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Problem Sheet 4

C7.6: General Relativity II

10 March, 2022

Section A: Introductory

Question 1. de Sitter spacetime

Consider (4+1)-dimensional Minkowski spacetime, i.e. $\mathbb{R}^5$ with standard Cartesian coordinates $\{v, w, x, y, z\}$ and metric $\eta = -\mathrm{d}v^2 + \mathrm{d}w^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$. For $\alpha > 0$, let $M \subseteq \mathbb{R}^5$ denote the level set

$$-v^2 + w^2 + x^2 + y^2 + z^2 = \alpha^2$$

Check that this is a timelike hypersurface. Can you sketch it (suppressing some dimensions)?

Section B: Core

Question 2. de Sitter spacetime

This is a continuation of problem 1 above, which considers a level set $-v^2 + w^2 + x^2 + y^2 + z^2 = \alpha^2 \neq 0$ as a hypersurface embedded in the five-dimensional Minkowski space with standard Cartesian coordinates $\{v, w, x, y, z\}$.

Question 2.(a)

By restricting the Minkowski metric to the tangent spaces of M we obtain a Lorentzian metric g on M. In fact, the Ricci curvature of the Lorentzian metric g on M satisfies $R_{ab} = \frac{3}{\alpha^2} g_{ab}$. The Einstein equations with cosmological constant Λ read $R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$. Therefore, (M, g) is a solution to the Einstein equations with cosmological constant $\Lambda = \frac{3}{\alpha^2}$ and $T_{ab} = 0$. It is called the **de Sitter** spacetime.

We now introduce coordinates on M by $(t, \chi, \theta, \varphi) \overset{\iota}{\mapsto} (v, w, x, y, z)$ with

$$\begin{aligned}
v &= \alpha \sinh(t/\alpha) \\
w &= \alpha \cosh(t/\alpha) \cos\chi \\
x &= \alpha \cosh(t/\alpha) \sin\chi \cos\theta \\
y &= \alpha \cosh(t/\alpha) \sin\chi \sin\theta \cos\varphi \\
z &= \alpha \cosh(t/\alpha) \sin\chi \sin\theta \sin\varphi
\end{aligned}$$

What is the range of these coordinates? Do they cover all of M?

Proof. The range of the coordinates are $t \in \mathbb{R}$, $\chi, \theta \in (0, \pi)$, and $\varphi \in (0, 2\pi)$. It covers a generic set of M and can be extended continuously to all of M. □

Question 2.(b)

Show that in these coordinates the metric g is given by

$$g = -\mathrm{d}t^2 + \alpha^2 \cosh^2(t/\alpha) \left(\mathrm{d}\chi^2 + \sin^2\chi \left[\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right]\right)$$

Draw the hypersurfaces of constant t in your above sketch. What is their topology, how does their geometry change with coordinate time t?

Proof. As a hypersurface of the Minkowski space $(\mathbb{R}^5, \eta)$, M has pullback metric g given by

$$g_{\mu\nu} = \eta(\partial_\mu, \partial_\nu) = \eta_{ab} \frac{\partial x^a}{\partial y^\mu} \frac{\partial x^b}{\partial y^\nu}$$

where $(x^a) = (v, w, x, y, z)$ and $(y^\mu) = (t, \chi, \theta, \varphi)$. From the equations we know, we can compute the metric by brute force. Alternatively, note that $\{\Sigma_t \subseteq M : t = \text{const}\}$ is a family of spacelike hypersurfaces which are (up to rescaling) isometric to S^3. So ∂_t is a timelike and hypersurface orthogonal. So

$$g = g_{tt}\,\mathrm{d}t^2 + a(t)\Omega$$

where $\Omega = \mathrm{d}\chi^2 + \sin^2\chi\,\mathrm{d}\theta^2 + \sin^2\chi\sin^2\theta\,\mathrm{d}\varphi^2$ is the round metric on S^3. We only need to compute

$$\begin{aligned} g_{tt} &= -\cosh^2(t/\alpha) + \sinh^2(t/\alpha)(\cos^2\chi + \sin^2\chi\cos^2\theta + \sin^2\chi\sin^2\theta\cos^2\varphi + \sin^2\chi\sin^2\theta\sin^2\varphi) \\ &= -\cosh^2(t/\alpha) + \sinh^2(t/\alpha) = -1 \\ a(t) = g_{\chi\chi} &= \alpha^2\cosh^2(t/\alpha)(\sin^2\chi + \cos^2\chi\cos^2\theta + \cos^2\chi\sin^2\theta\cos^2\varphi + \cos^2\chi\sin^2\theta\sin^2\varphi) \\ &= \alpha^2\cosh^2(t/\alpha) \end{aligned}$$

