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Problem Sheet 1

B8.1: Probability, Measure & Martingales

Grade: Alpha++
This is beauty. Your solutions are better than sample solutions.

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Section 1

Question 1

Let Ω be a set.

(a) Show that the collection of all sets $A \in \mathscr{P}(\Omega)$ such that either A or A^c is countable is a σ-algebra.

(b) Give an example of two algebras $\mathscr{F}_1$ and $\mathscr{F}_2$ on $\Omega = \{1,2,3\}$ whose union $\mathscr{F}_1 \cup \mathscr{F}_2$ is *not* an algebra.

(c) (Proof of Lemma 1.3) Suppose that $\{\mathscr{F}_j\}_{j\in J}$ is a non-empty family of σ-algebras on Ω. Prove that the intersection $\bigcap_{j\in J} \mathscr{F}_j$ is a σ-algebra on Ω.

Proof. (a) In this question **countable** means **at most countable**. Let $\mathscr{F}$ be the collection.

- Since $\varnothing$ is at most countable, $\varnothing \in \mathscr{F}$;
- For $A \in \mathscr{F}$, either A or $\Omega \setminus A$ is at most countable. Thus $\Omega \setminus A \in \mathscr{F}$;
- Let I be a countable index set. For $\{A_i : i \in I\} \subseteq \mathscr{F}$, let $I = J \cup K$ such that $\{A_i : i \in J\}$ and $\{\Omega \setminus A_i : i \in K\}$ are collections of at most countable sets. If $K = \varnothing$, then

$$\bigcup\{A_i : i \in I\} = \bigcup\{A_i : i \in J\}$$

is a countable union of at most countable sets and hence is at most countable. We deduce that $\bigcup\{A_i : i \in I\} \in \mathscr{F}$.

If $K \neq \varnothing$, then fix A_k such that $\Omega \setminus A_k$ is at most countable. We have

$$\Omega \setminus \bigcup\{A_i : i \in I\} = \bigcap\{\Omega \setminus A_i : i \in I\} \subseteq \Omega \setminus A_k$$

$\Omega \setminus \bigcup\{A_i : i \in I\}$ is a subset of an at most countable set and hence is at most countable. We deduce that $\bigcup\{A_i : i \in I\} \in \mathscr{F}$.

We conclude that $\mathscr{F}$ is a σ-algebra.

(b) On $\Omega = \{1,2,3\}$, the following are algebras:

$$\mathscr{F}_1 = \{\varnothing, \{1\}, \{2,3\}, \{1,2,3\}\} \qquad \mathscr{F}_2 = \{\varnothing, \{2\}, \{1,3\}, \{1,2,3\}\}$$

However, $\mathscr{F}_1 \cup \mathscr{F}_2 = \{\varnothing, \{1\}, \{2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$ is not an algebra, because $\{1\}, \{2\} \in \mathscr{F}_1 \cup \mathscr{F}_2$ and $\{1,2\} = \{1\} \cup \{2\} \notin \mathscr{F}_1 \cup \mathscr{F}_2$.

(c)
- For $j \in J$, $\varnothing \in \mathscr{F}_j$. Hence $\varnothing \in \bigcap\{\mathscr{F}_j : j \in J\}$.
- $A \in \bigcap\{\mathscr{F}_j : j \in J\} \iff \forall\, j \in J:\ A \in \mathscr{F}_j \iff \forall\, j \in J:\ \Omega \setminus A \in \mathscr{F}_j \iff \Omega \setminus A \in \bigcap\{\mathscr{F}_j : j \in J\}$.
- $\{A_i : i \in \mathbb{N}\} \subseteq \bigcap\{\mathscr{F}_j : j \in J\} \iff \forall\, j \in J:\ \{A_i : i \in \mathbb{N}\} \subseteq \mathscr{F}_j \iff \forall\, j \in J:\ \bigcup\{A_i : i \in \mathbb{N}\} \in \mathscr{F}_j \iff \bigcup\{A_i : i \in \mathbb{N}\} \in \bigcap\{\mathscr{F}_j : j \in J\}$.

