Peize Liu
St. Peter's College
University of Oxford

Problem Sheet 3

C3.1: Algebraic Topology

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Convention: All spaces are topological spaces. Maps of spaces are always continuous.

Question 1

Construct a degree d map $S^n \to S^n$ for any $n \geqslant 1$.

Proof.

- *Construct a degree d map $f : S^1 \to S^1$.*

  Let $f \colon S^1 \to S^1$ given by $f(z) = z^d$. For each $z \in S^1$, the local map $f|_z$ is an orientation-preserving homeomorphism. So $\deg_z f = 1$. Since f is d to one, we deduce that $\deg f = d$. ✓

- *For $g : S^n \to S^n$, construct a suspension map $\Sigma g : S^{n+1} \to S^{n+1}$ and prove that* $\deg g = \deg \Sigma g$.

  The suspension ΣS^n is the quotient CS^n/S^n, where CS^n is the cone of S^n. We note that $\Sigma S^n \cong S^{n+1}$ and $CS^n \cong \mathbb{D}^n$. Note that (CS^n, S^n) is a good pair. We have a long exact sequence

$$\cdots \longrightarrow H_{n+1}(S^n) \longrightarrow H_{n+1}(\mathbb{D}^n) \longrightarrow H_{n+1}(S^{n+1}) \xrightarrow{\delta_n} H_n(S^n) \longrightarrow H_n(\mathbb{D}^n) \longrightarrow \cdots$$

  Since $H_{n+1}(\mathbb{D}^n) = 0$ and $H_n(\mathbb{D}^n) = 0$, δ_n is in fact an isomorphism.

  The map $S^n \times [0,1] \mapsto g(S^n) \times [0,1]$ descends to a suspension of map $\Sigma g : S^{n+1} \to S^{n+1}$. We have a commutative diagram by naturality of LES

$$\begin{array}{ccc} H_{n+1}(S^{n+1}) & \xrightarrow{\delta_n} & H_n(S^n) \\ {\scriptstyle (\Sigma g)_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ H_{n+1}(S^{n+1}) & \xrightarrow[\delta_n]{} & H_n(S^n) \end{array}$$

  amazing

  Hence $\deg g = \deg(\Sigma g)$.

- *Combining the results above, deduce that $\Sigma^{n-1} f : S^n \to S^n$ is a map of degree d.*

  Inductively we have $\deg(\Sigma^{n-1} f) = \deg f = d$. ✓ □

  α

Question 2

Given finitely generated Abelian groups $A_1, \ldots, A_n$, construct a space with

$$H_\bullet(X) \cong \begin{cases} \mathbb{Z} & \bullet = 0 \\ A_k & \bullet = k \in \{1, \ldots, n\} \\ 0 & \text{otherwise} \end{cases}$$

Hint. CW-complex.

Proof.

- *For $m \geqslant 2$ and $n \geqslant 1$, construct a CW-complex X with $\widetilde{H}_n(X) \cong \mathbb{Z}/m$ and $\widetilde{H}_k(X) = 0$ for $k \neq n$.*

  Let $X := S^n \cup_\varphi \mathbb{D}^{n+1}$, where the attaching map $\varphi : S^n \to S^n$ has degree m. The cellular chain complex is given by

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_{n+1}(X, S^n) & \longrightarrow & \widetilde{H}(S^n) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \\ & & \mathbb{Z} & \xrightarrow{\deg \varphi} & \mathbb{Z} & & \end{array}$$

  Taking the homology. We have $H^{\mathrm{CW}}_{n+1}(X) = 0$ and $H^{\mathrm{CW}}_n(X) \cong \mathbb{Z}/\deg\varphi = \mathbb{Z}/m$. All other homology groups are zero obviously. good!

  The space $X = M(\mathbb{Z}/m, n)$ is called the **Moore space**.

- *For a finitely generated Abelian group A, construct a CW-complex with $\widetilde{H}_n(X) \cong A$ and $\widetilde{H}_k(X) = 0$ for $k \neq n$.*

  By the structure theorem for $\mathbb{Z}$-modules, we can write

  $$A \cong \mathbb{Z}^a \oplus \mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_b$$

  Then we take the space of wedge sums:

  $$X_n := \bigvee_{i=1}^{a} S^n \vee \bigvee_{i=1}^{b} M(\mathbb{Z}/d_i, n)$$

  Therefore

  $$\widetilde{H}_k(X_n) = \bigoplus_{i=1}^{a} \widetilde{H}_k(S^n) \oplus \bigoplus_{i=1}^{b} \widetilde{H}_k(M(\mathbb{Z}/d_i, n)) = \begin{cases} \mathbb{Z}^a \oplus \mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_b, & k = n \\ 0, & k \neq n \end{cases}$$

- *Back to the question.*

  For the group A_i, we have

  $$A_i \cong \mathbb{Z}^{a_i} \oplus \mathbb{Z}/d_{i,1} \oplus \cdots \oplus \mathbb{Z}/d_{i,b_i}$$

  We take the space

  $$X := \bigvee_{i=1}^{n} X_i = \bigvee_{i=1}^{n} \left(\bigvee_{j=1}^{a_i} S^i \vee \bigvee_{j=1}^{b_i} M(\mathbb{Z}/d_{i,j}, i) \right)$$

  Then

  $$\widetilde{H}_k(X) = \bigoplus_{i=1}^{n} \widetilde{H}_k(X_i) = \bigoplus_{i=1}^{n} A_i \delta_{ik} = \begin{cases} A_k, & k \in \{1, \dots, n\} \\ 0, & \text{otherwise} \end{cases}$$

  The homology group

  $$H_k(X) = \begin{cases} \mathbb{Z}, & k = 0 \\ A_k, & k \in \{1, \dots, n\} \\ 0, & \text{otherwise} \end{cases}$$

  good! □

Question 3

Let $f, g : S^n \to S^n$ satisfy $f(x) \neq g(x)$ for all $x \in S^n$. Prove that $f \simeq -\mathrm{id} \circ g$.

(Hint. Consider $\dfrac{\varphi_t}{\|\varphi_t\|}$ where $\varphi_t = tf - (1-t)g$.)

