ensures that any biholomorphism between the upper half plane and the polygonal area is expressed in terms of Schwarz-Christoffel Integral. However, the points $a_1, \ldots, a_n$ are often unknown when we want to find such mapping to a given polygonal area. The next proposition demonstrates the uniqueness of the mapping if we fix three points on the real line.

Proposition 4.68

Suppose that the open set $P \subseteq \mathbb{C}$ is a polygonal area whose boundary ∂P is a n-sided polygon ($n \geqslant 3$) with vertices (in order) $w_1, \ldots, w_n$. Given three points on the real line $-\infty < a_1 < a_2 < a_3 < +\infty$, there exists a unique biholomorphism $f : \mathbb{H} \to P$ such that $f(a_i) = w_i$ for $i = 1, 2, 3$.

Proof. We know that there exists a biholomorphism $g : \mathbb{H} \to P$. Suppose that for $i \in \{1, 2, 3\}$, $b_i = g^{-1}(w_i) \in \mathbb{R}$. By Proposition 4.21, there exists a Möbius transformation T such that $T(a_i) = b_i$ for $i = 1, 2, 3$. Moreover, $T \in \mathrm{Aut}(\mathbb{H})$ by Theorem 4.43 since $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$. Therefore $f := g \circ T$ is a biholomorphism from $\mathbb{H}$ to P such that $f(a_i) = w_i$.

Next we prove uniqueness. Suppose that $\bar{f}$ is another biholomorphism from $\mathbb{H}$ to P such that $\bar{f}(a_i) = w_i$. Then $\bar{f} \circ f \in \mathrm{Aut}(\mathbb{H})$. In particular, $\bar{f} \circ f$ is a Möbius transformation by Theorem 4.43 which fixes a_1, a_2, a_3. By Proposition 4.21, there is a unique Möbius transformation that fixes three points in $\mathbb{C}_\infty$, namely the identity mapping. Hence we must have $\bar{f} = f$ as claimed. □

Next we look at the case when the biholomorphism maps the infinity to a vertex of the polygon. We shall prove that the formula is obtained by deleting the last term $(z - a_n)^{-\beta_n}$ in the integral.

Theorem 4.69

Suppose that the open set $P \subseteq \mathbb{C}$ is a polygonal area whose boundary ∂P is a n-sided polygon with vertices (in order) $w_1, \ldots, w_n$. Suppose that the interior angle of ∂P at the vertex w_i is $\pi(1 - \beta_i)$ where $\beta_1 \in (-1, 1)$. $f : \mathbb{H} \to P$ is a biholomorphism. If there are $-\infty < a_1 < \cdots < a_{n-1} < \infty$ such that $f(a_i) = w_i$ for $i = 1, \ldots, n-1$ and $f(\infty) = w_n$, then

$$f(z) = C_1 \int_{z_0}^{z} (\zeta - a_1)^{-\beta_1} \cdots (\zeta - a_{n-1})^{-\beta_{n-1}}\, \mathrm{d}\zeta + C_2$$

where z_0, C_1 and C_2 are complex constants.

Proof. We choose $a < a_1$ and consider the Möbius transformation $T(z) = 1/(a - z) \in \mathrm{Aut}(\mathbb{H})$. T maps $a_1, \ldots, a_{n-1}$ and $a_n = \infty$ to $b_1, \ldots, b_{n-1}$ and $b_n = 0$. Now $g := f \circ T^{-1}$ is a biholomorphism from $\mathbb{H}$ to P which maps $b_1, \ldots, b_n$ to $w_1, \ldots, w_n$. By Schwarz-Christoffel Theorem, we have

$$g(z') = C_1 \int_{z'_0}^{z'} (\eta - b_1)^{-\beta_1} \cdots \eta^{-\beta_n}\, \mathrm{d}\eta + C_2$$

Change of variable:

$$\eta = \frac{1}{a - \zeta} \qquad \mathrm{d}\eta = \frac{1}{(a - \zeta)^2}\mathrm{d}\zeta$$