We conclude that $\bigcap\{A_j : j \in J\}$ is a σ-algebra. □

Question 2

Let $\mathscr{P} = \{P_j\}_{j\in J}$ be a partition of a set Ω (i.e., a collection of disjoint non-empty sets with union Ω). Show that the set $\mathscr{U}(\mathscr{P})$ consisting of all possible unions of sets P_j is a σ-algebra on Ω. Conversely, show that any σ-algebra $\mathscr{F}$ on a countable set Ω is of the form $\mathscr{U}(\mathscr{P})$ for some partition $\mathscr{P}$ of Ω.

[*Warning: the converse is very far from true when Ω is uncountable.*]

Proof. By definition, $\mathscr{U}(\mathscr{P}) = \left\{\bigcup\{P_j : j \in I\} :\ I \subseteq J\right\}$. First we show that $\mathscr{U}(\mathscr{P})$ is a σ-algebra on Ω:

- Since $\varnothing = \bigcup\{P_j : j \in \varnothing\}$, $\varnothing \in \mathscr{U}(\mathscr{P})$.
- For $A \in \mathscr{U}(\mathscr{P})$, there exists $I \subseteq J$ such that $A = \bigcup\{P_j : j \in I\}$. Since $\mathscr{P}$ is a partition of Ω,

$$\Omega = \bigcup\{P_j : j \in J\} = \bigcup\{P_j : j \in I\} \cup \bigcup\{P_j : j \in J \setminus I\}$$

Hence $\Omega\backslash A = \bigcup\{P_j : j \in J\backslash I\} \in \mathscr{U}(\mathscr{P})$.

- For $\{A_i : i \in \mathbb{N}\} \subseteq \mathscr{U}(\mathscr{P})$, $A_i = \bigcup\{P_{i,j} : j \in I_i\}$ for some $I_i \subseteq J$. Then

$$\bigcup\{A_i : i \in \mathbb{N}\} = \bigcup\{P_{i,j} : j \in I_i, i \in \mathbb{N}\} = \bigcup\left\{P_j : j \in \bigcup\{I_i : i \in \mathbb{N}\}\right\} \in \mathscr{U}(\mathscr{P})$$

Hence $\bigcup\{A_i : i \in \mathbb{N}\} \in \mathscr{U}(\mathscr{P})$.

We conclude that $\mathscr{U}(\mathscr{P})$ is a σ-algebra on Ω.

Conversely, let $\mathscr{F}$ be a σ-algebra on a countable set Ω. We define a equivalence relation on Ω: For $x, y \in \Omega$,

$$x \sim y \iff \forall A \in \mathscr{F}\ (x \in A \iff y \in A)$$

Let $\mathscr{P}$ be the partition defined by this equivalence relation. We claim that $\mathscr{F} = \mathscr{U}(\mathscr{P})$.

For $x \in \Omega$, let $[x]$ denotes the equivalence class of x. For $A \in \mathscr{F}$, if $x \in A$ then $[x] \subseteq A$ by definition. Hence A is a union of some equivalence classes. In particular $A \in \mathscr{U}(\mathscr{P})$. Hence $\mathscr{F} \subseteq \mathscr{U}(\mathscr{P})$. On the other hand, for $x \in \Omega$,

$$[x] = \bigcap\{A \in \mathscr{F} : x \in A\} \implies \Omega\backslash[x] = \bigcup\{A \in \mathscr{F} : x \notin A\}$$

Since $\Omega\backslash[x] \subseteq \Omega$, $\Omega\backslash[x]$ is at most countable. Write $\Omega\backslash[x] = \{y_i : i \in \mathbb{N}\}$ where $y_i \in A_i \in \mathscr{F}$ and $x \notin A_i$. Then

$$\Omega\backslash[x] = \bigcup\{\{y_i\} : i \in \mathbb{N}\} = \bigcup\{A_i : i \in \mathbb{N}\}$$

The union is now countable. Hence $\Omega\backslash[x] \in \mathscr{F}$ and $[x] \in \mathscr{F}$.

Finally, for $A \in \mathscr{U}(\mathscr{P})$, since $A \subseteq \Omega$ is at most countable, $A = \bigcup\{[x_j] : j \in I\}$ for some at most countable index set I. Hence $A \in \mathscr{F}$. $\mathscr{U}(\mathscr{P}) \subseteq \mathscr{F}$.