Deduce that

- if $f : S^n \to S^n$ has no fixed point then $f \simeq -\mathrm{id}$.
- if G is a group acting continuously and freely on S^{2n} then $G = 1$ or $\mathbb{Z}/2$. (Hint. Degree.)

Proof. Let $H(x,t) = \dfrac{tf(x) - (1-t)g(x)}{\|tf(x) - (1-t)g(x)\|}$. This is well-defined if $\|tf(x) - (1-t)g(x)\| \neq 0$ for all $t \in [0,1]$ and $x \in S^n$. If $\|tf(x) - (1-t)g(x)\| = 0$, then $tf(x) = (1-t)g(x)$. Taking the norm we have $t = (1-t)$ and $f(x) = g(x)$. This contradicts the assumption. Hence $H : S^n \times [0,1] \to S^n$ is well-defined.

We have $H(x,0) = f(x)$ and $H(x,1) = -g(x)$. H defines a homotopy from f to $-g$. $f \simeq -\mathrm{id} \circ g$.

- Take $g = \mathrm{id}$. f has no fixed points implies that $f(x) \neq g(x)$ for all $x \in S^n$. Hence $f \simeq -\mathrm{id}$.
- The action of G on S^{2n} defines a group homomorphism $G \to \mathrm{Homeo}(S^{2n})$. Each homeomorphism on S^{2n} has degree ± 1. So we have a group homomorphism $\deg : G \to \mathbb{Z}/2$. Since the action is free, each $g \in G \setminus \{e\}$ has no fixed point. Hence $g \simeq -\mathrm{id}$ and $\deg g = \deg(-\mathrm{id}) = (-1)^{2n+1} = -1$. In particular, $\ker \deg = \{e\}$. By first isomorphism theorem, G is isomorphic to a subgroup of $\mathbb{Z}/2$. Hence $G = \{e\}$ or $\mathbb{Z}/2$. □

Question 4

a) In the CW complex for $\mathbb{C}P^n$ from the course notes, show that the attaching maps commute with the obvious inclusions $S^{k-1} \subseteq S^k$ via $\mathbb{R}^k \equiv \mathbb{R}^k \times 0 \subseteq \mathbb{R}^{k+1}$, and $\mathbb{C}P^k \subseteq \mathbb{C}P^{k+1}$ via $\mathbb{C}^{k+1} \equiv \mathbb{C}^{k+1} \times 0 \subseteq \mathbb{C}^{k+2}$.

(You have to decide in which dimensions to consider these inclusions, and also recall $\mathbb{R}^2 \cong \mathbb{C}$, $(x, y) \mapsto x + \mathrm{i}y$.)

b) Explain why $\mathbb{R}P^n \cong \mathbb{D}^n/(\pm\mathrm{id}$ action on $\partial\mathbb{D}^n)$.

Under this identification, show that the i-th hyperplane $x_i = 0$ intersects $\mathbb{R}P^n$ in a copy of $\mathbb{R}P^{n-1}$, Show that the corresponding inclusion $\mathrm{ind}_i : \mathbb{R}P^{n-1} \to \mathbb{R}P^n$ induces isomorphisms $H_\bullet(\mathbb{R}P^{n-1}; \mathbb{Z}/2) \to H_\bullet(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$ for $\bullet \neq n$.

(Hint. "Homotope it".)

State and prove an analogous result for $\mathbb{C}P^n$ (using $\mathbb{Z}$).

c) Use the cultural remark on page 57 of the notes for this exercise. Compute the cup product to deduce

$$
\begin{aligned}
H^\bullet(\mathbb{C}P^n) &\cong \mathbb{Z}[x]/x^{n+1} && |x| = 2 \\
H^\bullet(\mathbb{R}P^n; \mathbb{Z}/2) &\cong (\mathbb{Z}/2)[y]/y^{n+1} && |y| = 1
\end{aligned}
$$

You may assume as known that $\mathbb{C}P^n$ and $\mathbb{R}P^n$ are compact connected smooth manifolds, and that $\mathbb{C}P^n$ is orientable.

Proof. a) We need to verify that the following diagram commutes:

$$
\begin{tikzcd}
S^{2n-1} \arrow[rrr] \arrow[dd, "\varphi_n"'] \arrow[dr, "\iota"] & & & S^{2n+1} \arrow[dl, "\iota"'] \arrow[dd, "\varphi_{n+1}"] \\
& \mathbb{R}^{2n} \cong \mathbb{C}^n \arrow[r] \arrow[dl, "\mathbb{P}"'] & \mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1} \arrow[dr, "\mathbb{P}"] & \\
\mathbb{C}P^{n-1} \arrow[rrr] & & & \mathbb{C}P^n
\end{tikzcd}
$$

The upper and lower trapezia in the diagram commute by definition. The left triangle commutes because both the attaching map φ_n and the projectivisation $\mathbb{P}$ are given by modulo S^1 action. Similarly the right triangle commutes. Hence the whole diagram commutes.

b) We have an isomorphism $\mathbb{D}^n / \langle x \sim -x \colon x \in \partial\mathbb{D}^n \rangle \cong S^n / \langle x \sim -x \colon x \in S^n \rangle =: \mathbb{R}P^n$ as follows. In $\mathbb{R}P^n$, the upper and lower hemisphere of S^n are identified. So we take the upper hemisphere $X \cong \mathbb{D}^n$. The equator ∂X is identified via the antipodal map. So we have the isomorphism as claimed above.

