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Problem Sheet 4

C2.2: Homological Algebra

Overall mark: βα

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Section A: Introductory

Question 1

Calculate $\operatorname{Tor}_\bullet^R(M,M)$ where $R=k[x]$, k is a field, and $M=\dfrac{k[x,x^{-1}]}{xk[x]}$.

Proof. We have a short exact sequence

$$0 \longrightarrow k[x] \xrightarrow{\cdot x} k[x,x^{-1}] \longrightarrow \frac{k[x,x^{-1}]}{xk[x]} \longrightarrow 0 \qquad (*)$$

By Flat Resolution Theorem 6.33, we can compute the Tor modules using a flat resolution. We claim that $k[x,x^{-1}]$ is a flat $k[x]$-module.

This is because $k[x,x^{-1}]=\bigcup_{n\in\mathbb{N}} x^{-n}k[x]$ is a filtered colimit of the modules $x^{-n}k[x]$. For a short exact sequence of $k[x]$-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

Tensoring $x^{-n}k[x]$ gives

$$0 \longrightarrow x^{-n}A \longrightarrow x^{-n}B \longrightarrow x^{-n}C \longrightarrow 0$$

which is still an exact sequence. So $x^{-n}k[x]$ is a flat $k[x]$-module. By Corollary 6.29, $k[x,x^{-1}]$ is flat. So $(*)$ provides a flat resolution for $M:=\dfrac{k[x,x^{-1}]}{xk[x]}$. Applying $(-\otimes_{k[x]} M)$ to $(*)$:

$$0 \longrightarrow k[x]\otimes_{k[x]} M \longrightarrow k[x,x^{-1}]\otimes_{k[x]} M \longrightarrow M\otimes_{k[x]} M \longrightarrow 0$$

By Question 4 of Sheet 2, we have $k[x,x^{-1}]\otimes_{k[x]} M \cong M[x^{-1}]$. Since $x=0$ in M, for $m\in M[x^{-1}]$, $m=x^{-1}\cdot x\cdot m=0$. Hence $M[x^{-1}]$. The (augmented) chain complex is given by

$$0 \longrightarrow M \longrightarrow 0 \longrightarrow M\otimes_{k[x]} M \longrightarrow 0$$

Taking the homology of this chain. We obtain[1]

$$\operatorname{Tor}_n^{k[x]}(M,M) \cong \begin{cases} M & n=1 \\ 0 & \text{otherwise} \end{cases}$$

✓ ☐

Section B: Core

Question 2 β+

Calculate the ring $\operatorname{Ext}_R^\bullet(k,k)$ where $R=\dfrac{k[x,y]}{(xy)}$, with k a field (viewed as $\dfrac{R}{(x,y)}$ as an R-module).

Proof. First we find a free resolution for k. Let $\pi: R\to k$ be the quotient map. Then $\ker\pi=\langle x,y\rangle$. Let $\alpha: R^2\to R$ be the homomorphism given by $(r,s)\mapsto xr+ys$. Then $\operatorname{im}\alpha=\langle x,y\rangle=\ker\pi$. We have $\ker\alpha=\langle y\rangle\oplus\langle x\rangle$. Let $\beta: R^2\to R^2$ be the homomorphism given by $(r,s)\mapsto(yr,xs)$. Then $\operatorname{im}\beta=\ker\alpha$, and $\ker\beta=\langle x\rangle\oplus\langle y\rangle$. Let $\gamma: R^2\to R^2$ be the homomorphism given by $(r,s)\mapsto(xr,ys)$. Then $\operatorname{im}\gamma=\ker\beta$ and $\ker\gamma=\operatorname{im}\beta$. So we have a free resolution

$$\cdots \xrightarrow{\begin{pmatrix} x & 0\\ 0 & y\end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & 0\\ 0 & x\end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & 0\\ 0 & y\end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & 0\\ 0 & x\end{pmatrix}} R^2 \xrightarrow{(x\ y)} R \xrightarrow{\pi} k \longrightarrow 0$$

Applying the functor $\operatorname{Hom}_R(-,k)$:

[1] This looks strange as we expect that $\operatorname{Tor}_0^{k[x]}(M,M)\cong M\otimes_{k[x]} M$. But it is also possible that we could show $M\otimes_{k[x]} M=0$ directly.

