Total: α

Good job.

Review computations in Q1.

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Problem Sheet 3

B3.2: Geometry of Surfaces

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Question 1

Let U be an open subset of $\mathbb{R}^2$ and let $\mathbf{r}: U \to \mathbb{R}^3$ be a smooth parametrisation of a surface $S = \mathbf{r}(U) \subseteq \mathbb{R}^3$. Let $E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2$ be its first fundamental form. A parametrisation is said to be *conformal* if it preserves angles between intersecting curves, and *equiareal* if it preserves areas.

- Show that the parametrisation is *conformal* if and only if $E = G$ and $F = 0$, and is *equiareal* if and only if $EG - F^2 = 1$
- What is the first fundamental form of the spherical coordinates local parametrisation of the unit sphere, given by $\mathbf{r}(\theta, \phi) = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)$? Show that this parametrisation is neither conformal nor equiareal. (In this familiar parametrisation, θ gives the longitude and ϕ the latitude.)
- *Mercator's projection* of the unit sphere minus the Date Line takes a point $\mathbf{r}(\theta, \phi)$ with latitude ϕ and longitude θ to $\left(\theta, \log\tan\left(\frac{\phi}{2} + \frac{\pi}{4}\right)\right)$ in $(-\pi, \pi) \times \mathbb{R}$. Show that this parametrisation is conformal but not equiareal.
- *Lambert's cylindrical projection* takes a point $\mathbf{r}(\theta, \phi)$ with latitude ϕ and longitude θ to $(\theta, \sin\phi)$ Show that this parametrisation is equiareal.

Proof.

- First we shall formulate an exact definition of a conformal parametrisation:

  Fix $\mathbf{p} \in U$. The map $\mathbf{r}: U \to S$ induces the differential map $\mathrm{d}\mathbf{r}_{\mathbf{p}}: \mathrm{T}_{\mathbf{p}}U \to \mathrm{T}_{\mathbf{r}(\mathbf{p})}S$ given by $\gamma' \mapsto (\mathbf{r} \circ \gamma)'$, where $\gamma: [-\varepsilon, \varepsilon] \to U$ is a curve with $\gamma(0) = \mathbf{p}$. Let $\mathbf{a}$ and $\mathbf{a}$ be tangent vectors of $\mathrm{T}_{\mathbf{p}}U$. We define the angle between $\mathbf{a}$ and $\mathbf{b}$ to be

  $$\angle(\mathbf{a}, \mathbf{b}) := \arccos\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}\right)$$

  Yes!

  We say that $\mathbf{r}$ is conformal at $\mathbf{p} \in U$, if $\angle(\mathbf{a}, \mathbf{b}) = \angle(\mathrm{d}\mathbf{r}_{\mathbf{p}}(\mathbf{a}), \mathrm{d}\mathbf{r}_{\mathbf{p}}(\mathbf{b}))$ for all $\mathbf{u}, \mathbf{v} \in \mathrm{T}_{\mathbf{p}}U$. We say that $\mathbf{r}$ is conformal in U if it is conformal at every point of U.

  "$\Longleftarrow$": Suppose that $E = G = \lambda(u, v)$ and $F = 0$. Then $g = \lambda(\mathrm{d}u^2 + \mathrm{d}v^2)$ is an isothermal coordinates. In particular we have $g(\mathrm{d}\mathbf{r}(\mathbf{a}), \mathrm{d}\mathbf{r}(\mathbf{b})) = \lambda\mathbf{a} \cdot \mathbf{b}$. Then

  $$\frac{\mathrm{d}\mathbf{r}(\mathbf{a}) \cdot \mathrm{d}\mathbf{r}(\mathbf{b})}{\|\mathrm{d}\mathbf{r}(\mathbf{a})\|\,\|\mathrm{d}\mathbf{r}(\mathbf{b})\|} = \frac{g(\mathrm{d}\mathbf{r}(\mathbf{a}), \mathrm{d}\mathbf{r}(\mathbf{b}))}{\sqrt{g(\mathrm{d}\mathbf{r}(\mathbf{a}), \mathrm{d}\mathbf{r}(\mathbf{a}))}\sqrt{g(\mathrm{d}\mathbf{r}(\mathbf{b}), \mathrm{d}\mathbf{r}(\mathbf{b}))}} = \frac{\lambda\mathbf{a} \cdot \mathbf{b}}{\sqrt{\lambda}\|\mathbf{a}\|\sqrt{\lambda}\|\mathbf{b}\|} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}$$

  Hence $\mathbf{r}$ is conformal.