Let $P = \{x_i = 0\}$ be a hyperplane in $\mathbb{R}^n$. Under this identification, $\mathbb{R}P^n$ is $\mathbb{D}^n \subseteq \mathbb{R}^n$ with $x \sim -x$ on S^{n-1}. Note that $\mathbb{D}^n \cap P = \mathbb{D}^{n-1} \subseteq P$ and $S^{n-1} \cap P = S^{n-2} \subseteq P$. Hence $\mathbb{R}P^n \cap P = \mathbb{R}P^{n-1}$. May be consider $\mathbb{R}^n/\pm\mathrm{id}$ and look at subspace of that instead of $\mathbb{R}^n$

We note that $(\mathbb{R}P^n, \mathbb{R}P^{n-1})$ is a good pair, and $\mathbb{R}P^n/\mathbb{R}P^{n-1} \simeq S^n$. We have the long exact sequence

$$
\begin{tikzcd}
\cdots \arrow[r] & \widetilde{H}_k(\mathbb{R}P^{n-1}; \mathbb{Z}/2) \arrow[r] & \widetilde{H}_k(\mathbb{R}P^n; \mathbb{Z}/2) \arrow[r] & \widetilde{H}_k(S^n; \mathbb{Z}/2) \arrow[dll, "\delta_k"'] & \\
& \widetilde{H}_{k-1}(\mathbb{R}P^{n-1}; \mathbb{Z}/2) \arrow[r] & \widetilde{H}_{k-1}(\mathbb{R}P^n; \mathbb{Z}/2) \arrow[r] & \widetilde{H}_{k-1}(S^n; \mathbb{Z}/2) \arrow[r] & \cdots
\end{tikzcd}
$$

We note that

$$
\widetilde{H}_k(S^n; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & k = n \\ 0, & \text{otherwise} \end{cases}
$$

typo?

Hence for $k \neq n$ ($k \neq n+1$), the above exact sequence breaks into an isomorphism $\widetilde{H}_k(\mathbb{R}P^{n-1}; \mathbb{Z}/2) \cong \widetilde{H}_k(S^n; \mathbb{Z}/2)$. For $k \neq n$, every non-zero homology group involved is isomorphic to $\mathbb{Z}/2$, and there is exactly a unique way to make this sequence exact:

If worked at the level of $C_*^{CW}(\mathbb{R}P^n; \mathbb{Z}/2)$ then one can say $\mathbb{R}P^{n-1} \to \mathbb{R}P^n$ is actually a cell map and get the right isomorphisms.

this should be 0 if k≠n

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "0"] & \mathbb{Z}/2 \arrow[r, "\mathrm{id}"] & \mathbb{Z}/2 \arrow[dll, "0"'] & & \\
& \mathbb{Z}/2 \arrow[r, "\mathrm{id}"'] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Hence $H_{n-1}(\mathbb{R}P^{n-1};\mathbb{Z}/2) \cong H_n(\mathbb{R}P^n;\mathbb{Z}/2)$. not what you were supposed to show

For cohomology, we can prove that the inclusion induces isomorphisms $H^k(\mathbb{R}P^n;\mathbb{Z}/2) \to H^k(\mathbb{R}P^{k-1};\mathbb{Z}/2)$ for $k \neq n$.

For $\mathbb{C}P^n$, we have a similar result: the inclusion $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ induces isomorphisms $H_k(\mathbb{C}P^{n-1}) \to H_k(\mathbb{C}P^n)$ and $H^k(\mathbb{C}P^n) \to H^k(\mathbb{C}P^{n-1})$ for $k \neq 2n$. ✓

c) For $\mathbb{C}P^n$, let α be a generator for $H^2(\mathbb{C}P^n) \cong \mathbb{Z}$. We use induction on n to prove that $H^{2i}(\mathbb{C}P^n)$ is generated by α^i for all $i \leqslant n$.

I think the question suggested a different approach! good though

Suppose that the result holds for $\mathbb{C}P^{n-1}$. By (b), we have the isomorphisms of cohomology groups $H^{2i}(\mathbb{C}P^{n-1}) \cong H^{2i}(\mathbb{C}P^n)$ for $i < n$. So $H^{2i}(\mathbb{C}P^n)$ is generated by α^i for all $i < n$. For $i = n$, since $\mathbb{C}P^n$ is compact, connected, and orientable, by Poincaré duality, there exists $\beta \in\in H^{2n-2}(\mathbb{C}P^n)$ such that $\alpha \smile \beta$ generates $H^{2n}(\mathbb{C}P^n)$. By induction hypothesis $\beta = m\alpha^{n-1}$. Hence $\alpha \smile \beta = m\alpha^n$. We must have $m = \pm 1$. Thus α^n generates $H^{2n}(\mathbb{C}P^n)$. This completes the induction. In particular, we have $H^\bullet(\mathbb{C}P^n) \cong \mathbb{Z}[x]/\langle x^{n+1}\rangle$ with $|x| = 2$.

For $\mathbb{R}P^2$, we note that it is compact, connected, and $\mathbb{Z}/2$-orientable. We can apply the same method to obtain that $H^\bullet(\mathbb{R}P^n) \cong (\mathbb{Z}/2)[y]/\langle y^{n+1}\rangle$ with $|y| = 1$. ✓ □

$\alpha \smile$

Question 5

Let $\mathbb{C}P^\infty = \bigcup_{n\geqslant 0} \mathbb{C}P^n$, $S^\infty = \bigcup_{n\geqslant 0} S^n$, and $\mathbb{R}P^\infty = \bigcup_{n\geqslant 0} \mathbb{R}P^n$, using the natural inclusions from 4.(a).

a) Describe a CW-complex structure on these spaces and compute $H_\bullet$.

b) Compute $H_\bullet(\mathbb{R}P^\infty;\mathbb{Z}/2)$.

c) Describe the ring structure on their cohomologies (for $\mathbb{R}P^\infty$ work over $\mathbb{Z}/2$).

Proof. a) $\mathbb{C}P^\infty$, S^∞ and $\mathbb{R}P^\infty$ are infinite CW-complexes.

