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Problem Sheet 1

B3.3: Algebraic Curves

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Question 1

Embed $\mathbb{R}^2$ in the projective plane $\mathbb{RP}^2$ by $(x, y) \mapsto [1, x, y]$. Find the point of intersection in $\mathbb{RP}^2$ of the projective lines corresponding to the parallel lines $y = mx$ and $y = mx + c$ in $\mathbb{R}^2$.

Proof. The embedding $(x, y) \mapsto [1 : x : y]$ maps $(0,0)$ to $[1 : 0 : 0]$, $(1, m)$ to $[1 : 1 : m]$, $(0, c)$ to $[1 : 0 : c]$, and $(1, m + c)$ to $[1 : 1 : m + c]$. The projective line corresponding to $y = mx$ is spanned by $(1,0,0)$ and $(1,0,m)$ in $\mathbb{R}^3$, and has homogeneous equation

$$mx - y = 0$$

The projective line corresponding to $y = mx + c$ is spanned by $(1,0,c)$ and $(1,1,m+c)$ in $\mathbb{R}^3$, and has homogeneous equation

$$cw + mx - y = 0$$

(In fact the line $ax + by + c = 0$ in $\mathbb{R}^2$ corresponds to the projective line $cw + ax + by = 0$ in $\mathbb{RP}^2$.)

Combining the two equations we deduce that the only point of intersection of the two projective lines is $[0 : 1 : m] \in \mathbb{RP}^2$. □

Question 2

Let $\mathbb{Z}_2$ be the field $\{0, 1\}$. Show that the number of points in n-dimensional projective space over $\mathbb{Z}_2$ is $2^{n+1} - 1$. How many projective lines are there in this space?

Proof. See Q3 for the general discussion.

For $p = 2$, $|\mathbb{Z}_2\mathbb{P}^n| = 2^{n+1} - 1$. The number of projective lines in $\mathbb{Z}_2\mathbb{P}^n$ is $\frac{1}{3}(2^{n+1} - 1)(2^n - 1)$. □

Question 3

What are the answers in Q2 if you instead work over the field $\mathbb{Z}_p$ with p elements, where p is prime?

Proof. First we count $|\mathbb{Z}_p\mathbb{P}^n|$. The $(n+1)$-dimensional vector space $\mathbb{Z}_p^{n+1}$ has cardinality p^{n+1}. Each vector $\mathbf{v} \in \mathbb{Z}_p^{n+1}$ spans a one-dimensional subspace:

$$\langle \mathbf{v} \rangle = \{0, \mathbf{v}, 2\mathbf{v}, \dots, (p-1)\mathbf{v}\}$$

Each one-dimensional subspace contains $p - 1$ non-zero vectors in $\mathbb{Z}_p^{n+1}$, and two distinct one-dimensional subspace only intersect at the origin. Therefore we deduce that

$$|\mathbb{Z}_p\mathbb{P}^n| = \frac{p^{n+1} - 1}{p - 1}$$

Next we count the projective lines in $\mathbb{Z}_p\mathbb{P}^n$. Note that every projective line is uniquely determined by a two-dimensional subspace of $\mathbb{Z}_p^{n+1}$. For any two distinct points $\langle \mathbf{v} \rangle, \langle \mathbf{w} \rangle$ in $\mathbb{Z}_p^{n+1}$, there is a unique projective line $\langle \mathbf{v}, \mathbf{w} \rangle$ that contains the two points. The total number of such choices is

$$\binom{|\mathbb{Z}_p\mathbb{P}^n|}{2} = \frac{p}{2} \frac{p^{n+1} - 1}{p - 1} \frac{p^n - 1}{p - 1}$$