We conclude that $\mathscr{U}(\mathscr{P}) = \mathscr{F}$. □

Question 3

Let (f_n) be a sequence of measurable functions on $(\Omega, \mathscr{F})$ taking values in $\overline{\mathbb{R}}$. Show that $f_1 + f_2, \max\{f_1, f_2\}$ and $\sup_n f_n$ are all measurable functions.

Proof. $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ is a linearly and densely ordered set with the topology generated by the basis:

$$\{(a,b) : a, b \in \mathbb{R}, a < b\} \cup \{[-\infty, a) : a \in \mathbb{R}\} \cup \{(a, +\infty] : a \in \mathbb{R}\}$$

For a map $f : \Omega \to \overline{\mathbb{R}}$, by Corollary 1.19 f is $\mathscr{F}$-measurable if and only if $\{x \in \overline{\mathbb{R}} : f(x) \leqslant t\} \in \mathscr{F}$ for all $t \in \mathbb{R}$.

1. $f_1 + f_2$ is measurable:

   For $t \in \mathbb{R}$, note that

$$\{x \in \Omega :\ (f_1 + f_2)(x) \leqslant t\} = \bigcup_{q \in \mathbb{Q}} \{x \in \Omega :\ f_1(x) \leqslant q \wedge f_2(x) \leqslant t - q\} = \bigcup_{q \in \mathbb{Q}} \left(\{x \in \Omega :\ f_1(x) \leqslant q\} \cap \{x \in \Omega :\ f_2(x) \leqslant t - q\}\right)$$

   For each $q \in \mathbb{Q}$, since f_1, f_2 are measurable, $\{x \in \Omega :\ f_1(x) \leqslant q\}, \{x \in \Omega :\ f_2(x) \leqslant t - q\} \in \mathscr{F}$. Hence $\{x \in \Omega :\ f_1(x) \leqslant q\} \cap \{x \in \Omega :\ f_2(x) \leqslant t - q\} \in \mathscr{F}$. As $\mathbb{Q}$ is countable, we deduce that the countable union $\{x \in \Omega :\ (f_1 + f_2)(x) \leqslant t\} \in \mathscr{F}$. Hence $f_1 + f_2$ is measurable.

2. We show that $h \circ f_1$ is measurable for any continuous map $h : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$.

   Since h is continuous, the pullback of a Borel set in $\overline{\mathbb{R}}$ under h is also Borel. Hence $(h \circ f_1)^{-1}(A) = f_1^{-1}\left(h^{-1}(A)\right) \in \mathscr{F}$ for any Borel set $A \subseteq \overline{\mathbb{R}}$. Hence $h \circ f_1$ is measurable.

3. $\max\{f_1, f_2\}$ is measurable:

   Note that $\max\{f_1, f_2\} = \dfrac{1}{2}\left((f_1 + f_2) + |f_1 - f_2|\right)$. Since f_2 is measurable, $-f_2 = h_1 \circ f_2$ is measurable, where $h_1(x) = -x$ is continuous. Hence $f_1 - f_2$ is measurable. Hence $|f_1 - f_2| = h_2 \circ (f_1 - f_2)$ is measurable, where $h_2(x) = |x|$ is continuous. Hence $g := (f_1 + f_2) + |f_1 - f_2|$ is measurable. Hence $\max\{f_1, f_2\} = h_3 \circ g$ is measurable, where $h_3(x) = \dfrac{1}{2}x$ is continuous.

4. $(\sup_n f_n)(x) := \sup\{f_n(x) : n \in \mathbb{N}\}$ is measurable:

   For $t \in \mathbb{R}$,
$$\left\{x \in \Omega : \sup_n f_n(x) \leqslant t\right\} = \bigcap_{n\in\mathbb{N}} \{x \in \Omega : f_n(x) \leqslant t\} = \Omega \setminus \left(\bigcup_{n\in\mathbb{N}} \Omega \setminus \{x \in \Omega : f_n(x) \leqslant t\}\right) \in \mathscr{F}$$
   since $\{x \in \Omega : f_n(x) \leqslant t\} \in \mathscr{F}$ for each $n \in \mathbb{N}$. Hence $\sup_n f_n$ is measurable. □