Therefore the metric is given by

$$g = -\mathrm{d}t^2 + \alpha^2\cosh^2(t/\alpha)\left(\mathrm{d}\chi^2 + \sin^2\chi\,\mathrm{d}\theta^2 + \sin^2\chi\sin^2\theta\,\mathrm{d}\varphi^2\right)$$

For constant t, the hypersurface Σ_t is a 3-sphere with radius $|\alpha\cosh(t/\alpha)|$. □

Question 2.(c)

We now construct the Penrose diagram. Choose a new time coordinate $\lambda(t)$ obeying

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{1}{\alpha\cosh(t/\alpha)}$$

Write the metric in the coordinates $(\lambda, \chi, \theta, \varphi)$ and show that the de Sitter spacetime is conformal to part of the Einstein static universe. Which boundary surfaces would you call past/future null infinity? Draw the Penrose diagram. Explain why an observer, even if they observe for an infinite time, cannot observe the entire spacetime. How does this compare to the situation in Minkowski spacetime?

Proof. Integrating the equation:

$$\lambda = \int \frac{\mathrm{d}t}{\alpha\cosh(t/\alpha)} + \lambda_0 = \arctan\sinh(t/\alpha) + \lambda_0 \implies t(\lambda) = \alpha\,\mathrm{arsh}\tan(\lambda - \lambda_0) \implies \frac{\mathrm{d}t}{\mathrm{d}\lambda} = \frac{\alpha}{|\cos(\lambda - \lambda_0)|}$$

Then the metric is given by

$$\begin{aligned} g &= -\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 \mathrm{d}\lambda^2 + a(t(\lambda))\left(\mathrm{d}\chi^2 + \sin^2\chi\,\mathrm{d}\theta^2 + \sin^2\chi\sin^2\theta\,\mathrm{d}\varphi^2\right) \\ &= \alpha^2\cosh^2(t/\alpha)\left(-\mathrm{d}\lambda^2 + \mathrm{d}\chi^2 + \sin^2\chi\,\mathrm{d}\theta^2 + \sin^2\chi\sin^2\theta\,\mathrm{d}\varphi^2\right) \\ &= \frac{\alpha^2}{\cos^2(\lambda - \lambda_0)}\left(-\mathrm{d}\lambda^2 + \mathrm{d}\chi^2 + \sin^2\chi\,\mathrm{d}\theta^2 + \sin^2\chi\sin^2\theta\,\mathrm{d}\varphi^2\right) \end{aligned}$$

Let $\widetilde{g} = \Omega^2 g$, where $\Omega = \alpha^{-1}\cos(\lambda - \lambda_0)$. Then the de Sitter spacetime is conformal to a spacetime with metric

$$\widetilde{g} = -\mathrm{d}\lambda^2 + \mathrm{d}\chi^2 + \sin^2\chi\,\mathrm{d}\theta^2 + \sin^2\chi\sin^2\theta\,\mathrm{d}\varphi^2 = -\mathrm{d}\lambda^2 + \Omega$$

This is the metric of the static closed universe in the FLRW cosmological model. □

Question 3. Conformal metric transformation

Let (M, g) be a Lorentzian manifold and let $\widetilde{g} = \Omega^2 g$ be a Lorentzian metric on M that is conformal to g, where Ω is a smooth function with $\Omega(x) \neq 0$ for all $x \in M$.