- For $X = \mathbb{C}P^\infty$, we have

$$X^0 = \mathrm{pt}, \quad X^{2n+1} = X^{2n} = \mathbb{C}P^n, \quad X^{2n} = X^{n-1} \cup_{\varphi_n} \mathbb{D}^{2n}, \text{ where } \varphi_n : S^{2n-1} \to \mathbb{C}P^{n-1} \cong S^{2n-1}/S^1$$ ✓

- For $X = S^\infty$, we have

$$X^0 = 2\text{ pts}, \quad X^n = (X^{n-1} \cup_{\varphi_n} \mathbb{D}^n) \cup_{\varphi_n} \mathbb{D}^n, \text{ where } \varphi_n = \mathrm{id} : S^{n-1} \to S^{n-1}$$ ✓

- For $X = \mathbb{R}P^\infty$, we have

$$X^0 = \mathrm{pt}, \quad X^n = X^{n-1} \cup_{\varphi_n} \mathbb{D}^n, \text{ where } \varphi_n : S^{n-1} \to \mathbb{R}P^{n-1} \cong S^{n-1}/(\mathbb{Z}/2)$$ ✓

$H_* = ?$

b) For each $n \in \mathbb{N}$, $\mathbb{R}P^\infty$ has exactly one n-cell. The cellular chain complex is given by

$$\cdots \longrightarrow \mathbb{Z}/2 \xrightarrow{\partial_4} \mathbb{Z}/2 \xrightarrow{\partial_3} \mathbb{Z}/2 \xrightarrow{\partial_2} \mathbb{Z}/2 \xrightarrow{\partial_1} \mathbb{Z}/2$$

For $n \in \mathbb{Z}_+$, $\partial_n : H_n(\mathbb{R}P^n, \mathbb{R}P^{n-1}) \to H_{n-1}(\mathbb{R}P^{n-1}, \mathbb{R}P^{n-2})$ is determined by the degree of the map

$$\partial\mathbb{D}^n = S^{n-1} \xrightarrow{\varphi_n} \mathbb{R}P^{n-1} \xrightarrow{q_n} \mathbb{R}P^{n-1}/\mathbb{R}P^{n-2} \cong S^{n-1}$$

which is $\deg \mathrm{id} + \deg(-\mathrm{id}) = 1 + (-1)^n$. Hence $\partial_n = 2$ for odd n and $\partial_n = 0$ for even n. Since we are using $\mathbb{Z}/2$-modules, $\partial_n = 0$ for all $n \in \mathbb{Z}_+$. The cellular chain complex is given by

$$\cdots \longrightarrow \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2$$

Hence the homology $\mathbb{Z}/2$-modules are given by

$$H_n(\mathbb{R}P^\infty; \mathbb{Z}/2) = \mathbb{Z}/2, \quad n \in \mathbb{N}$$

c) Note that the infinite-dimensional spaces are filtered colimits in Top: $\mathbb{C}P^\infty = \varinjlim_n \mathbb{C}P^n$, $S^\infty = \varinjlim_n S^n$, and $\mathbb{R}P^\infty = \varinjlim_n \mathbb{R}P^n$. The weak topology on the CW complexes implies that the the cochain complex $C^\bullet(\mathbb{C}P^\infty)$ is the filtered limit $\varprojlim_n C^\bullet(\mathbb{C}P^n)$ and similar for other spaces. We would like to invoke a general result:

Lemma 1. (Application 3.5.9 in *Weibel*)

Suppose that $\{X_k\}_{k\in\mathbb{N}}$ is an ascending chain of CW-complexes with $X = \varinjlim_k X_k = \bigcup_{k\in\mathbb{N}} X_k$. There is a short exact sequence

$$0 \longrightarrow \varprojlim\nolimits^1_k H^{n-1}(X_k; R) \longrightarrow H^n(X; R) \longrightarrow \varprojlim\nolimits_k H^n(X_k; R) \longrightarrow 0$$

where $\varprojlim^1$ is the first right derived functor of $\varprojlim$.

It's cool, but more elementary proofs can also be insightful! e.g. use naturality of cup product

In particular, $\varprojlim^1_k H^{n-1}(X_k; R) = 0$ and therefore $H^n(X; R) \cong \varprojlim_k H^n(X_k; R)$, provided that the following tower satisfies the **Mittag-Leffler condition**:

$$\cdots \longrightarrow H^{n-1}(X_{k+1}) \longrightarrow H^{n-1}(X_k) \longrightarrow H^{n-1}(X_{k-1}) \longrightarrow \cdots$$

be clearer what the condition is

We shall see that $X = \mathbb{C}P^\infty$, $\mathbb{R}P^\infty$ and S^∞ satisfies this condition trivially, because $H^{n-1}(X^k)$ is independent of n for $n > k$.

From Question 4 and from the lectures we have

$$H^k(\mathbb{C}P^n) \cong \mathbb{Z}, \qquad H^k(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2, \qquad H^k(S^n) \cong \begin{cases} \mathbb{Z}, & k = 0, n \\ 0, & \text{otherwise} \end{cases}$$

even k

Therefore we have

$$H^n(\mathbb{C}P^\infty) = \mathbb{Z}, \qquad H^n(\mathbb{R}P^\infty; \mathbb{Z}/2) = \mathbb{Z}/2, \qquad H^n(S^\infty) = 0 \quad (n > 0)$$

even n

The cohomology rings are given by

$$H^\bullet(\mathbb{C}P^\infty) \cong \mathbb{Z}[x], \qquad H^\bullet(\mathbb{R}P^\infty; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[x], \qquad H^\bullet(S^\infty) \cong \mathbb{Z} \qquad \square$$

$|x| = 2$ $\qquad$ $|x| = 1$ clarify grading of α

Question 6

Let $Y = X \cup_\varphi \mathbb{D}^m$ with the attaching map $\varphi : \partial\mathbb{D}^m \to X$. Prove that

$$H_\bullet(Y) \cong \begin{cases} H_\bullet(X) & \bullet \neq m-1, m \\ H_{m-1}(X)/\operatorname{im}\varphi_\bullet & \bullet = m-1 \\ H_m(X) \oplus \ker\varphi_\bullet & \bullet = m \end{cases}$$

(Hint. Consider $(Y, Y \setminus D)$ where $D \subseteq \mathbb{D}^m$ is a closed disc in the interior of $\mathbb{D}^m$.)