  "$\Longrightarrow$": Suppose that $\mathbf{r}$ is conformal. We have

  $$\mathrm{d}\mathbf{r}\left(\frac{\partial}{\partial u}\right) = \frac{\partial \mathbf{r}}{\partial u} = \mathbf{r}_u, \qquad \mathrm{d}\mathbf{r}\left(\frac{\partial}{\partial v}\right) = \frac{\partial \mathbf{r}}{\partial v} = \mathbf{r}_v$$

  Since $\mathbf{r}$ is conformal, we have

  $$\begin{aligned} \cos\angle(\mathbf{r}_u, \mathbf{r}_v) &= \cos\angle(\partial_u, \partial_v) = \cos\angle(\mathbf{e}_u, \mathbf{e}_v) = 0 \\ \cos\angle(\mathbf{r}_u + \mathbf{r}_v, \mathbf{r}_v) &= \cos\angle(\partial_u + \partial_v, \partial_v) = \cos\angle(\mathbf{e}_u + \mathbf{e}_v, \mathbf{e}_v) = 1/\sqrt{2} \\ \cos\angle(\mathbf{r}_u + \mathbf{r}_v, \mathbf{r}_u) &= \cos\angle(\partial_u + \partial_v, \partial_u) = \cos\angle(\mathbf{e}_u + \mathbf{e}_v, \mathbf{e}_u) = 1/\sqrt{2} \end{aligned}$$

  From $E = \mathbf{r}_u \cdot \mathbf{r}_u$, $F = \mathbf{r}_v \cdot \mathbf{r}_v$ and $G = \mathbf{r}_v \cdot \mathbf{r}_v$, we have

  $$\frac{F}{\sqrt{EG}} = 0, \qquad \frac{F + G}{\sqrt{E + 2F + G}\sqrt{G}} = \frac{1}{\sqrt{2}}, \qquad \frac{E + F}{\sqrt{E + 2F + G}\sqrt{E}} = \frac{1}{\sqrt{2}}$$

  The first equation implies that $F = 0$. Substituting into the second and the third ones we have

  $$\frac{\sqrt{E}}{\sqrt{E + G}} = \frac{\sqrt{G}}{\sqrt{E + G}} = \frac{1}{\sqrt{2}}$$

  which implies that $E = G$.

  Next we formulate an exact definition of equiareal parametrisation:

Let $K \subseteq U$ be a compact subset. The area of $\mathbf{r}(K) \subseteq S$ is given by

$$A[\mathbf{r}(K)] = \iint_{\mathbf{r}(K)} \mathrm{d}\sigma = \iint_K \sqrt{\det g}\,\mathrm{d}u\,\mathrm{d}v$$

where g is the metric tensor of S. We say that $\mathbf{r}: U \to S$ is equiareal, if $A[K] = A[\mathbf{r}(K)]$ for any compact subset $K \subseteq U$.

Now we observe that for any compact subset $K \subseteq U$,

$$A[K] = A[\mathbf{r}(K)] \iff \iint_K \sqrt{\det g}\,\mathrm{d}u\,\mathrm{d}v = \iint_K \mathrm{d}u\,\mathrm{d}v \iff \iint_K \left(\sqrt{\det g} - 1\right)\mathrm{d}u\,\mathrm{d}v = 0$$

Since $\sqrt{\det g} - 1$ is smooth, the above equation holds for all $K \subseteq U$ if and only if $\sqrt{\det g} - 1 = 0$. Finally $\det g = \det\begin{pmatrix} E & F \\ F & G \end{pmatrix} = EG - F^2$. We deduce that $\mathbf{r}$ is equiareal if and only if $EG - F^2 = 1$. Great argument!