Question 3.(a)

Show that the Christoffel symbols of $\widetilde{g}$ are given by

$$\widetilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \delta^\lambda_\mu \partial_\nu \log\Omega + \delta^\lambda_\nu \partial_\mu \log\Omega - g_{\mu\nu} g^{\lambda\rho} \partial_\rho \log\Omega$$

Proof. Koszul formula:

$$\widetilde{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2}\widetilde{g}^{\lambda\rho}\left(\partial_\mu \widetilde{g}_{\rho\nu} + \partial_\nu \widetilde{g}_{\rho\mu} - \partial_\rho \widetilde{g}_{\mu\nu}\right)$$

Since $\widetilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, then $\widetilde{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu}$, and

$$\begin{aligned}\widetilde{\Gamma}^\lambda_{\mu\nu} &= \frac{1}{2}\Omega^{-2} g^{\lambda\rho}\left(\Omega^2 \partial_\mu g_{\rho\nu} + 2\Omega g_{\rho\nu}\partial_\mu\Omega + \Omega^2 \partial_\nu g_{\rho\mu} + 2\Omega g_{\rho\mu}\partial_\nu\Omega - \Omega^2\partial_\rho g_{\mu\nu} - 2\Omega g_{\mu\nu}\partial_\rho\Omega\right) \\ &= \Gamma^\lambda_{\mu\nu} + \Omega^{-1} g^{\lambda\rho}\left(g_{\rho\nu}\partial_\mu\Omega + g_{\rho\mu}\partial_\nu\Omega - g_{\mu\nu}\partial_\rho\Omega\right) \\ &= \Gamma^\lambda_{\mu\nu} + \delta^\lambda_\mu \partial_\nu \log\Omega + \delta^\lambda_\nu \partial_\mu \log\Omega - g_{\mu\nu} g^{\lambda\rho}\partial_\rho \log\Omega\end{aligned}$$ □

The corresponding Levi-Civita connection is given by

$$g(\widetilde{\nabla}_X Y, Z) = g(\nabla_X Y, Z) + g(Y, Z)X(\log\Omega) + g(X, Z)Y(\log\Omega) - g(X, Y)Z(\log\Omega)$$

Question 3.(b)

Let $\gamma : \mathbb{R} \supseteq I \to M$ be a null geodesic with respect to g. Show that it is also a null geodesic with respect to $\widetilde{g}$ (but not necessarily affinely parametrised).

Proof. Suppose that γ is an affinely parametrised null geodesic in (M, g). Then we have $\nabla_{\dot\gamma}\dot\gamma = 0$ and $g(\dot\gamma, \dot\gamma) = 0$. Let $X = Y = \dot\gamma$ in the equation in 3.(a). We obtain that

$$g(\widetilde{\nabla}_{\dot\gamma}\dot\gamma, Z) = 2g(\dot\gamma, Z)\dot\gamma(\log\Omega) = 2g(\dot\gamma, Z)\frac{\mathrm{d}}{\mathrm{d}s}(\log\Omega(\gamma(s)))$$

This holds for all vector fields Z. Then we have

$$\widetilde{\nabla}_{\dot\gamma}\dot\gamma = 2\dot\gamma\frac{\mathrm{d}}{\mathrm{d}s}(\log\Omega(\gamma(s))) \tag{1}$$

Let α be a reparametrisation of γ such that $\alpha(s) = \gamma(f(s))$. Then

$$\widetilde{\nabla}_{\dot\alpha}\dot\alpha = \widetilde{\nabla}_{f'\dot\gamma}(f'\dot\gamma) = f'^2\widetilde{\nabla}_{\dot\gamma}\dot\gamma + f'\dot\gamma(f')\dot\gamma$$

We have

$$\widetilde{\nabla}_{\dot\alpha}\dot\alpha = 0 \iff \widetilde{\nabla}_{\dot\gamma}\dot\gamma = -\dot\gamma(\log f'(s))\dot\gamma = -\dot\gamma\frac{\mathrm{d}}{\mathrm{d}s}(\log f'(s)) \tag{2}$$

. Now we take $f(s)$ such that $f'(s) = \Omega(\gamma(s))^{-2}$. Then we note that (1) implies that (2) holds. So α is indeed a geodesic in $(M, \widetilde{g})$. Finally, note that

$$\widetilde{g}(\dot{\alpha}, \dot{\alpha}) = \Omega^2 g(\dot{\gamma}, \dot{\gamma}) f'^2 = 0$$

So α is a null geodesic in $(M, \widetilde{g})$. □

Question 3.(c)

Give a counterexample to the above for timelike/spacelike geodesics, i.e. give an explicit example of a Lorentzian manifold (M, g) together with a conformal metric $\widetilde{g}$ and a timelike/spacelike geodesic $\gamma : I \to \mathbb{R}$ with respect to g, which, however, is not a timelike/spacelike geodesic with respect to $\widetilde{g}$.