Proof. We note that (Y, X) is a good pair, because $Y \setminus D$ deformation retracts onto X. We have the long exact sequence:

$$\begin{tikzcd} \cdots \arrow[r] & \widetilde{H}_k(X) \arrow[r] & \widetilde{H}_k(Y) \arrow[r] & H_k(Y,X) \arrow[dll, "\delta_k"'] & \\ & \widetilde{H}_{k-1}(X) \arrow[r] & \widetilde{H}_{k-1}(Y) \arrow[r] & H_{k-1}(Y,X) \arrow[r] & \cdots \end{tikzcd}$$

We have

$$\widetilde{H}_k(Y,X) \cong \widetilde{H}_k(Y/X) \cong \widetilde{H}_k(S^m) = \begin{cases} \mathbb{Z}, & k = m \\ 0, & k \neq m \end{cases}$$

For $k \notin \{m, m-1\}$, we have the exact sequence

$$0 \longrightarrow \widetilde{H}_k(X) \longrightarrow \widetilde{H}_k(Y) \longrightarrow 0$$

which implies that $\widetilde{H}_k(X) \cong \widetilde{H}_k(Y)$. For $k \in \{m, m+1\}$, we have the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{H}_m(X) \arrow[r, "i_m"] & \widetilde{H}_m(Y) \arrow[r, "q_m"] & \widetilde{H}_m(Y/X) \arrow[dll, "\delta_m"'] \\ & \widetilde{H}_{m-1}(X) \arrow[r, "i_{m-1}"'] & \widetilde{H}_{m-1}(Y) \arrow[r] & 0 \end{tikzcd}$$

The connecting map $\delta_m : \widetilde{H}_m(Y/X) \cong \mathbb{Z} \to \widetilde{H}_{m-1}(X)$ is exactly the push-out $\varphi_{m-1} : \widetilde{H}_{m-1}(\partial\mathbb{D}^m) \cong \mathbb{Z} \to \widetilde{H}_{m-1}(X)$ of the attaching map $\varphi : \mathbb{D}^n \to X$. To prove this, we consider the map of the paired spaces: $(\mathbb{D}^m, \partial\mathbb{D}^m) \xrightarrow{\varphi} (Y,X)$ φ induces a map between the corresponding long exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & \widetilde{H}_m(\mathbb{D}^m) \arrow[r] \arrow[d, "0"] & H_m(\mathbb{D}^m, \partial\mathbb{D}^m) \arrow[r, "\delta'_m"] \arrow[d, "\varphi'_m"] & \widetilde{H}_{m-1}(\partial\mathbb{D}^m) \arrow[r] \arrow[d, "\varphi_{m-1}"] & \widetilde{H}_{m-1}(\mathbb{D}^m) \arrow[r] \arrow[d, "0"] & \cdots \\ \cdots \arrow[r] & \widetilde{H}_m(Y) \arrow[r] & H_m(Y,X) \arrow[r, "\delta_m"] & \widetilde{H}_{m-1}(X) \arrow[r] & \widetilde{H}_{m-1}(Y) \arrow[r] & \cdots \end{tikzcd}$$

Since $\widetilde{H}_m(\mathbb{D}^m) = 0$ and $\widetilde{H}_{m-1}(\mathbb{D}^m) = 0$, $\delta'_m : H_m(\mathbb{D}^m, \partial\mathbb{D}^m) \to H_{m-1}(\partial\mathbb{D}^m)$ is an isomorphism. By excision theorem, $\varphi'_m : H_m(\mathbb{D}^m, \partial\mathbb{D}^m) \to H_m(Y,X)$ is also an isomorphism. Hence $\varphi_{m-1} = \delta_m$ if we identify $H_m(Y,X) \cong \mathbb{Z}$ and $\widetilde{H}_{m-1}(\partial\mathbb{D}^m) \cong \mathbb{Z}$ by the corresponding isomorphisms.

Therefore $\widetilde{H}_{m-1}(Y) \cong \dfrac{\widetilde{H}_{m-1}(X)}{\ker i_{m-1}} = \dfrac{\widetilde{H}_{m-1}(X)}{\operatorname{im}\delta_m} = \dfrac{\widetilde{H}_{m-1}(X)}{\operatorname{im}\varphi_{m-1}}$.

For $\widetilde{H}_m(Y)$, we break the long exact sequence into short exact sequences. We have

$$0 \longrightarrow \widetilde{H}_m(X) \xrightarrow{i_m} \widetilde{H}_m(Y) \xrightarrow{q_m} \ker\delta_n \longrightarrow 0$$

Note that $\ker\delta_n \cong \ker\varphi_{m-1}$ is a submodule of the free $\mathbb{Z}$-module $\mathbb{Z}$. Since $\mathbb{Z}$ is a principal ideal domain, $\ker\varphi_{m-1}$ is also a free $\mathbb{Z}$-module, and hence is projective. This implies that the short exact sequence above splits. We have $\widetilde{H}_m(Y) \cong \widetilde{H}_m(X) \oplus \ker\varphi_{m-1}$. In conclusion:

$$H_k(Y) = \begin{cases} H_m(X) \oplus \ker\varphi_{m-1}, & k = m \\ H_{m-1}(X)/\operatorname{im}\varphi_{m-1}, & k = m-1 \\ H_k(X), & \text{otherwise} \end{cases} \qquad \square$$

Question 7

a) Prove that if each $x_i \in X_i$ has a contractible neighbourhood, then

$$H^\bullet\left(\bigvee_i X_i\right) \cong \prod_i H^\bullet(X_i), \quad \bullet \geqslant 1$$

is an isomorphism of rings.

b) Show that $S^1 \vee S^1 \vee S^2$ and T^2 have the same homology, but different cohomology rings.

Proof. a) *We have to assume that the index set is finite.*

First we prove that

$$\widetilde{H}^n\left(\bigvee_{i=1}^n X_i\right) \cong \bigoplus_{i=1}^n \widetilde{H}^n(X_i)$$

as $\mathbb{Z}$-modules. The proof is essentially the same as that of Question 4 of Sheet 2. We use the Mayer-Vietoris sequence for cohomology.

