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Problem Sheet 2

String Theory II

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Question 1. Torus-Partition Function and Modular Invariance: Free Boson

1. Fundamental Domain D of the torus: Let $\tau \in \mathbb{C}$ be the modular parameter (modulus) of the torus $T^2_\tau = \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$. Show that the torus T^2_τ and $T^2_{\gamma\tau}$, where $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, i.e.

$$\gamma\tau = \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1$$

describe the same space, i.e. the same identifications in $\mathbb{C}$. Using these $\mathrm{SL}_2(\mathbb{Z})$ transformations one can restrict τ to the fundamental domain

$$D: \quad -\frac{1}{2} \leqslant \mathrm{Re}(\tau) \leqslant \frac{1}{2}, \quad |\tau| \geqslant 1$$

Sketch D and determine the images of D under

$$\begin{aligned} T: &\quad \tau \to \tau + 1 \\ S: &\quad \tau \to -\tfrac{1}{\tau} \end{aligned}$$

2. For a single free boson $X^\mu(z, \overline{z})$ in d dimensions, the torus partition function is

$$Z(\tau) = \mathrm{Tr}\, q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}}$$

where $q = e^{2\pi i \tau}$. By performing the integral over momenta k and performing the sum over oscillator modes evaluate $Z(\tau)$ and show that it is modular invariant, i.e. $Z(\tau) = Z(\gamma\tau)$, with $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. You may use that $\eta(\tau + 1) = e^{i\pi/12}\eta(\tau)$ and $\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$. What is the relevance of this result?

Proof. 1. Pick $w_1, w_2 \in \mathbb{C}$ with $\frac{w_1}{w_2} \notin \mathbb{R}$

The lattice $\Lambda(w_1, w_2) := \{a w_1 + b w_2 : a, b \in \mathbb{Z}\}$

Tori $\mathbb{C}/\Lambda(w_1, w_2) \simeq \mathbb{C}/\Lambda(w_1', w_2') \Longleftrightarrow \Lambda(w_1, w_2) = \Lambda(w_1', w_2')$

If $\gamma \in SL_2(\mathbb{Z})$, then

This exhausts the conformal symmetry ?

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \Rightarrow w_1', w_2' \in \Lambda(w_1, w_2)$$

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} w_1' \\ w_2' \end{pmatrix} \Rightarrow w_1, w_2 \in \Lambda(w_1', w_2')$$

$\Rightarrow \Lambda(w_1, w_2) = \Lambda(w_1', w_2')$

Conversely, if γ preserves the lattices, then

$\det \gamma = ab - bc = 1 \Rightarrow \gamma \in SL_2(\mathbb{Z})$

Let $\tau := \frac{w_1}{w_2}$, then $\gamma(\tau) = \frac{a\tau + b}{c\tau + d}$

This shows that the modular group is $SL_2(\mathbb{Z})$.

2. $Z(\tau) = \mathrm{Tr}\left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}\right)$

where $L_0 = \frac{1}{2}\alpha_0 \cdot \alpha_0 + \underbrace{\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n}_{\text{Number operator } N}$

$L_0 |m, k\rangle = \alpha' k^2 + m |m, k\rangle$

$\mathrm{Tr}\, q^{L_0} = \prod_{n=1}^{\infty} \frac{1}{1 - q^n}$

$Z(\tau) = (q\bar{q})^{-1/24} \prod_{n=1}^{\infty} \frac{1}{1 - q^n} \prod_{m=1}^{\infty} \frac{1}{1 - \bar{q}^m} \int \frac{dk}{2\pi} (q\bar{q})^{k^2/2}$

$\propto \frac{1}{|\eta(\tau)|^2} \cdot \frac{1}{|\mathrm{Im}\,\tau|}$

η-function: $\eta(\tau + 1) = e^{i\pi/12}\eta(\tau)$,

$\eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\,\eta(\tau)$.

Modular invariance ✓

(Should work in covariant quantisation & include the ghost sector)

□

Question 2. Torus-Partition Function: RNS String

The one-loop or torus partition function for the closed RNS string is

$$Z_{T^2} = V_{10} \int_D \frac{\mathrm{d}^2\tau}{2\tau_2} \int \frac{\mathrm{d}^{10}k}{(2\pi)^{10}} \mathrm{Tr}_{\mathcal{H}_k} (-1)^{\mathcal{F}} q^{\alpha'(k^2+M^2)/4} \bar{q}^{\alpha'(k^2+\widetilde{M}^2)/4},$$

where $\mathcal{H}_k$ is the physical state (including GSO-projection) space with momentum k ground state, and $q = e^{2\pi i\tau}$, where τ is again the modular parameter (modulus) of the torus $T^2_\tau = \mathbb{C}/\mathbb{Z}\oplus\tau\mathbb{Z}$ and D the fundamental domain. The spacetime Fermion number operator is $\mathcal{F}$ (not to be confused with the worldsheet one F).

1. Evaluate the NS-sector and R-sector partition functions.
2. Using the results on theta-functions from the lecture, show that this partition function vanishes.