Since $b_i = \dfrac{1}{a - a_i}$, we have $\eta - b_i = \dfrac{1}{a - \zeta} - \dfrac{1}{a - a_i} = \dfrac{\zeta - a_i}{(a - \zeta)(a - a_i)}$. The integral becomes

$$f \circ T^{-1}(z') = C'_1 \int_{T^{-1}(z'_0)}^{T^{-1}(z')} (\zeta - a_1)^{-\beta_1} \cdots (\zeta - a_{n-1})^{-\beta_{n-1}} (a - \zeta)^{\sum_{i=1}^{n} \beta_i - 2}\, \mathrm{d}\zeta + C_2$$

where $C_1' = C_1(a-a_1)^{\beta_1}\cdots(a-a_{n-1})^{\beta_{n-1}}$. Since $\sum_{i=1}^n \beta_i = 2$, we have

$$f\circ T^{-1}(z') = C_1'\int_{T^{-1}(z_0')}^{T^{-1}(z')}(\zeta-a_1)^{-\beta_1}\cdots(\zeta-a_{n-1})^{-\beta_{n-1}}\,\mathrm{d}\zeta + C_2$$

Put $z = T^{-1}(z')$, $z_0 = T^{-1}(z_0')$. We obtain the desired formula

$$f(z) = C_1'\int_{z_0}^{z}(\zeta-a_1)^{-\beta_1}\cdots(\zeta-a_{n-1})^{-\beta_{n-1}}\,\mathrm{d}\zeta + C_2$$

□

Example 4.70. Trigonometric Functions Again.

Find a conformal mapping from $\mathbb{H}$ to the infinite half-strip $U := \{z\in\mathbb{C} : -\pi/2 < \operatorname{Re} z < \pi/2, \operatorname{Im} z > 0\}$.

Solution. We can consider U as a polygonal area with vertices $w_1 = -\pi/2$, $w_2 = \pi/2$ and $w_3 = \infty$. The interior angle at each vertex is $\pi/2$, $\pi/2$ and 0 respectively. Hence $\beta_1 = 1/2$, $\beta_2 = 1/2$, $\beta_3 = 1$. We choose $a_1 = -1$, $a_2 = 1$ and $a_3 = \infty$ on the real line. By Theorem 4.69, the mapping is given by

$$f(z) = \int_0^z (\zeta+1)^{-1/2}(\zeta-1)^{-1/2}\,\mathrm{d}\zeta + C_2 = \int_0^z \frac{1}{\sqrt{\zeta^2-1}}\,\mathrm{d}\zeta + C_2 = C_1\arcsin z + C_2$$

Since $f(-1) = -\pi/2$ and $f(1) = \pi/2$, we have $C_1 = 1$, $C_2 = 0$. Hence the desired mapping is given by $f(z) = \arcsin z$. We can compare this result with Example 4.35.

□

4.9 Harmonic Functions and Dirichlet Problem

In this section we shall investigate the properties of harmonic functions and solutions to Dirichlet boundary value problem with the help of conformal mappings. *For simply-connected domain U enclosed by simple closed curve* ∂U*, we can always transform the problem into Dirichlet BVP on a unit disk, by Riemann Mapping Theorem and Boundary Correspondence Theorem.* First, We repeat the definition of harmonic functions here.

4.9.1 Harmonic Functions.

Definition 0.12: Laplacian, Harmonic Functions.

The differential operator $\nabla^2 := \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ acting on twice differentiable functions in $\mathbb{R}^2$ is called the Laplacian. $f : U \to \mathbb{R}$ is called a harmonic function if $f \in \ker\nabla^2$.

For Dirichlet boundary value problem on an open set U, we need to find a harmonic function with prescribed value on the boundary. More formally:

Definition 4.71. Dirichlet Boundary Value Problem.

The Dirichlet boundary value problem consists of solving

$$\nabla^2 u = 0 \quad \text{in } U \qquad\qquad u = f \quad \text{on } \partial U$$

for some given function f defined on ∂U.

Remark. Recall that the real and imaginary parts of a holomorphic function are harmonic. On the contrary, any harmonic function is the real part of some holomorphic function.