Proof. We use the $(1+1)$-dimensional Minkowski spacetime: $M = \mathbb{R}^2$ and $g = \eta = -\mathrm{d}t^2 + \mathrm{d}x^2$. From the lectures we know that g is conformal to

$$\widetilde{g} = \Omega^2 g = -\mathrm{d}\widetilde{u}\mathrm{d}\widetilde{v}$$

where $\widetilde{u} = \arctan(t - x)$, $\widetilde{v} = \arctan(t + x)$, and $\Omega^2 = \cos^2 \widetilde{u} \cdot \cos^2 \widetilde{v}$. Clearly $\gamma(s) = (t(s), x(s)) := (2s, s)$ is a timelike geodesic in (M, g). This corresponds to $\widetilde{u}(s) = \arctan s$ and $\widetilde{v}(s) = \arctan(3s)$. Since $\widetilde{g}$ is constant in the coordinates $(\widetilde{u}, \widetilde{v})$, then geodesics in $(M, \widetilde{g})$ are straight lines in $(\widetilde{u}, \widetilde{v})$. But $\gamma(s) = (\arctan s, \arctan(3s))$ is not a straight line. So γ is not (and cannot be reparametrised into) a geodesic in $(M, \widetilde{g})$. □

Question 4. Surface gravity at Killing horizon

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Let (M, g) be a Lorentzian manifold and let $\Sigma \subseteq M$ be a Killing horizon of a Killing vector field T. Show that the surface gravity κ, given by $\nabla_T T|_\Sigma = \kappa T|_\Sigma$, satisfies

$$\kappa^2 = -\frac{1}{2}\left[(\nabla_a T_b)\left(\nabla^a T^b\right)\right]\Big|_\Sigma .$$

[*Hint: Use that T is hypersurface-orthogonal on Σ.*]

Proof. Since T is hypersurface orthogonal on Σ, by Proposition 1.34, we have $T_{[\mu}\nabla_\nu T_{\lambda]} = 0$. We expand this and use the Killing equation $\nabla_\mu T_\nu + \nabla_\nu T_\mu = 0$:

$$\begin{aligned} 0 &= T_{[\mu}\nabla_\nu T_{\lambda]} \\ &= T_\mu (\nabla_\nu T_\lambda - \nabla_\lambda T_\nu) - T_\nu (\nabla_\mu T_\lambda - \nabla_\lambda T_\mu) + T_\lambda (\nabla_\mu T_\nu - \nabla_\nu T_\mu) \\ &= 2(T_\mu \nabla_\nu T_\lambda + T_\nu \nabla_\lambda T_\mu + T_\lambda \nabla_\mu T_\nu) \end{aligned}$$

Contracting the equation with $\nabla^\mu T^\nu$:

$$\begin{aligned} 0 &= (\nabla^\mu T^\nu)(T_\mu \nabla_\nu T_\lambda + T_\nu \nabla_\lambda T_\mu + T_\lambda \nabla_\mu T_\nu) \\ &= T_\mu (\nabla^\mu T^\nu)(\nabla_\nu T_\lambda) + T_\nu (\nabla^\nu T^\mu)(\nabla_\mu T_\lambda) + T_\lambda (\nabla^\mu T^\nu)(\nabla_\mu T_\nu) \end{aligned}$$