Question 4

(Lemma 1.29) Let $(\Omega, \mathscr{F})$ be the product space of two measurable spaces $(\Omega_i, \mathscr{F}_i)$, $i = 1, 2$. Show that if $f : \Omega \to \mathbb{R}$ is measurable then

- for each $\omega_1 \in \Omega_1$, $\Omega_2 \ni \omega_2 \to f(\omega_1, \omega_2)$ is $\mathscr{F}_2$-measurable and
- for each $\omega_2 \in \Omega_2$, $\Omega_1 \ni \omega_1 \to f(\omega_1, \omega_2)$ is $\mathscr{F}_1$-measurable.

(Exercise 1.13) Deduce that if $D \in \mathscr{F}$ and $D(\omega_1) := \{\omega_2 : (\omega_1, \omega_2) \in D\}$ is its section for a fixed $\omega_1 \in \Omega_1$ then $D(\omega_1) \in \mathscr{F}_2$.

Proof. In this question we use $\mathscr{F}_1 \times \mathscr{F}_2$ to denote $\{A_1 \times A_2 : A_1 \in \mathscr{F}_1, A_2 \in \mathscr{F}_2\}$ and use $\mathscr{F}_1 \otimes \mathscr{F}_2$ to denote the σ-algebra generated by $\mathscr{F}_1 \times \mathscr{F}_2$.

First we prove the result in Exercise 1.13.

Let $\mathscr{A}(\omega_1) := \{D \in \mathscr{F}_1 \otimes \mathscr{F}_2 :\ D(\omega_1) \in \mathscr{F}_2\} \subseteq \mathscr{F}_1 \otimes \mathscr{F}_2$ for a fixed $\omega_1 \in \Omega_1$. First we show that $\mathscr{A}(\omega_1)$ is a σ-algebra:

- Since $\varnothing(\omega_1) = \varnothing \in \mathscr{F}_2$, $\varnothing \in \mathscr{A}(\omega_1)$.
- For $D \in \mathscr{A}(\omega_1)$:
$$D(\omega_1) \in \mathscr{F}_2 \implies \Omega \setminus D(\omega_1) = ((\Omega_1 \times \Omega_2) \setminus D)(\omega_1) \in \mathscr{F}_2 \implies (\Omega_1 \times \Omega_2) \setminus D \in \mathscr{A}(\omega_1)$$
- For $\{D_i : i \in \mathbb{N}\} \subseteq \mathscr{A}(\omega_1)$:
$$\forall\, i \in \mathbb{N}\ D_i(\omega_1) \in \mathscr{F}_2 \implies \bigcup_{i\in\mathbb{N}} D_i(\omega_1) = \left(\bigcup_{i\in\mathbb{N}} D_i\right)(\omega_1) \in \mathscr{F}_2 \implies \bigcup_{i\in\mathbb{N}} D_i \in \mathscr{A}(\omega_1)$$

Next, for $D \in \mathscr{F}_1 \times \mathscr{F}_2$, $D = A_1 \times A_2$ for some $A_1 \in \mathscr{F}_1$ and $A_2 \in \mathscr{F}_2$. Hence $D(\omega_1) = A_2 \in \mathscr{F}_2$. In particular $\mathscr{F}_1 \times \mathscr{F}_2 \subseteq \mathscr{A}(\omega_1)$. Therefore $\mathscr{F}_1 \otimes \mathscr{F}_2 = \sigma(\mathscr{F}_1 \times \mathscr{F}_2) \subseteq \sigma(\mathscr{A}(\omega_1)) = \mathscr{A}(\omega_1)$. We deduce that $\mathscr{A}(\omega_1) = \mathscr{F}_1 \otimes \mathscr{F}_2$.

Now we prove Lemma 1.29. By symmetry it suffices to prove one result: $\omega_2 \mapsto f(\omega_1, \omega_2)$ is $\mathscr{F}_2$-measurable for each $\omega_1 \in \Omega_1$.