Let Y_1, Y_2 be contractible neighbourhoods of x_1, x_2 in X_1, X_2 respectively. With abuse of notation we may set $A := X_1 \vee Y_2$ and $B := X_2 \vee Y_1$, so that $A \cup B = X_1 \vee X_2$ and $A \cap B = Y_1 \vee Y_2$. The Mayor-Vietoris sequence is given by

$$\cdots \to H^{n-1}(Y_1 \vee Y_2) \to H^n(X_1 \vee X_2) \to H^n(X_1 \vee Y_2) \oplus H^n(X_2 \vee Y_1) \to H^n(Y_1 \vee Y_2) \to \cdots$$

If you use excision theorem then can work with infinite wedgesum. Need only $H^*(\bigvee X_i) \cong \prod_i H^*(X_i)$ (extra sheet2)

Since $Y_1 \vee Y_2$ is contractible, $\widetilde{H}^n(Y_1 \vee Y_2) = 0$ for all $n \in \mathbb{N}$. We have

$$H^n(X_1 \vee X_2) \cong H^n(X_1 \vee Y_2) \oplus H^n(X_2 \vee Y_1) \cong H^n(X_1) \oplus H^n(X_2)$$

Inductively we have

$$\widetilde{H}^n\left(\bigvee_{i=1}^n X_i\right) \cong \bigoplus_{i=1}^n \widetilde{H}^n(X_i)$$

Next we consider the cup product on these two groups. For each i, the inclusion of spaces $\iota_i : X_i \hookrightarrow \bigvee_{i=1}^n X_i$ induces the ring homomorphism $\iota_i^* : \widetilde{H}^\bullet(\bigvee_{i=1}^n X_i) \to \widetilde{H}^\bullet(X_i)$ by the naturality of the cup product. Therefore we have a ring homomorphism

$$\prod_{i=1}^n \iota_i^* : \widetilde{H}^\bullet\left(\bigvee_{i=1}^n X_i\right) \to \prod_{i=1}^n \widetilde{H}^\bullet(X_i)$$

It is an ring isomorphism because it is bijective as a group homomorphism.

b) In Question 4.(c) of Sheet 2 we have proven that $S^1 \vee S^1 \vee S^2$ and T^2 has the same homology groups.

By (a), we have the ring isomorphism

$$\widetilde{H}^\bullet(S^1 \vee S^1 \vee S^2) \cong \widetilde{H}^\bullet(S^1) \times \widetilde{H}^\bullet(S^1) \times \widetilde{H}^\bullet(S^2)$$

Suppose that $f : \widetilde{H}^\bullet(S^1) \times \widetilde{H}^\bullet(S^1) \times \widetilde{H}^\bullet(S^2) \to \widetilde{H}^\bullet(T^2)$ is a graded ring isomorphism.

Let a, b be the generators of the two $\widetilde{H}^1(S^1) \cong \mathbb{Z}$ respectively. Then $a \smile b = b \smile a$. On the other hand, $f(a), f(b) \in \widetilde{H}^1(T^2)$. Therefore $f(a) \smile f(b) = (-1)^{1\cdot 1} f(b) \smile f(a) = -f(b) \smile f(a)$. Hence we must have $f(a) \smile f(b) = 0$. But in the lectures we have known that $f(a) \smile f(b)$ is a generator of $\widetilde{H}^2(T^2)$. This is a contradiction. Hence the cohomology graded rings of $S^1 \vee S^1 \vee S^2$ and T^2 are not isomorphic. □

Question 8

a) Let X be the **Moore space** $M(\mathbb{Z}/m, n) = S^n \cup_\varphi \mathbb{D}^{n+1}$, where the attaching map $\varphi : \partial\mathbb{D}^{n+1} = S^n \to S^n$ has degree m. Compute $H_\bullet^{\mathrm{CW}}(X)$ and $H^\bullet_{\mathrm{CW}}(X)$.

b) Let $Y = \mathbb{C}P^2 \cup_\varphi \mathbb{D}^3$, where the attaching map $\varphi : \partial\mathbb{D}^3 = S^2 \to S^2 \cong \mathbb{C}P^1 \subseteq \mathbb{C}P^2$ has degree p. Compute $H^\bullet_{\mathrm{CW}}(Y)$.

c) For $X = M(\mathbb{Z}/p, 2)$, show that $H^\bullet(Y) \cong H^\bullet(X \vee S^4)$ as rings but $H^\bullet(Y; \mathbb{Z}/p) \ncong H^\bullet(X \vee S^4; \mathbb{Z}/p)$.

Proof. a) In Question 2, we have shown that the cellular chain complex is given by

$$
\begin{tikzcd}
0 \arrow[r] & C^{\mathrm{CW}}_{n+1}(X) \arrow[r] \arrow[d, "\simeq"] & C^{\mathrm{CW}}_{n}(X) \arrow[r] \arrow[d, "\simeq"] & 0 \\
& \mathbb{Z} \arrow[r, "\deg\varphi"] & \mathbb{Z} &
\end{tikzcd}
$$

and the homology groups are given by

$$
H_k^{\mathrm{CW}}(M(\mathbb{Z}/m, n)) = \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z}/m, & k = n \\ 0, & \text{otherwise} \end{cases}
$$

The cellular cochain complex is obtained by dualising the cellular chain complex:

$$
\begin{tikzcd}
0 \arrow[r] & C^{n}_{\mathrm{CW}}(X) \arrow[r] \arrow[d, "\simeq"] & C^{n+1}_{\mathrm{CW}}(X) \arrow[r] \arrow[d, "\simeq"] & 0 \\
& \mathbb{Z} \arrow[r, "\deg\varphi"] & \mathbb{Z} &
\end{tikzcd}
$$

Therefore the cohomology groups are given by

$$
H^k_{\mathrm{CW}}(M(\mathbb{Z}/m, n)) = \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z}/m, & k = n+1 \\ 0, & \text{otherwise} \end{cases}
$$

b) We note that Y is a CW-complex with

$$
Y^0 = Y^1 = \mathrm{pt}, \qquad Y^2 = \mathbb{C}P^1, \qquad Y^3 = \mathbb{C}P^1 \cup_\varphi \mathbb{D}^3, \qquad Y = Y^4 = Y^3 \cup_\psi \mathbb{D}^4
$$

We shall calculate the cellcular chain complex. It is obvious that

$$
H_1(Y_1, Y_0) = 0, \qquad H_2(Y^2, Y^1) = \tilde{H}_2(\mathbb{C}P^1) \cong \mathbb{Z}, \qquad H_3(Y^3, Y^2) \cong H_3(S^3) \cong \mathbb{Z}, \qquad H_4(Y, Y^3) \cong H_4(S^4) \cong \mathbb{Z}
$$