Proposition 4.72

Suppose that $U \subseteq \mathbb{C}$ is a simply-connected domain and $u \in C^2(U)$ is a harmonic function. Then there exists a holomorphic function $f : U \to \mathbb{C}$ such that $\operatorname{Re} f = u$. Moreover, u is analytic.

Proof. Consider $g : U \to \mathbb{C}$ defined by $g(z) = \frac{\partial u}{\partial x} - \mathrm{i}\frac{\partial u}{\partial y}$. g is real-differentiable as $u \in C^2(U)$ and it is easy to check that g satisfies Cauchy-Riemann equations. Therefore by Proposition 0.11, g is holomorphic. Since U is simply-connected, by Proposition 1.38 g has a primitive G on U.

Suppose that $G(z) = a(z) + \mathrm{i}b(z)$. Then

$$\frac{\partial a}{\partial x} - \mathrm{i}\frac{\partial a}{\partial y} = G' = g = \frac{\partial u}{\partial x} - \mathrm{i}\frac{\partial u}{\partial y} \quad \text{for } z \in U$$

Hence $\frac{\partial a}{\partial x} = \frac{\partial u}{\partial x}$ and $\frac{\partial a}{\partial y} = \frac{\partial u}{\partial y}$. In particular $\nabla(a - u) = 0$. By chain rule, a and u differ by a constant on U. Then $f(z) := G(z) + (a(z_0) - u(z_0))$ is a holomorphic function that satisfies $\operatorname{Re} f = u$ on U. By Theorem 1.19, f is analytic. It follows that $u = \operatorname{Re} f$ is analytic. □

Theorem 4.73. Mean Value Property of Harmonic Functions.

Suppose that $u : B(z_0, r) \to \mathbb{R}$ is harmonic. Then

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + r\,\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta$$

Proof. The mean value property of harmonic functions follows directly from the mean value property of holomorphic functions, which is a direct corollary of Cauchy's Integral Formula:

By Proposition 4.72, there exists a holomorphic function $f : U \to \mathbb{C}$ such that $\operatorname{Re} f = u$. By Cauchy's Integral Formula, we have

$$f(z_0) = \frac{1}{2\pi\mathrm{i}}\oint_{\gamma(z_0,r)} \frac{f(z)}{z - z_0}\,\mathrm{d}z = \frac{1}{2\pi\mathrm{i}}\int_0^{2\pi} \frac{f(z_0 + r\,\mathrm{e}^{\mathrm{i}\theta})}{r\,\mathrm{e}^{\mathrm{i}\theta}}\,\mathrm{i}r\,\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + r\,\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta$$

Take the real part:

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + r\,\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta$$

□

Remark. We say that $u : U \to \mathbb{R}$ satisfies the **mean value property**, if for all $B(z_0, r) \subseteq U$ we have

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + r\,\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta$$

We shall prove a converse of Theorem 4.73 in Theorem 4.80, which states that any function satisfying the mean value property is harmonic.

Theorem 4.74. Extreme Value Property of Harmonic Functions.

Suppose that $U \subseteq \mathbb{C}$ is a domain and $u : U \to \mathbb{C}$ is harmonic and non-constant. Then u cannot attain maximum or minimum value in U.

Proof. It suffices to prove that u cannot attain maximum in U. Suppose for contradiction that u attains maximum value at $z_0 \in U$. Choose $r > 0$ such that $B(z_0, r) \subseteq U$. Since u is harmonic and satisfies mean value property, we have

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + r\,\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta \leqslant \sup_{\theta \in [0,2\pi]} u(z_0 + r\,\mathrm{e}^{\mathrm{i}\theta})$$

Hence $u(z_0) = \sup_{\theta \in [0,2\pi]} u(z_0 + r\,\mathrm{e}^{\mathrm{i}\theta})$ and u is constant on $\partial B(z_0, r)$. This holds for any $r > 0$. By continuity u must be constant on $\overline{B}(z_0, r)$. Notice that $B(z_0, r)$ has limit points in U. Since u is analytic, by identity theorem u is constant on the whole U, which is a contradiction. □

Lemma 4.75

Suppose that $f : U \to V$ is holomorphic and $u : V \to \mathbb{R}$ is harmonic. Then $u \circ f : U \to \mathbb{R}$ is also harmonic.