For $D \in \mathscr{F}_1 \otimes \mathscr{F}_2$, the indicator function $\mathbf{1}_D$ is $\mathscr{F}$-measurable. The map $\omega_2 \mapsto \mathbf{1}_D(\omega_1, \omega_2)$ is in fact the indicator function $\mathbf{1}_{D(\omega_1)}$ on Ω_2. By the previous proof we have shown that $D(\omega_1) \in \mathscr{F}_2$. Therefore $\mathbf{1}_{D(\omega_1)}$ is $\mathscr{F}_2$-measurable.

For a simple function $\varphi = \sum_i c_i \mathbf{1}_{D_i}$, $\omega_2 \mapsto \varphi(\omega_1, \omega_2)$ is $\sum_i c_i \mathbf{1}_{D_i(\omega_1)}$ and hence is $\mathscr{F}_2$-measurable.

For a $\mathscr{F}$-measurable function $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$, by Lemma 1.26 there exists a increasing sequence of simple function $\{\varphi_n\}$ such that $\varphi_n \uparrow f$. Then $\omega_2 \mapsto f(\omega_1, \omega_2)$ is the limit of $\omega_2 \mapsto \varphi_n(\omega_1, \omega_2)$ as $n \to \infty$. Again by Lemma 1.26 we deduce that $\omega_2 \mapsto f(\omega_1, \omega_2)$ is $\mathscr{F}_2$-measurable. □

Question 5. Proof of Theorem 2.10

Let μ_1, μ_2 be two finite measures on $(\Omega, \mathscr{F})$ with $\mu_1(\Omega) = \mu_2(\Omega)$. Verify that $\{A \in \mathscr{F} : \mu_1(A) = \mu_2(A)\}$ is a λ-system.

Proof. Let $\mathscr{A} := \{A \in \mathscr{F} : \mu_1(A) = \mu_2(A)\}$

- $\mu_1(\Omega) = \mu_2(\Omega)$ implies that $\Omega \in \mathscr{A}$.
- For $A, B \in \mathscr{A}$ with $A \subseteq B$, we have $\mu_1(A) = \mu_2(A)$ and $\mu_1(B) = \mu_2(B)$. By additivity of μ_1, $\mu_1(B) = \mu_1(A \cup B \setminus A) = \mu_1(A) + \mu_1(B \setminus A)$. Similarly $\mu_2(B) = \mu_2(A) + \mu_2(B \setminus A)$. Hence $\mu_1(B \setminus A) = \mu_2(B \setminus A)$. $B \setminus A \in \mathscr{A}$.
- Let $\{A_n : n \in \mathbb{N}\} \subseteq \mathscr{A}$ be an ascending chain. By continuity of μ_1 and μ_2,
$$\mu_1\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{n\to\infty} \mu_1(A_n) = \lim_{n\to\infty} \mu_2(A_n) = \mu_2\left(\bigcup_{n\in\mathbb{N}} A_n\right)$$

Hence $\bigcup\{A_n : n \in \mathbb{N}\} \in \mathscr{A}$.

We conclude that $\mathscr{A}$ is a λ-system. ✓ $\square$

Question 6

Use π-λ systems Lemma to prove the Monotone Class Theorem (Theorem 1.28) in the case when $\mathscr{C} = \{\mathbf{1}_A : A \in \mathscr{A}\}$ for a π-system $\mathscr{A}$.

Proof. Consider $\mathscr{E} := \{E \subseteq \Omega : \mathbf{1}_E \in \mathscr{H}\}$. First we show that $\mathscr{E}$ is a λ-system.

- $\mathbf{1}_\Omega \in \mathscr{H}$ implies that $\Omega \in \mathscr{E}$. ✓
- For $A, B \in \mathscr{E}$ with $A \subseteq B$, $\mathbf{1}_A, \mathbf{1}_B \in \mathscr{H}$. As $\mathscr{H}$ is a vector space, $\mathbf{1}_{B\setminus A} = \mathbf{1}_B - \mathbf{1}_A \in \mathscr{H}$. Hence $B \setminus A \in \mathscr{E}$. ✓
- Let $\{A_n : n \in \mathbb{N}\} \subseteq \mathscr{E}$ be an ascending chain. Then $\{\mathbf{1}_{A_n} : n \in \mathbb{N}\}$ is an ascending chain in $\mathscr{H}$ with limit $\mathbf{1}_{\bigcup\{A_n : n\in\mathbb{N}\}}$. Hence $\mathbf{1}_{\bigcup\{A_n:n\in\mathbb{N}\}} \in \mathscr{H}$. Hence $\bigcup\{A_n : n \in \mathbb{N}\} \in \mathscr{E}$. ✓

Since $\mathscr{C} \subseteq \mathscr{H}$, $\mathscr{A} \subseteq \mathscr{E}$. By π-λ systems Lemma, $\sigma(\mathscr{A}) \subseteq \mathscr{E}$.