You can, and it's not very difficult.

$$k \xrightarrow{\text{id}} k \xrightarrow{0} k^2 \xrightarrow{0} k^2 \xrightarrow{0} k^2 \longrightarrow \cdots$$

Taking cohomology we obtain the Ext modules

$$\operatorname{Ext}^n_R(k,k) \cong \begin{cases} k & n = 0 \\ k^2 & n \geqslant 1 \end{cases}$$

Next we shall prove that the Yoneda product on $\operatorname{Ext}^\bullet_R(k,k)$ makes it a graded ring

$$\operatorname{Ext}^\bullet_R(k,k) \cong \frac{k[x,y]}{\langle xy \rangle} = R, \qquad |x| = |y| = 1$$

This is a reasonable guess,
but it actually isn't the correct ring!

To prove this, we compute the Ext in another way. We use $P_\bullet$ to denote the free resolution for k obtained above. From the discussion in Section 9.1, $\operatorname{Ext}^n_R(k,k)$ is isomorphic to the quotient of the module of the chain maps $P_\bullet \to P_\bullet[-n]$ by the submodule of null-homotopic chain maps.

We compute $\operatorname{Ext}^1_R(k,k)$. A chain map $f_\bullet : P_\bullet \to P_\bullet[-1]$ is the commutative diagram:

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\gamma} & R^2 & \xrightarrow{\beta} & R^2 & \xrightarrow{\gamma} & R^2 & \xrightarrow{\beta} & R^2 & \xrightarrow{\alpha} & R & \longrightarrow & 0 \\
 & & \downarrow f_4 & & \downarrow f_3 & & \downarrow f_2 & & \downarrow f_1 & & & & \\
\cdots & \xrightarrow{\beta} & R^2 & \xrightarrow{\gamma} & R^2 & \xrightarrow{\beta} & R^2 & \xrightarrow{\alpha} & R & \longrightarrow & 0 & &
\end{array}$$

To make this diagram commutative, we must have

$$f_1 = \begin{pmatrix} \varphi_1 & \psi_1 \end{pmatrix}, \qquad f_n = \begin{pmatrix} \varphi_n & 0 \\ 0 & \psi_n \end{pmatrix}, \; n \geqslant 2$$

It isn't true that the components of f_1 must lie in the ideals (x) and (y).

where $\varphi_n, \psi_n \in R$ satisfy

Try writing them as 2x2 matrices of R-elements.

$$y\varphi_{2n-1} = x\varphi_{2n}, \qquad x\psi_{2n-1} = y\psi_{2n}$$

Hence $\varphi_{2n-1} \in \langle x \rangle$, $\varphi_{2n} \in \langle y \rangle$, $\psi_{2n-1} \in \langle y \rangle$, and $\psi_{2n} \in \langle x \rangle$. Since $xy = 0$, the equations above are equal to zero.

If $f_\bullet$ is chain null-homotopic, then there exists $h_\bullet : P_\bullet \to P_\bullet$ such that $f_n = h_{n-1} \circ \partial_n + \partial_n \circ h_n$. If we write

$$h_1 = c, \qquad h_n = \begin{pmatrix} f_n & 0 \\ 0 & g_n \end{pmatrix}, \; n \geqslant 2$$

Then $f_n = h_{n-1} \circ \partial_n + \partial_n \circ h_n$ implies that

$$\varphi_1 = (c + f_2)x, \qquad \psi_1 = (c + g_2)y$$

and for $n \geqslant 1$,

$$\begin{aligned} \varphi_{2n-1} &= (f_{2n-1} + f_{2n})x, & \psi_{2n-1} &= (g_{2n-1} + g_{2n})y \\ \varphi_{2n} &= (f_{2n} + f_{2n+1})y, & \psi_{2n} &= (g_{2n} + g_{2n+1})x \end{aligned}$$

We note that as long as $\varphi_{2n-1} \in \langle x \rangle$, $\varphi_{2n} \in \langle y \rangle$, $\psi_{2n-1} \in \langle y \rangle$, and $\psi_{2n} \in \langle x \rangle$, the chain map $h_\bullet$ can be solved by the above equations successively. So *every chain map $f_\bullet : P_\bullet \to P_\bullet[-1]$ is null-homotopic. This is absurd. I don't know where has gone wrong...*

It is indeed wrong, but is it necessarily absurd? See above.