- From $\mathbf{r}(\theta,\phi) = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)$, we have

$$\mathbf{r}_\theta = (-\sin\theta\cos\varphi, \cos\theta\cos\varphi, 0), \qquad \mathbf{r}_\varphi = (-\cos\theta\sin\varphi, -\sin\theta\sin\varphi, \cos\varphi)$$

Then

$$\begin{aligned} E &= \mathbf{r}_\theta \cdot \mathbf{r}_\theta = \sin^2\theta\cos^2\varphi + \cos^2\theta\cos^2\varphi = \cos^2\varphi \\ F &= \mathbf{r}_\theta \cdot \mathbf{r}_\varphi = \sin\theta\cos\theta\cos\varphi\sin\varphi - \sin\theta\cos\theta\cos\varphi\sin\varphi = 0 \\ G &= \mathbf{r}_\varphi \cdot \mathbf{r}_\varphi = \cos^2\theta\sin^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\varphi = 1 \end{aligned}$$

Great!

Hence the metric tensor is given by

$$g = E\,\mathrm{d}\theta^2 + 2F\,\mathrm{d}\theta\,\mathrm{d}\varphi + G\,\mathrm{d}\varphi^2 = \cos^2\varphi\,\mathrm{d}\theta^2 + \mathrm{d}\varphi^2$$

Yes!

(This is not the standard spherical coordinates.)

We observe that g is not conformal because $E \neq G$, and g is not equiareal because $EG - F^2 = \cos^2\varphi \neq 1$. Indeed!

- The result proven in the beginning can be generalised as follows: Let $f: S_1 \to S_2$ be a map between smooth surfaces. Then f induces a pull-back metric $f^*(g_2)$ on S_1 given by

$$f^*(g_2)(\mathbf{a},\mathbf{b}) := g_2(\mathrm{d}f(\mathbf{a}), \mathrm{d}f(\mathbf{b}))$$

for all $\mathbf{a},\mathbf{b} \in \mathrm{T}_\mathbf{p}(S_1)$ and $\mathbf{p} \in S_1$.

f is conformal if and only if there exists a continuous function λ on S_1 such that $f^*(g_2) = \lambda g_1$. f is equiareal if and only if $\det f^*(g_2) = \det g_1$. I see you know some Riemannian Geometry :)

Let $\mathbf{r}(\theta,\varphi) = \left(\theta, \log\tan\left(\frac{\phi}{2} + \frac{\pi}{4}\right)\right)$. We have

Here you are pulling back the euclidean metric of one parameter domain to another. Why?

$$\mathbf{r}_\theta = (1,0), \qquad \mathbf{r}_\varphi = \left(0, \frac{1}{\cos\varphi}\right)$$

Hence the metric tensor is given by

$$g_2 = \mathrm{d}\theta^2 + \frac{1}{\cos^2\varphi}\mathrm{d}\varphi^2$$

This is the Euclidean metric on $r(U)$ pulled to back to (θ,φ) domain. Why care?

Comparing to the metric tensor of the sphere:

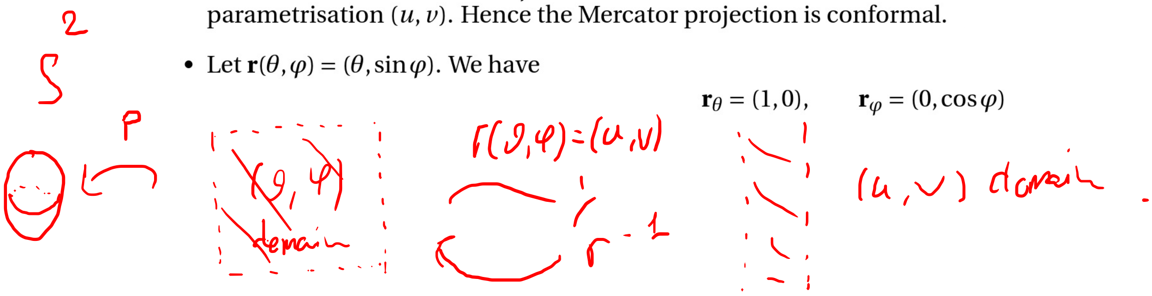
$$g_1 = \cos^2\varphi\,\mathrm{d}\theta^2 + \mathrm{d}\varphi^2$$

We find that $f^*(g_2) = \dfrac{1}{\cos^2\varphi} g_1$, where f is the map from the sphere (minus a line) to $(-\pi,\pi) \times \mathbb{R}$ induced by the parametrisation (u,v). Hence the Mercator projection is conformal.