Let $f : \Omega \to \mathbb{R}$ be a bounded $\sigma(\mathscr{C})$-measurable function. By Lemma 1.26 there exists a sequence of simple functions

$$\varphi_n = \sum_{i=1}^{k_n} c_{i,n} \mathbf{1}_{E_{i,n}}$$

such that $\varphi_n \uparrow f$, where $E_{i,n} \in \sigma(\mathscr{C})$.

As $\sigma(\mathscr{C})$ is the smallest σ-algebra such that every function in $\mathscr{C}$ is measurable and $\mathscr{C} = \{\mathbf{1}_A : A \in \mathscr{A}\}$, we have $\sigma(\mathscr{C}) = \sigma(\mathscr{A})$. $E_{i,n} \in \sigma(\mathscr{A}) \subseteq \mathscr{E}$ implies that $\mathbf{1}_{E_{i,n}} \in \mathscr{H}$. Since $\mathscr{H}$ is a vector space, $\varphi_n \in \mathscr{H}$. Since $\varphi_n \uparrow f$, we deduce that $f \in \mathscr{H}$.

We conclude that $\mathscr{H}$ contains all bounded $\sigma(\mathscr{C})$-measurable functions. ✓ $\square$

Question 7. Lemma 2.4

Let $\mu : \mathscr{A} \to [0, \infty)$ be an additive set function on an algebra $\mathscr{A}$ taking only finite values. Show that μ is countably additive iff for every sequence (A_n) of sets in $\mathscr{A}$ with $A_n \downarrow \emptyset$ we have $\mu(A_n) \to 0$.

Proof. "$\Longrightarrow$": Suppose that μ is countably additive. By Proposition 2.3(v), for any descending chain $\{A_n : n \in \mathbb{N}\}$ with $\bigcap\{A_n : n \in \mathbb{N}\} = \emptyset$,

$$\lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \mu(\emptyset) = 0$$

"$\Longleftarrow$": Suppose that for any descending chain $\{A_n : n \in \mathbb{N}\}$ with $\bigcap\{A_n : n \in \mathbb{N}\} = \emptyset$ we have $\lim_{n\to\infty} \mu(A_n) = 0$.

Consider a countable collection of disjoint sets $\{A_n : n \in \mathbb{N}\}$. Let $A := \bigcup_{n\in\mathbb{N}} A_n$ and $E_k := A \setminus \bigcup_{n=0}^{k} A_n$. Then $E_n \downarrow 0$. From finite additivity we have

$$0 = \lim_{n\to\infty} \mu(E_n) = \lim_{k\to\infty} \mu\left(A \setminus \bigcup_{n=0}^{k} A_n\right) = \mu(A) - \lim_{k\to\infty} \sum_{n=0}^{k} \mu(A_n) = \mu(A) - \sum_{n=0}^{\infty} \mu(A_n)$$

Hence $\sum_{n=0}^{\infty} \mu(A_n) = \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right)$. We conclude that μ is countably additive. $\square$

Question 8

On $\mathbb{R}$ consider the σ-algebra $\mathscr{A}$ of sets which are either countable or have a countable complement. Let $\mu(A) = 0$ for countable A and $\mu(A) = 1$ otherwise , $A \in \mathscr{A}$. Show that μ is a probability measure on $\mathscr{A}$.

Proof. Since $\mathbb{R}$ is uncountable, $\mu(\mathbb{R}) = 1$. So it suffices to shown that μ is a measure on $(\mathbb{R}, \mathscr{A})$.

- Since $\emptyset$ is at most countable, $\mu(\emptyset) = 0$. ✓

- Suppose that $\{A_n : n \in \mathbb{N}\} \subseteq \mathscr{A}$ is a countable collection of disjoint sets.