On Σ, $\nabla_T T = \kappa T$ implies in local coordinates that $T^\mu \nabla_\mu T^\nu = \kappa T^\mu$. Hence

$$T_\mu (\nabla^\mu T^\nu)(\nabla_\nu T_\lambda) = \kappa T^\nu \nabla_\nu T_\lambda = \kappa^2 T_\lambda, \qquad T_\nu (\nabla^\nu T^\mu)(\nabla_\mu T_\lambda) = \kappa T^\mu \nabla_\mu T_\lambda = \kappa^2 T_\lambda$$

Substituting back to the equation:

$$2\kappa^2 T_\lambda + T_\lambda(\nabla^\mu T^\nu)(\nabla_\mu T_\nu) = 0$$

Since $T \neq 0$ and λ is free, we have

$$\kappa^2 = -\frac{1}{2}(\nabla^\mu T^\nu)(\nabla_\mu T_\nu)\Big|_\Sigma$$ □

(I am sorry that I did not have time to go through the rest of the questions.)

Question 5. Geometric optics in curved space

This problem guides you through the derivation of the laws of geometric optics in curved spacetime. Let (M, g) be a Lorentzian manifold and $F \in \Omega^2(M)$ a smooth two-form, the Faraday tensor. The source-free Maxwell equations read

$$\mathrm{d}F = 0 \quad \text{and} \quad \nabla^\mu F_{\mu\nu} = 0. \tag{1}$$

Since $\mathrm{d}F = 0$, one can locally[1] find a potential $A \in \Omega^1(M)$ such that $\mathrm{d}A = F$.

Question 5.(a)

Show that F satisfies $\nabla^\mu F_{\mu\nu} = 0$ iff A satisfies

$$\nabla^\mu \nabla_\mu A_\nu - \nabla_\nu \nabla^\mu A_\mu - R_{\mu\nu} A^\mu = 0 \tag{2}$$

Proof. Since $F = \mathrm{d}A$, the components satisfy $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. Then

$$\begin{aligned} \nabla^\mu F_{\mu\nu} &= \nabla^\mu (\nabla_\mu A_\nu - \nabla_\nu A_\mu) \\ &= \nabla^\mu \nabla_\mu A_\nu - \nabla^\mu \nabla_\nu A_\mu \\ &= \nabla^\mu \nabla_\mu A_\nu - \nabla_\nu \nabla^\mu A_\mu - R_{\mu\nu} A^\mu \end{aligned}$$

So LHS $= 0 \iff$ RHS $= 0$. □

Question 5.(b)

Recall the gauge freedom $\widetilde{A}_\mu = A_\mu + \partial_\mu \chi$. Show that any solution A_μ can be put into the Lorentz gauge $\nabla^\mu \widetilde{A}_\mu = 0$ by solving an inhomogeneous wave equation for χ (note that $\Box_g \chi := \nabla^\mu \nabla_\mu \chi$ is the wave operator in curved spacetime).

Proof. The correct guage freedom should be $\widetilde{A}_\mu = A_\mu + \nabla_\mu \chi$. Note that $\nabla^\mu \widetilde{A}_\mu = \nabla^\mu A_\mu + \nabla^\mu \nabla_\mu \chi = 0$ if and only if χ satisfies that $\Box_g \chi = -\nabla^\mu A_\mu$.

Remark. If we can solve $\Box_g \chi = -\nabla^\mu A_\mu$, then (M, g) is called a **globally hyperbolic spacetime**. □

[1] Or in fact in any simply connected domain — so for instance in all of the Schwarzschild spacetime.

Question 5.(c)

We now construct approximate solutions of (2) in the Lorentz gauge, i.e. of

$$\nabla^\mu \nabla_\mu A_\nu - R_{\mu\nu} A^\mu = 0 \quad \text{and} \quad \nabla^\mu A_\mu = 0 \tag{3}$$