Computed in part (a)

- Let $\mathbf{r}(\theta,\varphi) = (\theta, \sin\varphi)$. We have

$$\mathbf{r}_\theta = (1,0), \qquad \mathbf{r}_\varphi = (0, \cos\varphi)$$



You need to pull $p^* g_{S^2}$ back by r^{-1}

Hence the metric tensor is given by

$$g_3 = \mathrm{d}\theta^2 + \cos^2\varphi\,\mathrm{d}\varphi^2$$

Same issue as before.

Let $\tilde{f}$ be the map from the sphere (minus the poles) to $(-\pi,\pi)\times(-1,1)$ induced by the parametrisation (u,v). We find that $\det g_1 = \det \tilde{f}^*(g_3) = \cos^2\varphi$. Hence the Lambert's cylindrical projection is equiareal. □

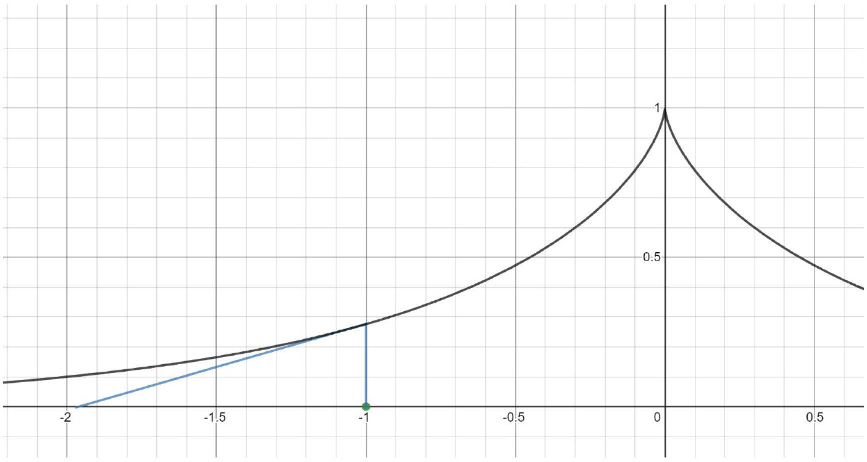
(a)

Question 2

The *tractrix* is a curve in $\mathbb{R}^2$ such that the distance along any tangent line from its point of contact with the curve to its point of intersection with the x-axis is 1. If θ is the angle the tangent line makes with the x -axis, show that the surface of revolution (the tractoid) obtained by rotating the tractrix about the x -axis has first fundamental form $\cot^2\theta\,\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}v^2$ where v is the angle of rotation of the surface of revolution. By making a suitable change of coordinates between (v,θ) and (x,y), show that the tractoid is locally isometric to the hyperbolic plane with first fundamental form $(\mathrm{d}x^2+\mathrm{d}y^2)/y^2$.

Proof. First we consider the tractrix in the (x,w)-plane parametrised by θ. From the diagram below we find that

$$w = \sin\theta, \qquad \frac{\mathrm{d}w}{\mathrm{d}x} = \tan\theta, \qquad \frac{\mathrm{d}x}{\mathrm{d}\theta} = \frac{\mathrm{d}x}{\mathrm{d}w}\frac{\mathrm{d}w}{\mathrm{d}\theta} = \frac{\cos\theta}{\tan\theta} = \frac{\cos^2\theta}{\sin\theta}$$



Then the tractoid is parametrised by

$$\mathbf{r}(\theta,v) = (x(\theta), w\cos v, w\sin v) = (x(\theta), \sin\theta\cos v, \sin\theta\sin v)$$

(I guessed what $x(\theta)$ is, you should write it).