□

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Question 3

Consider the functor $F : \mathbb{Z}\text{-Mod} \to \mathbb{Z}\text{-Mod}$, $A \mapsto A[2^\infty] = \{a \in A : \exists n \in \mathbb{N} \text{ s.t. } 2^n a = 0\}$. Prove F is left exact and calculate the groups and maps in the LES of derived functors associated to the SES $0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}/2 \to 0$

Proof. Suppose that $f : A \to B$ is a $\mathbb{Z}$-module homomorphism. Then the restriction $f|_{F(A)} : F(A) \to F(B)$ is a well-defined $\mathbb{Z}$-module homomorphism, as

$$a \in F(A) \implies 2^n a = 0 \implies 2^n f(a) = 0 \implies f(a) \in F(B)$$

We write $F(f) := f|_{F(A)}$. It is not hard to verify that F is indeed a functor from $\mathbb{Z}$-Mod to $\mathbb{Z}$-Mod.

Consider the short exact sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

Applying the functor F:

$$0 \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \longrightarrow 0$$

By definition, $\ker F(f) \subseteq \ker f = 0$. Hence $F(f)$ is a monomorphism. Also,

$$\ker F(g) = \ker g \cap F(B) = \operatorname{im} f \cap F(B) \supseteq \operatorname{im} F(f)$$

For $b \in \operatorname{im} f \cap F(B)$, there exists $n \in \mathbb{N}$ such that $2^n b = 0$ and $a \in A$ such that $f(a) = b$. Therefore $f(2^n a) = 2^n b = 0$ and hence $a \in A$. Thus $b \in \operatorname{im} F(f)$. So we have $\ker F(g) = \operatorname{im} F(f)$. This indicates that F is left exact. ✓

To compute the long exact sequence associated with the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

We need injective resolutions for $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$. They are given by

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{1/2} \mathbb{Q}/\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

Apply the functor F. Since $\mathbb{Z}$ and $\mathbb{Q}$ are torsion-free, $F(\mathbb{Z}) = F(\mathbb{Q}) = 0$. $F(\mathbb{Z}/2) = \mathbb{Z}/2$. $F(\mathbb{Q}/\mathbb{Z}) = \mathbb{Z}[2^{-1}]/\mathbb{Z} = \left\{\frac{m}{2^n} : n \in \mathbb{Z}_+, 0 \leqslant m < 2^n\right\}$.

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}[2^{-1}]/\mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{1/2} \mathbb{Z}[2^{-1}]/\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}[2^{-1}]/\mathbb{Z} \longrightarrow 0$$

Taking the cohomology, we have

$$\mathrm{R}^n F(\mathbb{Z}) = \begin{cases} \mathbb{Z}[2^{-1}]/\mathbb{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases}, \qquad \mathrm{R}^n F(\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & n = 0 \\ \dfrac{\mathbb{Z}[2^{-1}]/\mathbb{Z}}{2\mathbb{Z}[2^{-1}]/\mathbb{Z}} & n = 1 \\ 0 & n \geqslant 2 \end{cases}$$

✓

this is also 0 (why?)

We put these modules into the long exact sequence of the derived functors:

$$
\begin{array}{lccccc}
n=0 & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2 \\
 & & & & \swarrow_{1/2} & \\
n=1 & \mathbb{Z}[2^{-1}]/\mathbb{Z} & \xrightarrow[\cdot 2]{} & \mathbb{Z}[2^{-1}]/\mathbb{Z} & \xrightarrow{\pi} & \frac{\mathbb{Z}[2^{-1}]/\mathbb{Z}}{2\mathbb{Z}[2^{-1}]/\mathbb{Z}} \\
 & & & & \swarrow & \\
n=2 & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}
$$

Correct, but...

0

□

Section C: Optional

Question 4

k : field , $R := k[x]/x^3, M := k[x]/x, N := k[x]/x^2$.

Write down explicitly (all objects and morphisms in) a SES of R-projective resolutions using the horseshoe lemma on the following Mod_R-SES: $0 \to M \xrightarrow{\alpha} N \xrightarrow{\beta} M \to 0$