Proof. Being harmonic is a local property. It suffices to consider any $B(z_0, r) \subseteq U$. Suppose that $w_0 := f(z_0)$. Fix $\varepsilon > 0$ such that $B(w_0, \varepsilon) \subseteq V$. By continuity of f we can find $\delta > 0$ such that $f(B(z_0, \delta)) \subseteq B(w_0, \varepsilon)$. Since $B(w_0, \varepsilon)$ is simply-connected, we can find a holomorphic function $g : B(w_0, \varepsilon) \to \mathbb{C}$ such that $\operatorname{Re} g = u$. Hence on $B(z_0, \delta)$ we have $u \circ f = \operatorname{Re}(g \circ f)$. $u \circ f$ is harmonic on $B(z_0, r)$ and hence on the whole U. □

4.9.2 Poisson Kernel.

Now we turn to the simplest case of Dirichlet BVP. We can express the value of a harmonic function in the unit disk in terms of its value on the boundary by the so-called Poisson formula. All we need is the mean value property of harmonic functions.

Theorem 4.76. Poisson Formula for Harmonic Functions on the Unit Disk.

Suppose that $u:\overline{B}(0,1)\to\mathbb{C}$ is harmonic on $\mathbb{D}$. For $z\in\mathbb{D}$, we have:

$$u(z_0)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z_0|^2}{|\,e^{i\theta}-z_0|^2}\,u(e^{i\theta})\,d\theta$$

Proof. For any $z_0\in\mathbb{D}$, the automorphism

$$\varphi_{z_0}(z)=\frac{z_0-z}{1-\bar{z}_0 z}$$

of the unit disk exchanges 0 and z_0. Let $v:=u\circ\varphi_{z_0}$. Then v is also harmonic on $\mathbb{D}$ and $v(0)=u(z_0)$. By mean value property, we have

$$v(0)=\frac{1}{2\pi}\int_0^{2\pi}v(e^{i\beta})\,d\beta$$

Notice that φ_{z_0} maps $\partial\mathbb{D}$ to $\partial\mathbb{D}$ by

$$e^{i\theta}=\frac{z_0-e^{i\beta}}{1-\bar{z}_0\,e^{i\beta}}\quad\Longrightarrow\quad e^{i\beta}=\frac{z_0-e^{i\theta}}{1-\bar{z}_0\,e^{i\theta}}$$

Take the differential and modulus on both sides:

$$i\,e^{i\beta}\,d\beta=i\,e^{i\theta}\,d\theta\frac{|z_0|^2-1}{(1-\bar{z}_0\,e^{i\theta})^2}\quad\Longrightarrow\quad d\beta=d\theta\frac{1-|z_0|^2}{|1-\bar{z}_0\,e^{i\theta}\,|^2}=d\theta\frac{1-|z_0|^2}{|\,e^{i\theta}-z_0|^2}$$

Substitute back to the integral and we have:

$$u(z_0)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z_0|^2}{|\,e^{i\theta}-z_0|^2}\,u(e^{i\theta})\,d\theta$$

□

Remark. The factor $\dfrac{1-|z_0|^2}{|\,e^{i\theta}-z_0|^2}$ is called the **Poisson kernel** of the unit disk:

$$P(\zeta,z):=\frac{1-|z|^2}{|\zeta-z|^2}=\mathrm{Re}\left(\frac{\zeta+z}{\zeta-z}\right)$$

The factor $(\zeta+z)/(\zeta-z)$ is called **Schwarz kernel**.

The Poisson Integral Formula is also written in the following form:

$$u(z)=\frac{1}{2\pi}\int_0^{2\pi}P(\zeta,z)\,u(\zeta)\,d\theta\qquad\text{where }\zeta=e^{i\theta}$$

Alternatively, in the polar coordinates, the Poisson kernel is given by

$$P(\zeta,z)=P(\phi-\theta,r)=\frac{1-r^2}{1-2r\cos(\phi-\theta)+r^2}\qquad\text{where }\zeta=e^{i\theta},\ z=r\,e^{i\phi}$$

$$u(r,\phi)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(\phi-\theta)+r^2}\,u(1,\theta)\,d\theta$$

Corollary 4.77. Properties of Poisson Kernel.