Hence

$$\mathbf{r}_\theta = \left(\frac{\cos^2\theta}{\sin\theta}, \cos\theta\cos v, \cos\theta\sin v\right), \qquad \mathbf{r}_v = (0, -\sin\theta\sin v, \sin\theta\cos v)$$

The metric tensor is given by

$$g = \left(\frac{\cos^4\theta}{\sin^2\theta} + \cos^2\theta\right)\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}v^2 = \cot^2\theta\,\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}v^2$$

Change of variable: $x = v$, $y = \csc\theta$. Then $\mathrm{d}x = \mathrm{d}v$ and $\mathrm{d}y = -\dfrac{\cos\theta}{\sin^2\theta}\mathrm{d}\theta$. We have

$$g = \sin^2\theta\left(\frac{\cos^2\theta}{\sin^4\theta}\mathrm{d}\theta^2 + \mathrm{d}v^2\right) = \frac{1}{y^2}(\mathrm{d}x^2+\mathrm{d}y^2)$$

We deduce that the tractoid is locally isometric to the hyperbolic plane, because they have the same Riemannian metric. □

Great!

Question 3

Show that the Gaussian curvature of a surface which is the graph of a smooth function $z = f(x, y)$ is given by

$$K = \frac{f_{xx} f_{yy} - f_{xy}^2}{\left(1 + f_x^2 + f_y^2\right)^2}$$

Calculate K when $f(x, y) = xy$ and sketch the surface.

Proof. $\mathbf{r}(x, y) = (x, y, f(x, y))$ gives a global parametrisation of the surface. The tangent vectors are given by

$$\mathbf{r}_x = (1, 0, f_x), \qquad \mathbf{r}_y = (0, 1, f_y)$$

The metric tensor is given by

$$g = \mathrm{d}s^2 = \mathbf{r}_i \cdot \mathbf{r}_j \,\mathrm{d}u^i \,\mathrm{d}u^j = f_x^2 \,\mathrm{d}x^2 + 2 f_x f_y \,\mathrm{d}x \,\mathrm{d}y + f_y^2 \,\mathrm{d}y^2$$

The normal vector is given by

$$\mathbf{n} = \frac{\mathbf{r}_x \wedge \mathbf{r}_y}{\|\mathbf{r}_x \wedge \mathbf{r}_y\|} = \frac{(f_x, f_y, -1)}{\|(f_x, f_y, -1)\|} = \frac{(f_x, f_y, -1)}{\sqrt{1 + f_x^2 + f_y^2}}$$

The second fundamental form is given by

$$\mathrm{II} = \mathrm{d}^2\mathbf{r} \cdot \mathbf{n} = -\mathbf{r}_{ij} \cdot \mathbf{n} \,\mathrm{d}u^i \,\mathrm{d}u^j = -\frac{1}{\sqrt{1 + f_x^2 + f_y^2}} (f_{xx} \,\mathrm{d}x^2 + 2 f_{xy} \,\mathrm{d}x \,\mathrm{d}y + f_{yy} \,\mathrm{d}y^2)$$

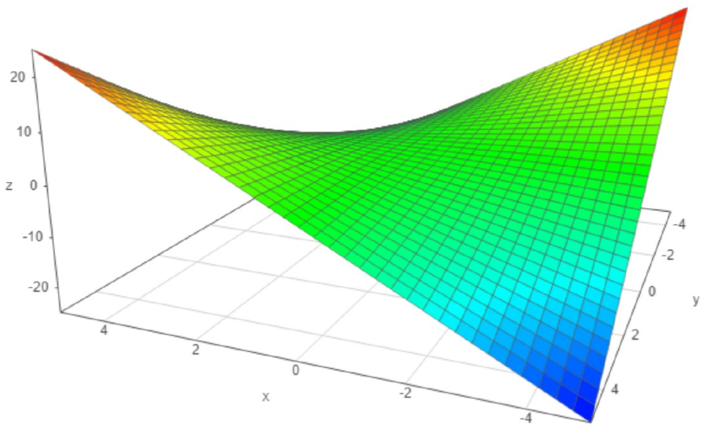
By definition the Gaussian curvature is given by

$$K = \frac{\det \mathrm{II}}{\det g} = \frac{f_{xx} f_{yy} - f_{xy}^2}{\left(1 + f_x^2 + f_y^2\right)^2}$$