We make the **geometric optics ansatz**

$$A_\nu^{\text{approx}} = \frac{1}{\lambda} a_\nu \, \mathrm{e}^{\mathrm{i}\lambda\phi}, \tag{4}$$

where $a_\nu \in \Omega^1(M), \phi \in C^\infty(M)$, and $\lambda > 0$ is a large parameter. Compute $\nabla^\mu \nabla_\mu A_\nu^{\text{approx}} - R^\mu_\nu A_\mu^{\text{approx}}$ and $\nabla^\mu A_\mu^{\text{approx}}$, group the terms according to their power in λ, and show that the equations (3) are satisfied by (4) up to order $\mathcal{O}(1/\lambda)$ iff a_μ and ϕ satisfy

$$\nabla^\mu \phi \cdot a_\mu = 0, \quad \nabla^\mu \phi \cdot \nabla_\mu \phi = 0, \quad \nabla^\mu \phi \cdot \nabla_\mu a_\nu + \frac{1}{2} \Box_g \phi \cdot a_\nu = 0. \tag{5}$$

Also infer that if the large parameter λ is large compared to covariant derivatives of a_ν and the spacetime curvature $R_{\mu\nu}$, then (4) with a_ν and ϕ satisfying (5) is a good approximate solution of (3).

Proof.

$$\nabla^\mu A_\mu^{\text{approx}} = \mathrm{i} a_\mu \, \mathrm{e}^{\mathrm{i}\lambda\phi} \, \nabla^\mu \phi + \frac{1}{\lambda} \mathrm{e}^{\mathrm{i}\lambda\phi} \, \nabla^\mu a_\mu$$

$$\nabla^\mu \nabla_\mu A_\nu^{\text{approx}} - R^\mu_\nu A_\mu^{\text{approx}} = \nabla^\mu \left(\mathrm{i} a_\mu \, \mathrm{e}^{\mathrm{i}\lambda\phi} \, \nabla^\mu \phi + \frac{1}{\lambda} \mathrm{e}^{\mathrm{i}\lambda\phi} \, \nabla^\mu a_\mu \right) - \frac{1}{\lambda} \mathrm{e}^{\mathrm{i}\lambda\phi} \, R_{\mu\nu} a^\mu$$

$$= -\lambda a_\nu \, \mathrm{e}^{\mathrm{i}\lambda\phi} (\nabla^\mu \phi)(\nabla_\mu \phi) + \mathrm{i} \mathrm{e}^{\mathrm{i}\lambda\phi} \left(2 (\nabla^\mu a_\nu)(\nabla_\mu \phi) + a_\nu \Box_g \phi \right) + \frac{1}{\lambda} \mathrm{e}^{\mathrm{i}\lambda\phi} \left(\nabla^\mu \nabla_\mu a_\nu - R_{\mu\nu} a^\mu \right)$$

We impose that $\dfrac{\|a_\mu\|}{\lambda} = \mathcal{O}(1)$ and $\dfrac{\|\nabla_\mu a_\nu\|}{\lambda} \ll 1$. □

Question 5.(d)

The vector $k := (\mathrm{d}\phi)^\sharp$ is called the **wave vector**. Can you justify this terminology? Consider an observer following a timelike curve γ parametrised by proper time who carries with himself an orthonormal basis $\{E_0 = \dot{\gamma}, E_1, \dots, E_n\}$ of the tangent space which forms his local reference frame. Show that they would interpret the quantity $-\frac{1}{2\pi} \lambda \cdot E_0 \phi \big|_p = -\frac{1}{2\pi} \lambda \cdot g(E_0, k) \big|_p$ as the frequency of the electromagnetic wave (4) at a point p on his worldline.

Proof. The components of the electromagnetic tensor corresponding to the ansatz are given by

$$F_{\mu\nu}^{\text{approx}} = \mathrm{i} \, \mathrm{e}^{\mathrm{i}\lambda\phi} \big(\underbrace{(\nabla_\mu \phi)}_{k_\mu} a_\nu - \underbrace{(\nabla_\nu \phi)}_{k_\nu} a_\mu \big) + \mathcal{O}\left(\frac{\|\nabla a\|}{\lambda} \right)$$

The light fronts are surfaces of constant phase ϕ. $k = (\mathrm{d}\phi)^\sharp$ is the wave vector.