For each projective line in $\mathbb{Z}_p\mathbb{P}^n$, the number of points on the line is

$$|\mathbb{Z}_p\mathbb{P}^1| = \frac{p^2 - 1}{p - 1} = p + 1$$

Hence each projective line is counted

$$\binom{p+1}{2} = \frac{1}{2}p(p+1)$$

times repeatedly. We deduce that the number of projective lines in $\mathbb{Z}_p\mathbb{P}^n$ is

$$\frac{(p^n - 1)(p^{n+1} - 1)}{(p - 1)^2(p + 1)}$$

□

Question 4

Show that

$$f : ([z_0, z_1], [w_0, w_1]) \mapsto [z_0 w_0, -z_0 w_1 - z_1 w_0, z_1 w_1]$$

is a well-defined map from $\mathbb{CP}^1 \times \mathbb{CP}^1$ to $\mathbb{CP}^2$. Also show that this map is surjective. Is the corresponding map $f : \mathbb{RP}^1 \times \mathbb{RP}^1 \to \mathbb{RP}^2$ surjective?

Proof. For $\lambda \in \mathbb{C} \setminus \{0\}$, we have $[z_0 : z_1] = [\lambda z_0 : \lambda z_1]$ and $[w_0 : w_1] = [\lambda w_0 : \lambda w_1]$.

$$f([\lambda z_0 : \lambda z_1], [w_0 : w_1]) = [\lambda z_0 w_0 : -\lambda z_0 w_1 - \lambda z_1 w_0 : \lambda z_1 w_1] = [z_0 w_0 : -z_0 w_1 - z_1 w_0 : z_1 w_1] = f([z_0 : z_1], [w_0 : w_1]);$$
$$f([z_0 : z_1], [\lambda w_0 : \lambda w_1]) = [\lambda z_0 w_0 : -\lambda z_0 w_1 - \lambda z_1 w_0 : \lambda z_1 w_1] = [z_0 w_0 : -z_0 w_1 - z_1 w_0 : z_1 w_1] = f([z_0 : z_1], [w_0 : w_1])$$

Hence $f : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^2$ is well-defined.

We shall show that f is surjective. That is, for any $[a : b : c] \in CP^2$, there exists $[z_0 : z_1], [w_0 : w_1] \in \mathbb{CP}^1$ such that $f([z_0 : z_1], [w_0 : w_1]) = [a : b : c]$.

- If $ac \neq 0$:

  By definition, we have $z_0 w_0 = a$, $z_1 w_1 = c$, $z_0 w_1 + z_1 w_0 = -b$. Let $\alpha := z_0 w_1$ and $\beta := z_1 w_0$. Then $\alpha + \beta = -b$ and $\alpha\beta = ac$. Therefore α, β be the roots of the quadratic equation

  $$x^2 + bx + ac = 0$$

  which has two complex roots

  $$\alpha, \beta = \frac{-b \pm \sqrt{b^2 + 4ac}}{2}$$

  Now we can take $[z_0 : z_1] = \left[1 : \frac{\beta}{a}\right] = [a : \beta]$, $[w_0 : w_1] = \left[1 : \frac{\alpha}{a}\right] = [a : \alpha]$. Then

  $$f([z_0 : z_1], [w_0 : w_1]) = [a^2 : -a(\alpha + \beta) : \alpha\beta] = [a^2 : ab : ac] = [a : b : c]$$

  as required.

- If either $a = 0$ or $c = 0$:

  By symmetry we consider $a \neq 0$ and $c = 0$. Note that the above construction where $[z_0 : z_1] = [a : \beta]$, $[w_0 : w_1] = [a : \alpha]$ still works.

- If $a = c = 0$:

  Here $b \neq 0$. We take $[z_0 : z_1] = [0 : \sqrt{b}\mathrm{i}]$, $[w_0 : w_1] = [\sqrt{b}\mathrm{i} : 0]$. Then

  $$f([z_0 : z_1], [w_0 : w_1]) = [0 : b : 0]$$

  as required.

The corresponding map from real projective spaces is not surjective. Consider the point $[1 : 1 : 1] \in \mathbb{RP}^2$. As above, we see that $\alpha := z_0 w_1$ and $\beta := z_1 w_0$ are the roots of the quadratic equation

$$x^2 + x + 1 = 0$$

But the equation has no real solutions. Hence we cannot find $[z_0 : z_1], [w_0 : w_1] \in \mathbb{RP}^1$ such that $f([z_0 : z_1], [w_0 : w_1]) = [1 : 1 : 1]$. □

Question 5

Adapt the ideas from the lectures to show that complex projective space $\mathbb{CP}^n$ is compact. What is the relationship between this space and a sphere of appropriate dimension?