For $f(x, y) = xy$, we have $f_x = y$, $f_y = x$, $f_{xx} = f_{yy} = 0$ and $f_{xy} = 1$. Then the Gaussian curvature is given by

$$K = -\frac{1}{(1 + x^2 + y^2)^2}$$

Sketch of the surface $z = xy$:



Great

□

Question 4

Let $\mathbf{r}(u, v)$ be a parametrized surface in $\mathbb{R}^3$ with $(u, v) \in U$, a connected open set in $\mathbb{R}^2$. Let S^2 denote the sphere of radius 1 with centre the origin in $\mathbb{R}^3$ and let $\mathbf{n} : U \to S^2$ be the mapping defined by assigning to each point of the surface the unit normal. Suppose that the restriction of $\mathbf{n}$ to U is a bijection onto $\mathbf{n}(U)$ and that the Gaussian curvature K is nowhere zero in U. Show that the area of $\mathbf{n}(U)$ equals the absolute value of $\int_U K \, \mathrm{d}A$.

Proof. The map $\mathrm{N} : \mathbf{r}(U) \to S^2$ given by $\mathrm{N}(\mathbf{r}(u,v)) = \mathbf{n}(u,v)$ is called the **Gauss map**. The surface element on $\mathbf{n}(U)$ is $\mathrm{d}\sigma_0 = |\mathbf{n}_u \wedge \mathbf{n}_v| \, \mathrm{d}u \, \mathrm{d}v$. The surface element on $\mathbf{r}(U)$ is given by $\mathrm{d}\sigma = |\mathbf{r}_u \wedge \mathbf{r}_v| \, \mathrm{d}u \, \mathrm{d}v$. So the area of $\mathbf{n}(U)$ is given by

$$\iint_U |\mathbf{n}_u \wedge \mathbf{n}_v| \, \mathrm{d}u \, \mathrm{d}v = \iint_{\mathbf{r}(U)} \frac{|\mathbf{n}_u \wedge \mathbf{n}_v|}{|\mathbf{r}_u \wedge \mathbf{r}_v|} \, \mathrm{d}\sigma$$

So it suffices to prove that the Gaussian curvature is given by

absolute value

$$|K| = \frac{|\mathbf{n}_u \wedge \mathbf{n}_v|}{|\mathbf{r}_u \wedge \mathbf{r}_v|}$$

This result is a part of the proof of Theorema Egregium (Page 65 in Hitchin's notes). Instead I shall prove it using the Weingarten map.

Let $X = \mathbf{r}(U)$. Gauss map induces the differential map $\mathrm{dN} : \mathrm{T}_{\mathbf{p}} X \to \mathrm{T}_{\mathrm{N}(\mathbf{p})} S^2$. Note that $\mathrm{T}_{\mathrm{N}(\mathbf{p})} S^2$ and $\mathrm{T}_{\mathbf{p}} X$ can be naturally identified as subspaces of $\mathbb{R}^3$. The Weingarten map is then defined as $W = -\mathrm{dN} : \mathrm{T}_{\mathbf{p}} X \to \mathrm{T}_{\mathbf{p}} X$. The Weingarten map also gives an alternative definition of the second fundamental form $\mathrm{II} \in \mathrm{T}^* X \otimes \mathrm{T}^* X$:

$$\mathrm{II}(\mathbf{a}, \mathbf{b}) := g(W(\mathbf{a}), \mathbf{b})$$

Consider the matrix representation of W in the basis $\{\mathbf{r}_u, \mathbf{r}_v\}$. Since $W(\mathbf{r}_u) = -\mathbf{n}_u$ and $W(\mathbf{r}_v) = -\mathbf{n}_v$, we have

$$-\begin{pmatrix} \mathbf{n}_u \\ \mathbf{n}_v \end{pmatrix} = W \begin{pmatrix} \mathbf{r}_u \\ \mathbf{r}_v \end{pmatrix} \implies -\begin{pmatrix} \mathbf{n}_u \\ \mathbf{n}_v \end{pmatrix} \begin{pmatrix} \mathbf{r}_u & \mathbf{r}_v \end{pmatrix} = W \begin{pmatrix} \mathbf{r}_u \\ \mathbf{r}_v \end{pmatrix} \begin{pmatrix} \mathbf{r}_u & \mathbf{r}_v \end{pmatrix} \implies \mathrm{II} = Wg$$