We pick coordinates centred at p such that $E_0 = \partial_t$ and $E_i = \partial_i$. We expand ϕ as

$$\phi(t, \boldsymbol{x}) = \phi(0) + t \partial_t \phi(t, \boldsymbol{x}) + x^i \partial_i \phi(t, \boldsymbol{x}) + \cdots$$

Then

$$\mathrm{e}^{\mathrm{i}\lambda\phi(t,\boldsymbol{x})} \approx \mathrm{e}^{\mathrm{i}\lambda\phi(0)} \, \mathrm{e}^{\mathrm{i}\lambda(t\partial_t \varphi(0) + x^i \partial_i \phi(0))}$$

$e^{i\lambda t \partial_t \varphi(0)} = e^{-i \cdot 2\pi f t}$ is the frequency term. So

$$f = -\frac{1}{2\pi}\lambda \partial_t \phi(0) = -\frac{\lambda}{2\pi} g_p(E_0, k)$$

is the frequency of the wave. □

Question 5.(e)

The equation $\nabla^\mu \phi \cdot \nabla_\mu \phi = 0$ is known as the **Eikonal equation**. It can be always solved locally. Show that it implies that the wave vector k is null and that it satisfies $\nabla_k k = 0$, i.e., it is propagated affinely along null geodesics.

Proof. k is null precisely because $(\nabla^\mu \phi)(\nabla_\mu \phi) = 0$.

$$\nabla_\mu (k_\nu k^\nu) = 2(\nabla_\mu k_\nu) k^\nu = 2(\nabla_\mu \nabla_\nu \phi) k^\nu = 2(\nabla_\nu \nabla_\mu \phi) k^\nu = 2(\nabla_\nu k_\mu) k^\nu = 2(\nabla_k k)_\mu = 0$$

Hence $\nabla_k k = 0$. □

Question 5.(f)

Let us now decompose the covector amplitude a_ν in (4) as $a_\nu = \alpha \cdot f_\nu$, with the **amplitude** $\alpha \in C^\infty(M)$ and the **polarisation covector** $f_\nu \in \Omega^1(M)$. It follows from the first equation in (5) that $f_\nu k^\nu = 0$, i.e., the polarisation vector is orthogonal to the wave vector, i.e., it must be tangent to the null hypersurfaces $\phi =$ const. Show that to leading order in λ the electric and magnetic fields do not change by adding a multiple of k_ν to f_ν.

Thus, only if f is spacelike do we have a non-vanishing electromagnetic field. Without loss of generality we can thus normalise the polarisation covector by $f_\nu f^\nu = 1$. Show that the third equation in (5) implies the propagation equation

$$\nabla_k \alpha + \frac{1}{2} \nabla^\mu k_\mu \cdot \alpha = 0 \tag{6}$$

for the amplitude along the integral curves of k and that the polarisation covector is parallel-propagated along k, i.e.,

$$\nabla_k f = 0$$

Note that (6) in particular implies that if α vanishes on some point on an integral curve of k (which are null geodesics by $\nabla_k k = 0$), then it vanishes along the whole curve. *This makes precise in which sense and under what conditions 'light propagates along null geodesics in general relativity'.*

Proof. $a_\nu = \alpha f_\nu$ and $a_\mu \nabla^\mu \phi = 0$ implies that $g(f^\sharp, k) = 0$.

We make the gauge transformation $f_\nu \mapsto f_\nu + \xi k_\nu$. Then the change in $F^{\text{approx}}_{\mu\nu}$ is given by

$$\frac{1}{2} F^{\text{approx}}_{\mu\nu} = \frac{\xi}{\lambda} e^{i\lambda\phi} \left(k_\nu \nabla_\mu \alpha + \alpha \nabla_{[\mu} k_{\nu]} + i\lambda (\nabla_{[\mu} \phi) k_{\nu]} \right) = \underbrace{\frac{\xi\alpha}{2\lambda} e^{i\lambda\phi} (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\phi}_{=0} + \mathcal{O}\left(\frac{\|\nabla a\|}{\lambda} \right)$$

Hence the leading order of $F^{\text{approx}}_{\mu\nu}$ is left invariant.