(i) $P(\zeta,z)>0$ for all $z\in\mathbb{D}$ and $\zeta\in\partial\mathbb{D}$;

(ii) $\displaystyle\int_0^{2\pi}P(\zeta,z)\,d\theta=2\pi$ for all $z\in\mathbb{D}$;

(iii) Fix $\zeta\in\partial\mathbb{D}$. Then $P(\zeta,z)$ is harmonic in $\mathbb{D}$.

Proof. (i). Trivial by definition.

(ii). $\displaystyle\int_0^{2\pi} P(\zeta, z)\,\mathrm{d}\theta = \int_0^{2\pi} \frac{1-|z_0|^2}{|\mathrm{e}^{\mathrm{i}\theta}-z_0|^2}\,\mathrm{d}\theta = \int_0^{2\pi}\mathrm{d}\beta = 2\pi.$

(iii). $P(\zeta, z) = \mathrm{Re}\left(\dfrac{\zeta+z}{\zeta-z}\right)$ and the Schwarz kernel is holomorphic in $\mathbb{D}$. □

4.9.3 Dirichlet Boundary Value Problems.

Now we can discuss the existence and uniqueness of the solution to Dirichlet BVP on the unit disk.

Lemma 4.78

Suppose that $f : \partial\mathbb{D} \to \mathbb{R}$ is continuous. Then $u : \mathbb{D} \to \mathbb{R}$ defined by the Poisson integral

$$u(z) = \int_0^{2\pi} P(\zeta, z) f(\zeta)\,\mathrm{d}\theta$$

is a bounded harmonic function. Moreover, $u(z) \to f(\zeta)$ as $z \to \zeta$ at any $\zeta \in \partial\mathbb{D}$.

Proof. Since $\partial\mathbb{D}$ is compact, and $P(\zeta, z)$ and $f(\zeta)$ are continuous, $P(\zeta, z) f(\zeta)$ is bounded. Hence $u(z)$ is bounded. u is harmonic because

$$\nabla^2 u(z) = \nabla^2\left(\int_0^{2\pi} P(\zeta, z) f(\zeta)\,\mathrm{d}\theta\right) = \int_0^{2\pi} f(\zeta)\nabla^2 P(\zeta, z)\,\mathrm{d}\theta = 0$$

Now we fix $\zeta = \mathrm{e}^{\mathrm{i}\theta_0} \in \partial\mathbb{D}$ and $\varepsilon > 0$. Since f is continuous at ζ, there exists $\delta > 0$ such that $|f(\mathrm{e}^{\mathrm{i}\theta}) - f(\mathrm{e}^{\mathrm{i}\theta_0})| \leqslant \varepsilon$ whenever $|\theta - \theta_0| \leqslant \delta$. Then

$$\begin{aligned}\left|u(z) - f(\mathrm{e}^{\mathrm{i}\theta_0})\right| &= \left|\frac{1}{2\pi}\int_0^{2\pi} P(\mathrm{e}^{\mathrm{i}\theta}, z)\left(f(\mathrm{e}^{\mathrm{i}\theta}) - f(\mathrm{e}^{\mathrm{i}\theta_0})\right)\mathrm{d}\theta\right| \\ &\leqslant \frac{1}{2\pi}\int_{|\theta-\theta_0|<\delta} P(\mathrm{e}^{\mathrm{i}\theta}, z)\left|f(\mathrm{e}^{\mathrm{i}\theta}) - f(\mathrm{e}^{\mathrm{i}\theta_0})\right|\mathrm{d}\theta + \frac{1}{2\pi}\int_{|\theta-\theta_0|\geqslant\delta} P(\mathrm{e}^{\mathrm{i}\theta}, z)\left|f(\mathrm{e}^{\mathrm{i}\theta}) - f(\mathrm{e}^{\mathrm{i}\theta_0})\right|\mathrm{d}\theta \\ &=: I_1 + I_2\end{aligned}$$