  If there exists A_i, A_j uncountable with $A_i \cap A_j = \varnothing$, then $\mathbb{R} \backslash A_i \supseteq A_j$ is not countable, contradiction. Hence $\bigcup\{A_n : n \in \mathbb{N}\}$ can have at most one uncountable element. ✓

  If every A_n is countable, then $\bigcup\{A_n : n \in \mathbb{N}\}$ is a countable union of countable sets, and hence is countable. We have:

$$\sum_{n=0}^{\infty} \mu(A_n) = \sum_{n=0}^{\infty} 0 = 0 = \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right)$$

✓

  Suppose that A_i is uncountable and the rest are countable. Then $\bigcup\{A_n : n \in \mathbb{N}\}$ is uncountable and

$$\sum_{n=0}^{\infty} \mu(A_n) = \mu(A_i) + \sum_{n \neq i} \mu(A_n) = 1 = \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right)$$

✓

  Hence μ is countably additive. □

Question 9

Let Ω be a set and $\mathscr{I}$ a π-system on Ω. Let $\mathscr{F} = \sigma(\mathscr{I})$. Suppose that μ_1 and μ_2 are two measures on $(\Omega, \mathscr{F})$ with $\mu_1(\Omega) = \mu_2(\Omega) < \infty$ and $\mu_1 = \mu_2$ on $\mathscr{I}$. Then Theorem 2.10 on uniqueness of extension says that $\mu_1 = \mu_2$ on $\mathscr{F}$. Find an example on $\Omega = \{1,2,3,4\}$ where this fails if we drop the assumption that $\mathscr{I}$ is a π-system.

Proof. Let $\mathscr{I} = \{\{1,2\},\{2,3\}\}$. Then $\mathscr{I}$ is not a π-system and $\sigma(\mathscr{I}) = \mathscr{P}(\Omega)$. Let $\mu_1 : \mathscr{P}(\Omega) \to [0,+\infty)$ be a measure such that

$$\mu_1(\{1\}) = 0, \qquad \mu_1(\{2\}) = 1, \qquad \mu_1(\{3\}) = 0, \qquad \mu_1(\{4\}) = 1$$

Let $\mu_2 : \mathscr{P}(\Omega) \to [0,+\infty)$ be a measure such that

$$\mu_2(\{1\}) = 1, \qquad \mu_2(\{2\}) = 0, \qquad \mu_2(\{3\}) = 1, \qquad \mu_2(\{4\}) = 0$$

Then

$$\mu_1(\{1,2\}) = \mu_2(\{1,2\}) = 1, \qquad \mu_1(\{1,2\}) = \mu_2(\{1,2\}) = 1, \qquad \mu_1(\Omega) = \mu_2(\Omega) = 2$$

But it is clear that $\mu_1 \neq \mu_2$. ✓ □

Question 10

Let $\Psi = \Phi^{-1}$ the be inverse the CDF of a standard normal random variable. On the probability space $((0,1), \mathscr{B}((0,1)), \texttt{Leb})$ define $X(\omega) = \Psi(\omega)$ and

$$Y_1(\omega) = \mathbf{1}_{(0,0.5)}(\omega) - \mathbf{1}_{[0.5,1)}(\omega)$$
$$Y_2(\omega) = \mathbf{1}_{(0,0.25)}(\omega) + \mathbf{1}_{[0.75,1)}(\omega) - \mathbf{1}_{[0.25,0.75)}(\omega)$$

- Show that $X \sim \mathcal{N}(0,1)$ and $Y_1 \sim Y_2$ are distributed according to $\nu = (\delta_1 + \delta_{-1})/2$.
- Describe the joint distributions of (X, Y_1) and (X, Y_2) on $\mathbb{R}^2$. Are they the same?
- Let μ be the distribution of $|X|$. Give an example of a couple of random variables distributed according to $\mu \otimes \nu$.

Proof.

1. Let F_X be the distribution function of X. Then

$$F_X(x) = m(\{\omega \in (0,1) : X(\omega) \leqslant x\}) = m\left(\{\omega \in (0,1) : \Phi^{-1}(\omega) \leqslant x\}\right) = m(\{\omega \in (0,1) : \omega \leqslant \Phi(x)\}) = \Phi(x)$$

   ✓ where m is the Lebesgue measure on $(0,1)$. Since Φ is the distribution function of $N(0,1)$, we deduce that $X \sim N(0,1)$.