□

Question 5.(g)

Consider now the Schwarzschild spacetime with an observer γ_A following a timelike curve of constant $r = r_A > 2M, \theta = \theta_0, \varphi = \varphi_0$ and another observer γ_B following a timelike curve of constant $r = r_B > r_A, \theta = \theta_0, \varphi = \varphi_0$. Make precise, using the laws of geometric optics derived in this exercise, that a high-frequency light signal of frequency f_A as measured by observer A, sent from A to B, arrives red-shifted at observer B with a frequency

$$f_B = \sqrt{\frac{1 - \frac{2M}{r_A}}{1 - \frac{2M}{r_B}}} f_A$$

Proof. Use the Eddington-Finkelstein coordinates (u, r, θ, φ) such that $g = -\left(1 - \frac{2M}{r}\right) du^2 - 2dudr + r^2\Omega$. Then $\gamma(r) = (u_0, r, \theta_0\, \varphi_0)$ is a null geodesic. ∂_t in Schwarzschild is equal to ∂_u in Eddington-Finkelstein. $\dot\gamma = \partial_r \propto k$ is the wave vector.

$$E_0^{(A)} = \left(1 - \frac{2M}{r_A}\right)^{-1/2} \partial_t, \qquad E_0^{(B)} = \left(1 - \frac{2M}{r_B}\right)^{-1/2} \partial_t$$

$$-g(E_0^{(A,B)}, \partial_r) = \left(1 - \frac{2M}{r_{A,B}}\right)^{-1/2}$$

$$\lambda k|_{r_A} = 2\pi f_A \sqrt{1 - \frac{2M}{r_A}} \partial_r$$

k is parallelly transported: $\nabla_k k = \nabla_{\partial_r} k = 0$.

$$f_B = -\frac{1}{2\pi} g_{r_B}(E_0^{(B)}, \lambda k) = \sqrt{\frac{1 - 2M/r_A}{1 - 2M/r_B}} f_A$$

□

Question 6. Kerr observers

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Let $M = \mathbb{R} \times (r_+, \infty) \times S^2$ with the standard Boyer-Lindquist coordinates $\{t, r, \theta, \varphi\}$, where $r_+ = M + \sqrt{M^2 - a^2}$, $M > 0$, and $0 < a < M$. We define the Kerr metric g as

$$g = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{2Mra\sin^2\theta}{\rho^2}(dt \otimes d\varphi + d\varphi \otimes dt) + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\varphi^2$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. Consider a stationary observer A with velocity $u\,(\partial_t + \Omega\partial_\varphi)$ at some value of $r_0 \in (r_+, \infty)$ and some value of $\theta_0 \in (0, \pi)$, where $u > 0$ is chosen such that the velocity is normalised. Show that Ω corresponds to the angular frequency of A *as seen by an observer B with velocity ∂_t at infinity who is at rest with respect to the asymptotic Lorentz frame.*

Therefore, an observer with $\Omega = 0$ appears static from infinity 'with respect to the fixed stars'.

[*Hint: The movement of A as seen by B depends on the null geodesics connecting A's worldline with B's. Use the symmetries of the Kerr spacetime to answer this question without actually computing the null geodesics.*]

Proof. $\dot{\gamma}_A = u(\partial_t + \Omega\partial_\varphi)$. $\dot{\gamma}_B = \partial_t$. The worldline of A is $\gamma_A(t) = (t, r_A, \theta_A, \varphi_0 + \Omega t)$. Fix t_B. There is a null geodsic backward in time intersecting γ_A at $t = t_A$. Since ∂_t is a Killing vector field, $t_A \mapsto t_A + 2\pi/\Omega$ corresponds to $t_B \mapsto t_B + 2\pi/\Omega$. So B observes A moving in the period of $2\pi/\Omega$. Hence Ω is the angular frequency of A observed by B. □

Section C: Optional

Question 7. Limits of Kerr metric

Show that the Kerr metric (6) from the last problem reduces to

(a) the Schwarzschild metric for $a = 0$;

(b) the Minkowski metric in spheroidal coordinates for $M = 0$, but $a \neq 0$. Here, the spheroidal coordinates in Minkowski spacetime are given by

$$
\begin{aligned}
x &= \sqrt{r^2 + a^2}\sin\theta\cos\varphi \\
y &= \sqrt{r^2 + a^2}\sin\theta\sin\varphi \\
z &= r\cos\theta
\end{aligned}
$$