Using the result of Question 6, we can patch the

$$
\begin{tikzcd}
\cdots \arrow[r] & H_4(Y) \arrow[r, "q_4"] & H_4(Y, Y^3) \arrow[r, "\psi_*"] & \tilde{H}_3(Y^3) \arrow[r, "i_3"] \arrow[d, equal] & \tilde{H}_3(Y) \arrow[r] & \cdots & \\
& \cdots \arrow[r] & 0 \arrow[r] & \tilde{H}_3(Y^3) \arrow[r, "q_3"] & H_3(Y^3, \mathbb{C}P^1) \arrow[r, "\varphi_*"] & \tilde{H}_2(\mathbb{C}P^1) \arrow[r] & \cdots
\end{tikzcd}
$$

which gives the cellular chain complex

$$
H_4(Y, Y^3) \xrightarrow{\deg\psi} H_3(Y^3, \mathbb{C}P^1) \xrightarrow{\deg\varphi} \tilde{H}_2(\mathbb{C}P^1) \longrightarrow 0 \longrightarrow \mathbb{Z}
$$

We know that $\deg\varphi = p$. Note that $\psi : \partial\mathbb{D}^4 = S^3 \to \mathbb{C}P^1 = S^2 \subseteq S^3$ is not surjective onto S^3. So $\deg\psi = 0$. So we have:

$$
\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z}
$$

Taking the dual, we obtain the cellular cochain complex:

$$
\mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \xrightarrow{0} \mathbb{Z}
$$

Taking the cohomology:

$$
H^k_{\mathrm{CW}}(Y) = \begin{cases} \mathbb{Z}, & k = 0, 4 \\ \mathbb{Z}/p, & k = 3 \\ 0, & \text{otherwise} \end{cases}
$$

c) We consider the cup product structure on $H^\bullet(Y)$. Let α be a generator for $H^4(Y)$ and β be a generator for

$H^3(Y)$. Then $\alpha \smile \alpha = 0$, $\beta \smile \beta = 0$, and $\alpha \smile \beta = 0$ for degree reason. Therefore the cohomology ring is given by

$$H^\bullet(Y) \cong \frac{\mathbb{Z}[x,y]}{\langle x^2, y^2, xy, py\rangle}$$

just say trivial ring structure apart from H^0 rescaling (for both

On the other hand, for $H^\bullet(X \vee S^4)$, we have the ring isomorphism

$$H^\bullet(X \vee S^4) \cong H^\bullet(X) \times H^\bullet(S^4) \cong \frac{\mathbb{Z}[x]}{\langle x^2\rangle} \times \frac{\mathbb{Z}[x]}{\langle y^2, py\rangle}$$

Finally, we note that there exists a ring isomorphism

$$f\colon \frac{\mathbb{Z}[x,y]}{\langle x^2, y^2, xy, py\rangle} \to \frac{\mathbb{Z}[x]}{\langle x^2\rangle} \times \frac{\mathbb{Z}[x]}{\langle y^2, py\rangle}, \qquad 1 \mapsto (1,1), \quad x \mapsto (x,0), \quad y \mapsto (0,y)$$

Hence $H^\bullet(Y) \cong H^\bullet(X \vee S^3)$ as cohomology rings.

Next we consider the cohomologies in $\mathbb{Z}/p$-coefficients.

For Y, we have the cochain complex

$$\mathbb{Z}/p \longrightarrow 0 \longrightarrow \mathbb{Z}/p \xrightarrow{\cdot p} \mathbb{Z}/p \xrightarrow{0} \mathbb{Z}/p$$

Taking the cohomology:

$$H^k(Y;\mathbb{Z}/p) = \begin{cases} \mathbb{Z}/p, & k = 0,2,3,4 \\ 0, & \text{otherwise} \end{cases}$$

For X, we have the cochain complex

$$\mathbb{Z}/p \longrightarrow 0 \longrightarrow \mathbb{Z}/p \xrightarrow{\cdot p} \mathbb{Z}/p \longrightarrow 0$$

Taking the cohomology:

$$H^k(X;\mathbb{Z}/p) = \begin{cases} \mathbb{Z}/p, & k = 0,2,3 \\ 0, & \text{otherwise} \end{cases}$$

For S^4, we have

$$H^k(S^4;\mathbb{Z}/p) = \begin{cases} \mathbb{Z}/p, & k = 0,4 \\ 0, & \text{otherwise} \end{cases}$$

good but justify

Let α be a generator of $H^2(Y;\mathbb{Z}/p)$. Suppose that $f : H^\bullet(Y;\mathbb{Z}/p) \to H^\bullet(X \vee S^3;\mathbb{Z}/p)$ is a graded ring isomorphism. Then $f(\alpha)$ is a generator of $H^2(X;\mathbb{Z}/p)$. For degree reason $f(\alpha) \smile f(\alpha) = 0$. But since Y is a compact, connected, orientable 4-dimensional manifold, by Poincaré duality, $\alpha \smile \alpha \neq 0 \in H^4(Y;\mathbb{Z}/p)$. This is contradictory. Hence $H^\bullet(Y;\mathbb{Z}/p) \not\cong H^\bullet(X \vee S^3;\mathbb{Z}/p)$ as graded rings. □

Question 9

Compute directly the cup product structure on $H^\bullet(K)$ and $H^\bullet(K;\mathbb{Z}/2)$, where K is the Klein bottle.

(Do not use the intersection theory, only use CW-complexes and the definition of $\smile$.)