Hence $W = \mathrm{II} g^{-1}$, where II and g are the local matrix of the second and first fundamental forms. In particular we find that

$$\det W = \det(\mathrm{II}) \left(\det g\right)^{-1} =: K$$

So the Gaussian curvature K is the determinant of the Weingarten map W. We therefore deduce that

$$\|\mathbf{n}_u \wedge \mathbf{n}_v\| = |\det W| \, \|\mathbf{r}_u \wedge \mathbf{r}_v\| \implies |K| = \frac{|\mathbf{n}_u \wedge \mathbf{n}_v|}{|\mathbf{r}_u \wedge \mathbf{r}_v|}$$

which finishes the proof. You forgot absolute values. Need to argue how to remove them using $K \neq 0$, K cts, U connected. □

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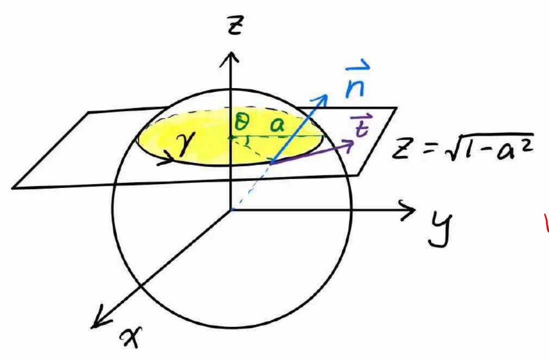
Question 5

Let S be the unit sphere in $\mathbb{R}^3$ and γ the circle obtained by intersecting S with the plane $z = \sqrt{1-a^2}$. Calculate the geodesic curvature of γ and the area of the smaller region of the sphere bounded by γ, and use these results to illustrate the Gauss-Bonnet theorem.

Proof. A diagram is drawn below:



The curve γ, parametrised by arc length, is given by

$$\gamma(s) = \left(a\cos\frac{s}{a}, a\sin\frac{s}{a}, \sqrt{1-a^2}\right)$$

Then the tangent vector along γ is

$$\mathbf{t} = \frac{\gamma'(s)}{\|\gamma'(s)\|} = \left(-\sin\frac{s}{a}, \cos\frac{s}{a}, 0\right)$$

and the derivative of tangent vector is given by

$$\mathbf{t}' = -\left(\frac{1}{a}\cos\frac{s}{a}, \frac{1}{a}\sin\frac{s}{a}, 0\right)$$

The normal vector is

$$\mathbf{n} = \frac{\gamma(s)}{\|\gamma(s)\|} = \left(a\cos\frac{s}{a}, a\sin\frac{s}{a}, \sqrt{1-a^2}\right)$$

By definition, the geodesic curvature is given by

$$\kappa_g := \mathbf{t}' \cdot (\mathbf{n} \wedge \mathbf{t}) = -\left(\frac{1}{a}\cos\frac{s}{a}, \frac{1}{a}\sin\frac{s}{a}, 0\right) \cdot \left(-\sqrt{1-a^2}\cos\frac{s}{a}, -\sqrt{1-a^2}\sin\frac{s}{a}, a\right) = \frac{\sqrt{1-a^2}}{a} = \frac{z}{a}$$

The smaller region bounded by γ has area:

$$A = \int_{\theta=0}^{\arccos z} \int_{\varphi=0}^{2\pi} \sin\theta \, \mathrm{d}\varphi \, \mathrm{d}\theta = 2\pi(1-z)$$

To illustrate the local Gauss-Bonnet Theorem, we still need to compute the Gaussian curvature of the sphere. But if we note that $\mathbf{n} = \mathbf{r}$ on the unit sphere, then we immediately find that the first and the second fundamental form of the unit sphere coincide. Hence the sphere has Gaussian curvature $K = 1$.

Finally,

$$\oint_\gamma \kappa_g \, \mathrm{d}s + \iint_A K \, \mathrm{d}\sigma = 2\pi a \cdot \frac{z}{a} + 2\pi(1-z) = 2\pi$$

which is consistent with the local form of the Gauss-Bonnet Theorem. $\square$