For I_1, since $|f(\mathrm{e}^{\mathrm{i}\theta}) - f(\mathrm{e}^{\mathrm{i}\theta_0})| \leqslant \varepsilon$, we have

$$I_1 := \frac{1}{2\pi}\int_{|\theta-\theta_0|<\delta} P(\mathrm{e}^{\mathrm{i}\theta}, z)\left|f(\mathrm{e}^{\mathrm{i}\theta}) - f(\mathrm{e}^{\mathrm{i}\theta_0})\right|\mathrm{d}\theta \leqslant \frac{1}{2\pi}\int_{|\theta-\theta_0|<\delta} \varepsilon P(\mathrm{e}^{\mathrm{i}\theta}, z)\,\mathrm{d}\theta < \varepsilon$$

For I_2, for $|\theta - \theta_0| \geqslant \delta$ the Poisson kernel

$$P(\mathrm{e}^{\mathrm{i}\theta}, z) = P(\phi - \theta, r) = \frac{1-r^2}{1-2r\cos(\phi-\theta)+r^2} \leqslant \frac{1-r^2}{1-2r\cos\delta+r^2} \to 0$$

as $r \to 1$. Hence for fixed ε and δ we can find $\eta > 0$ such that $P(\phi-\theta, r) < \varepsilon$ whenever $|\theta - \theta_0| \geqslant \delta$ and $|z - \mathrm{e}^{\mathrm{i}\theta_0}| < \eta$. Then

$$I_2 := \frac{1}{2\pi}\int_{|\theta-\theta_0|\geqslant\delta} P(\mathrm{e}^{\mathrm{i}\theta}, z)\left|f(\mathrm{e}^{\mathrm{i}\theta}) - f(\mathrm{e}^{\mathrm{i}\theta_0})\right|\mathrm{d}\theta < \frac{1}{2\pi}\int_{|\theta-\theta_0|\geqslant\delta} \varepsilon\left|f(\mathrm{e}^{\mathrm{i}\theta}) - f(\mathrm{e}^{\mathrm{i}\theta_0})\right|\mathrm{d}\theta < M\varepsilon$$

where $M := \sup\limits_{\theta\in[0,2\pi]} |f(\mathrm{e}^{\mathrm{i}\theta})|$. Hence $\left|u(z) - f(\mathrm{e}^{\mathrm{i}\theta_0})\right| < (1+M)\varepsilon$. The convergence hence follows. □

Theorem 4.79. Dirichlet BVP on the Unit Disk.

Suppose that $f : \partial\mathbb{D} \to \mathbb{R}$ is continuous. Then $u : \mathbb{D} \to \mathbb{R}$ defined by the Poisson integral

$$u(z) = \int_0^{2\pi} P(\zeta, z) f(\zeta)\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z_0|^2}{|\mathrm{e}^{\mathrm{i}\theta}-z_0|^2} f(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta$$

is the unique solution to the Dirichlet BVP:

$$\nabla^2 u = 0 \quad \text{in } \mathbb{D} \qquad u = f \quad \text{on } \partial\mathbb{D}$$

Proof. By the previous lemma, the Poisson integral $u(z)$ is a solution to the Dirichlet BVP. Suppose there is another solution v. Then $u-v$ is a harmonic with $u-v=0$ on $\partial\mathbb{D}$. By extreme value property $u-v$ is constant in $\mathbb{D}$ and by continuity $u=v$ as claimed. □

Remark. Very often we are dealing with **piecewise continuous functions** as boundary value condition instead. In this case the uniqueness does not hold. However, if we restrict to **bounded harmonic functions** on $\mathbb{D}$ which agrees with f at those points on $\partial\mathbb{D}$ where f is continuous, Dirichlet BVP still has unique solution.

Remark. As an application of Theorem 4.79, we can prove the converse of Theorem 4.73.

Theorem 4.80

Suppose that $U\subseteq\mathbb{C}$ is a domain and $u:U\to\mathbb{R}$ satisfies the mean value property. Then u is a harmonic function.