Proof. The proof will depend on the definition of $\mathbb{CP}^n$. Here we starts from the algebraic definition: $\mathbb{CP}^n$ is the set of one-dimensional linear subspaces of $\mathbb{C}^{n+1}$.

First we note that every one-dimensional subspace of $\mathbb{C}^{n+1}$ is $\langle \mathbf{v} \rangle$ for some $\mathbf{v} \in \mathbb{C}^{n+1} \setminus \{0\}$, and $\langle \mathbf{v} \rangle = \langle \mathbf{w} \rangle$ if and only if there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\mathbf{v} = \lambda \mathbf{w}$. Then $\mathbb{CP}^n$ is equivalent to the quotient set $(\mathbb{C}^{n+1} \setminus \{0\}) / \sim$, where $\mathbf{v} \sim \mathbf{w}$ if and only if there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that $\mathbf{v} = \lambda \mathbf{w}$. In this way $\mathbb{CP}^n$ has the quotient topology induced from the Euclidean topology of $\mathbb{C}^{n+1}$.

Let $\pi : \mathbb{C}^{n+1} \to \mathbb{CP}^n$ be the quotient map. Consider the restriction $\pi|_{S^{2n+1}} : S^{2n+1} \to \mathbb{CP}^n$, where S^{2n+1} is the unit sphere in $\mathbb{C}^{n+1}$. $\pi|_{S^{2n+1}}$ is surjective, because for any $\langle \mathbf{v} \rangle \in \mathbb{CP}^n$, $\langle \mathbf{v} \rangle = p\left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right)$. By Heine-Borel Theorem S^n is compact. Hence $\mathbb{CP}^n = \pi(S^{2n+1})$ is compact, because π preserves compactness.

In fact, $\mathbb{CP}^n$ is homeomorphic to $S^{2n+1} / \sim$. The sphere S^{2n+1} is a $(2n+1)$-dimensional real manifold. □

Question 6

Prove Pappus's theorem by using the general position ideas outlined in lectures. First prove the theorem in the degenerate case when A, B, C', B' are not in general position. Then assume these points are in general position and take them to be $[1,0,0], [0,1,0], [0,0,1], [1,1,1]$. Calculate the three intersections explicitly, verify they are collinear, and explain why this proves the theorem in general.

Proof. *The is the same question as Question 8 in ASO Projective Geometry Sheet 1.*

If A, B, C', B' are not in general position, we may consider the case that C' lies in the projective line ABC. The other cases are similar.

If $C' \in ABC$, then $BC' \cap B'C = CA' \cap C'A = C$. Then C and $AB' \cap A'B$ are of course on the same projective line.

It follows from general position theorem that there exists a unique projective transformation such that

$$A \mapsto [1,0,0], \quad B \mapsto [0,1,0], \quad C' \mapsto [0,0,1] \quad B' \mapsto [1,1,1].$$

Clearly projective transformations preserve projective lines. So without loss of generality we can take

$$A = [1,0,0], \quad B = [0,1,0], \quad C' = [0,0,1] \quad B' = [1,1,1].$$

Since $C \in AB$, $C = [a, b, 0]$ for some $a, b \in F$. Since $A' \in C'B'$, $A' = [c : c : d]$ for some $c, d \in F$. A direct calculation shows that:

$$\langle \mathbf{x} \rangle = AB' \cap A'B = [c : d : d] \quad \langle \mathbf{y} \rangle = BC' \cap B'C = [0 : b-a : -a] \quad \langle \mathbf{z} \rangle = CA' \cap C'A = [(a-b)c : 0 : -bd]$$

Then we have $(b-a)\mathbf{x} - d\mathbf{y} + \mathbf{z} = 0$. Hence $AB' \cap A'B$, $BC' \cap B'C$ and $CA' \cap C'A$ are collinear. □