2. Let F_{Y_1} and F_{Y_2} be the distribution functions of Y_1 and Y_2 respectively. Then for Y_1, $\mathbf{1}_{(0,1/2)}(\omega) - \mathbf{1}_{[1/2,1)}(\omega)$ is equal to -1

for $\omega \in [1/2, 1)$ and 1 for $\omega \in [0, 1/2)$. Hence

$$F_{Y_1}(x) = m\left(\left\{\omega \in (0,1) : \mathbf{1}_{(0,1/2)}(\omega) - \mathbf{1}_{[1/2,1)}(\omega) \leqslant x\right\}\right) = \begin{cases} 0, & x \in (-\infty, -1) \\ 1/2, & x \in [-1, 1) \\ 1, & x \in [1, \infty) \end{cases}$$

For Y_2, $\mathbf{1}_{(0,1/4)}(\omega) + \mathbf{1}_{[3/4,1)}(\omega) - \mathbf{1}_{[1/4,3/4)}(\omega)$ is equal to -1 for $\omega \in [1/4, 3/4)$ and 1 for $\omega \in [0, 1/4) \cup [3/4, 1)$. Hence

$$F_{Y_2}(x) = m\left(\left\{\omega \in (0,1) : \mathbf{1}_{(0,1/4)}(\omega) + \mathbf{1}_{[3/4,1)}(\omega) - \mathbf{1}_{[1/4,3/4)}(\omega) \leqslant x\right\}\right) = \begin{cases} 0, & x \in (-\infty, -1) \\ 1/2, & x \in [-1, 1) \\ 1, & x \in [1, \infty) \end{cases}$$

Then $F_{Y_1} = F_{Y_2}$ and hence $Y_1 \sim Y_2$. In particular, the distribution function induces the push-forward measure $\nu = (\delta_1 + \delta_{-1})/2$.

3. Let F_{X,Y_1} be the joint distribution function of X and Y_1. Then

$$F_{X,Y_1}(x, y) = \begin{cases} 0, & y \in (-\infty, -1) \\ m\left((0, \Phi(x)) \cap [1/2, 1)\right), & y \in [-1, 1) \\ m\left((0, \Phi(x)) \cap (0, 1/2)\right), & y \in [1, \infty) \end{cases}$$

Let F_{X,Y_2} be the joint distribution function of X and Y_2. Then

$$F_{X,Y_2}(x, y) = \begin{cases} 0, & y \in (-\infty, -1) \\ m\left((0, \Phi(x)) \cap [1/4, 3/4)\right), & y \in [-1, 1) \\ m\left((0, \Phi(x)) \cap ([0, 1/4) \cup [3/4, 1))\right), & y \in [1, \infty) \end{cases}$$

Note that $F_{X,Y_1}(1/4, 1) = 1/2$ and $F_{X,Y_2}(1/4, 1) = 1/4$. So the two distruibutions are not the same.

4. Observe that

$$F_{|X|}(x) = m\left(\{\omega \in (0,1) : |X(\omega)| \leqslant x\}\right) = m\left(\{\omega \in (0,1) : -x \leqslant X(\omega) \leqslant x\}\right) = \Phi(x) - \Phi(-x)$$

Consider the probability space $\left((0,1)^2, \mathscr{B}\left((0,1)^2\right), \texttt{Leb}\right)$. Let $A, B : (0,1)^2 \to \mathbb{R}$ be random variables defined by

$$A(\omega_1, \omega_2) := F_{|X|}^{-1}(\omega_1)$$
$$B(\omega_1, \omega_2) := \mathbf{1}_{(0,0.5)}(\omega_2) - \mathbf{1}_{[0.5,1)}(\omega_2)$$

Then A and B are independent, and $A \sim |X|$ and $B \sim Y_1$. Hence (A, B) induces the push-forward measure $\mu \otimes \nu$ on $\left(\mathbb{R}^2, \mathscr{B}\left(\mathbb{R}^2\right)\right)$. $\square$