Proof. 1. (GSO-projection : See BLT §9.1) □

$$Z_{RNS} = V_{10} \int_D \frac{d^2\tau}{2\tau_2} \int \frac{d^{10}k}{(2\pi)^{10}} \mathrm{Tr}_{\mathcal{H}_k} (-1)^{\mathcal{F}} q^{\alpha'(k^2+M^2)/4} \bar{q}^{\alpha'(k^2+\widetilde{M}^2)/4}$$

GSO projector : $\mathbb{P}^{GSO}_{NS} = \frac{1}{2}(1+(-1)^{\mathcal{F}})$

$\mathbb{P}^{GSO}_{R} = \frac{1}{2}(1+\eta(-1)^{\mathcal{F}})$ $\qquad \mathcal{F}$: chirality

States :

NS $\qquad \alpha^{i_1}_{-n_1} \cdots \alpha^{i_N}_{-n_N} b^{j_1}_{-r_1} \cdots b^{j_M}_{-r_M} |0\rangle_{NS}$

R $\qquad \alpha^{i_1}_{-n_1} \cdots \alpha^{i_N}_{-n_N} b^{j_1}_{-r_1} \cdots b^{j_M}_{-r_M} |0\rangle_{R}$ (or $|\dot{0}\rangle_R$)

Mass : $M^2 = \frac{2}{\alpha'}(N^{(\alpha)}_{tr} + N^{(b)}_{tr} + a) = \frac{2}{\alpha'}(\widetilde{N}^{(\alpha)}_{tr} + \widetilde{N}^{(b)}_{tr} + a)$

↳ transverse

$a_{NS} = -\frac{1}{2}, \quad a_R = 0$

R-sector :

$$\underbrace{\tfrac{1}{2}(-1)}_{(-1)^{\mathcal{F}}} \underbrace{(8+8)}_{\text{chirality degenercy}} \sum_n \sum_m \sum_r \sum_s \langle m,n,r,s| q^{N^{(\alpha)}_{tr}} \bar{q}^{\widetilde{N}^{(\alpha)}_{tr}} q^{N^{(b)}_{tr}} \bar{q}^{\widetilde{N}^{(b)}_{tr}} |m,n,r,s\rangle$$

$\alpha \quad \tilde{\alpha} \quad b \quad \tilde{b}$

• Bosons : $\sum_{m=0}^{\infty} q^{\sum_n \alpha_{-n}\cdot\alpha_n} = \sum_m d(m) q^m = \prod_{n=0}^{\infty} \frac{1}{1-q^n}$

• Fermions : $\sum_{m=0}^{1} \prod_r q^{b_{-r}\cdot b_r} = \prod_r (1-q^r)$

$$\Rightarrow \quad -8 \prod_i \left(\frac{1+q^i}{1-q^i}\right)^8 \prod_j \left(\frac{1+\bar{q}^j}{1-\bar{q}^j}\right)^8 \quad \text{(only contribution from R)}$$

NS-sector : No degenercy in vacuum

$$\frac{1}{2} \prod_i \left(\frac{1+q^i}{1-q^i}\right)^8 \prod_j \left(\frac{1+\bar{q}^j}{1-\bar{q}^j}\right)^8 \cdot \frac{1}{q^{1/2}} \frac{1}{\bar{q}^{1/2}} - \frac{1}{2} \prod_i \left(\frac{1-q^i}{1-q^i}\right)^8 \prod_j \left(\frac{1-\bar{q}^j}{1-\bar{q}^j}\right)^8 \cdot \frac{1}{q^{1/2}} \frac{1}{\bar{q}^{1/2}}$$

$\frac{1}{2}(\underline{1} + \cdots) \qquad \frac{1}{2}(\cdots + \underline{(-1)^{\mathcal{F}}})$

Question 3. Ghosts!

Let b and c be anticommuting fields, β and γ commuting fields, with action

$$S = \frac{1}{2\pi} \int \mathrm{d}^2 z (b\bar{\partial} c + \beta \bar{\partial} \gamma).$$

The OPE algebra is

$$b(z)c(0) \sim \frac{1}{z}, \quad c(z)b(0) \sim \frac{1}{z}, \quad \beta(z)\gamma(0) \sim -\frac{1}{z}, \quad \gamma(z)\beta(0) \sim \frac{1}{z}.$$

Define

$$\begin{aligned} T_{\text{ghost}}(z) &= (\partial b)c - \lambda \partial(bc) + (\partial \beta)\gamma - \frac{1}{2}(2\lambda - 1)\partial(\beta\gamma) \\ J_{\text{ghost}}(z) &= -\frac{1}{2}(\partial\beta)c + \frac{2\lambda - 1}{2}\partial(\beta c) - 2b\gamma \end{aligned}$$

1. Compute the conformal weights of b and c, β and γ.
2. Furthermore compute the OPE of TT and determine the central charge.
3. This system of ghosts can be used in the Faddeev-Popov gauge-fixing for $\lambda = 2$. Check that for this value the total central charge of T_{ghost} and T_{RNS} vanishes for $d = 10$.

Proof. 1.

□