Proof. To compute the cup products it is easier to use Δ-complexes rather than CW-complexes. From Question 6 of Sheet 1, the simplicial chain complex of K is given by

not just easier, it's not intuitive how to even define $\cup$ on CW-cx

$$0 \longrightarrow \mathbb{Z}U \oplus \mathbb{Z}L \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}} \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c \xrightarrow{0} \mathbb{Z}v \longrightarrow 0$$

Taking the dual we obtain the simplicial cochain complex

$$0 \longrightarrow \mathbb{Z}v^\vee \xrightarrow{0} \mathbb{Z}a^\vee \oplus \mathbb{Z}b^\vee \oplus \mathbb{Z}c^\vee \xrightarrow{\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}} \mathbb{Z}U^\vee \oplus \mathbb{Z}L^\vee \longrightarrow 0$$

Taking the cohomology, we have

	$H^n_\Delta(K)$	Generators
$n=0$	$\mathbb{Z}$	$v^\vee$
$n=1$	$\mathbb{Z}$	$\tilde{a}^\vee := (b^\vee + c^\vee)$
$n=2$	$\mathbb{Z}/2$	$U^\vee$

We note that $H^\bullet(K)$ has an obviously unique cup product structure, where $v^\vee$ is the identity, $\tilde{a}^\vee \smile \tilde{a}^\vee = 0$ (by graded commutativity), $U^\vee \smile U^\vee = 0$, $\tilde{a}^\vee \smile U^\vee = 0$ (by the degree). In summary, we have

$$H^\bullet(K) \cong \frac{\mathbb{Z}[x,y]}{\langle x^2, y^2, xy, 2y\rangle}$$

specify grading

(just say trivial cup product)

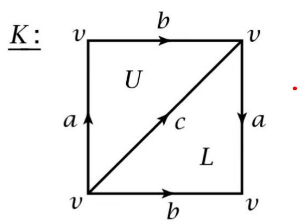
For $\mathbb{Z}/2$-coefficients, we have the cochain complex[1]

$$0 \longrightarrow \mathbb{Z}_2 v^\vee \xrightarrow{0} \mathbb{Z}_2 a^\vee \oplus \mathbb{Z}_2 b^\vee \oplus \mathbb{Z}_2 c^\vee \xrightarrow{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}} \mathbb{Z}_2 U^\vee \oplus \mathbb{Z}_2 L^\vee \longrightarrow 0$$

Taking the cohomology, we have

	$H^n_\Delta(K;\mathbb{Z}/2)$	Generators
$n=0$	$\mathbb{Z}/2$	$v^\vee$
$n=1$	$(\mathbb{Z}/2)^2$	$A := a^\vee - b^\vee$, $B := b^\vee - c^\vee$
$n=2$	$\mathbb{Z}/2$	$U^\vee$

To determine the cup product structure on $H^\bullet(K;\mathbb{Z}/2)$ it suffices to compute $A \smile A$, $B \smile B$, and $A \smile B$. From the diagram:



We have

$$(A \smile B)(U) = A(U|_{[e_0,e_1]})B(U|_{[e_1,e_2]}) = (a^\vee - b^\vee)(a)(b^\vee - c^\vee)(b) = 1$$
$$(A \smile B)(L) = A(L|_{[e_0,e_1]})B(L|_{[e_1,e_2]}) = (a^\vee - b^\vee)(b)(b^\vee - c^\vee)(-a) = 0$$
$$(A \smile A)(U) = A(U|_{[e_0,e_1]})A(U|_{[e_1,e_2]}) = (a^\vee - b^\vee)(a)(a^\vee - b^\vee)(b) = 1$$
$$(A \smile A)(L) = A(L|_{[e_0,e_1]})A(L|_{[e_1,e_2]}) = (a^\vee - b^\vee)(b)(a^\vee - b^\vee)(-a) = 1$$
$$(B \smile B)(U) = B(U|_{[e_0,e_1]})B(U|_{[e_1,e_2]}) = (b^\vee - c^\vee)(a)(b^\vee - c^\vee)(b) = 0$$
$$(B \smile B)(L) = B(L|_{[e_0,e_1]})B(L|_{[e_1,e_2]}) = (b^\vee - c^\vee)(b)(b^\vee - c^\vee)(-a) = 0$$

Hence in the cohomology $H^2(K;\mathbb{Z}/2)$ we have

$$A \smile B = U^\vee, \qquad A \smile A = U^\vee + L^\vee = \cancel{0}, \qquad B \smile B = 0$$

In summary, we have

$$H^\bullet(K;\mathbb{Z}/2) \cong \frac{(\mathbb{Z}/2)[x,y,z]}{\langle x^2, y^2, xy - z, z^2\rangle} \cong \frac{(\mathbb{Z}/2)[x,y]}{\langle x^2, y^2\rangle} \qquad \square$$

specify grading

also not quite right

α -

[1] For typographical reason we use $\mathbb{Z}_2$ to denote $\mathbb{Z}/2$.

Question 10

Let $I = [0,1]$. Build orientation-preserving homeomorphisms of pairs

$$(\mathbb{D}^n, S^{n-1}) \cong (I^n, \partial I^n) \cong (I^\ell \times I^k, \partial I^\ell \times I^k \cup I^\ell \times \partial I^k) \cong (\mathbb{D}^\ell \times \mathbb{D}^k, \partial\mathbb{D}^\ell \times \mathbb{D}^k \cup \mathbb{D}^\ell \times \partial\mathbb{D}^k)$$

where $\ell + k = n$.

Proof. The canonical isomorphism $\sigma\colon \mathbb{R}^n \to \mathbb{R}^\ell \times \mathbb{R}^k$ restricts to an homeomorphism $\sigma'\colon I^n \to I^\ell \times I^k$, which is obviously orientation-preserving. σ maps the boundary ∂I^n to $\partial(I^\ell \times I^k) = \partial I^\ell \times I^k \cup I^\ell \times \partial I^k$.

On the other hand, for each $m \geqslant 1$, we have a orientation-preserving homeomorphism $\varphi_m : \mathbb{D}^m \to I^m$ given by

$$\varphi_m(x) = \begin{cases} 0, & x = 0 \\ \dfrac{\|x\|_2}{\|x\|_1} x, & x \neq 0 \end{cases}$$

with $\varphi_m(\partial\mathbb{D}^m) = \partial I^m$. Since the norms $\|-\|_1$ and $\|-\|_2$ are equivalent, φ_m is a homeomorphism. Now, $(\varphi_\ell^{-1}, \varphi_k^{-1}) \circ \sigma' \circ \varphi_n$ is an orientation-preserving homeomorphism from the pair $(\mathbb{D}^n, S^{n-1})$ to $(\mathbb{D}^\ell \times \mathbb{D}^k, \partial\mathbb{D}^\ell \times \mathbb{D}^k \cup \mathbb{D}^\ell \times \partial\mathbb{D}^k)$. $\square$