Proof. For any $B(z_0,r)\subseteq U$, u satisfies the mean value property on $B(z_0,r)$. By Theorem 4.79, (with an appropriate linear mapping) there exists a unique harmonic function $v:B(z_0,r)\to\mathbb{R}$ such that $u=v$ on $\partial B(z_0,r)$. Both u and v satisfies the mean value property and hence the extrem value property. It follows that $u=v$ in $B(z_0,r)$. Since $B(z_0,r)$ is arbitrary, $u=v$ in U. Therefore u is a harmonic function. □

Remark. The Dirichlet BVP on any simply-connected domain with well-behaved boundary can be transformed to the Dirichlet BVP on the unit disk. As an example we give the explicit formula for the solution to Dirichlet BVP on the upper half plane.

Theorem 4.81. Dirichlet BVP on the Upper Half Plane.

Suppose that $f:\mathbb{R}\to\mathbb{R}$ is continuous and both $\lim\limits_{x\to-\infty}f(x)$ and $\lim\limits_{x\to+\infty}f(x)$ exists and are finite. Then $u:\mathbb{H}\to\mathbb{R}$ defined by the integral

$$u(z)=\operatorname{Re}\left(\frac{1}{\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{f(t)}{t-z}\,\mathrm{d}t\right)$$

is the unique solution to the Dirichlet BVP:

$$\nabla^2u=0\quad\text{in }\mathbb{H}\qquad u=f\quad\text{on }\partial\mathbb{H}$$

Proof. For any $z_0\in\mathbb{H}$, consider the biholomorphism

$$\psi(z)=\frac{z-z_0}{z-\bar{z}_0}$$

which maps $\mathbb{H}$ onto $\mathbb{D}$, $(-\infty,+\infty)$ onto $\partial\mathbb{D}\setminus\{1\}$, and z_0 to 0. Then $\psi\circ f$ is continuous on $\partial\mathbb{D}\setminus\{1\}$ and bounded on $\partial\mathbb{D}$. By Theorem 4.79 and the remark after it, there exists a unique bounded harmonic function $v:\mathbb{D}\to\mathbb{R}$ such that $v=\psi\circ f$ on $\partial\mathbb{D}\setminus\{1\}$. Hence $u:=\psi^{-1}\circ v$ is the unique solution to the Dirichlet BVP on $\mathbb{H}$.

Now we turn to the computation of the explicit formula of the solution. By mean value property we have

$$v(0)=\frac{1}{2\pi}\int_0^{2\pi}v(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta$$

Since ψ maps $(-\infty,+\infty)$ onto $\partial\mathbb{D}\setminus\{1\}$,

$$\mathrm{e}^{\mathrm{i}\theta}=\frac{t-z_0}{t-\bar{z}_0}\quad\Longrightarrow\quad\theta=-\mathrm{i}\log\left(\frac{t-z_0}{t-\bar{z}_0}\right)$$

Take the differential:

$$\mathrm{d}\theta=-\mathrm{i}\frac{t-\bar{z}_0}{t-z_0}\cdot\left(-\frac{\bar{z}_0-z_0}{(t-\bar{z}_0)^2}\right)\mathrm{d}t=\mathrm{i}\frac{\bar{z}_0-z_0}{(t-z_0)(t-\bar{z}_0)}\mathrm{d}t=\frac{2\operatorname{Im}z_0}{t^2-2t\operatorname{Re}z_0+|z_0|^2}\mathrm{d}t=\operatorname{Re}\left(\frac{2}{\mathrm{i}(t-z_0)}\right)\mathrm{d}t$$

Since $v(0)=u(z_0)$, we have

$$u(z_0)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}f(t)\operatorname{Re}\left(\frac{2}{\mathrm{i}(t-z_0)}\right)\mathrm{d}t=\operatorname{Re}\left(\frac{1}{\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{f(t)}{t-z}\,\mathrm{d}t\right)$$ □

Remark. In the Cartesian coordinates, the result can be written as

$$u(x,y)=\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{f(t)}{(t-x)^2+y^2}\,\mathrm{d}t$$