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# **Problem Sheet 4**
# B1.1: Logic

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**Question 1** β

(i) Let $\mathcal{L}$ be a first-order language, let $\mathcal{A}$ be an $\mathcal{L}$-structure, let $v$ be an assignment in $\mathcal{A}$ and let $u$ and $t$ be terms in $\mathcal{L}$. Define a new assignment $v'$ by

$$v'(x_j) := \begin{cases} v(x_j) & \text{if } j \neq i \\ \tilde{v}(t) & \text{if } j = i \end{cases}$$

Let $u[t/x_i]$ be the term obtained by replacing each occurrence of $x_i$ in $u$ by $t$. Show that then $\tilde{v'}(u) = \tilde{v}(u[t/x_i])$.

(ii) Prove that for any closed (i.e. variable free) terms $t_1, t_2, t_3$ one has $\vdash (t_1 \doteq t_2 \to t_2 \doteq t_1)$ and $\{t_1 \doteq t_2, t_2 \doteq t_3\} \vdash t_1 \doteq t_3$.

*Proof.* (i) We process by induction on the length of $u$.

What about constant symbols?

Base case: Suppose that $u$ has length 1. Then $u = x_k$ for some $i \in \mathbb{N}$. If $i = k$, then $u[t/x_i] = t$, and $\tilde{v'}(u) = v'(x_i) = \tilde{v}(t) = \tilde{v}(u[t/x_i])$. If $i \neq k$, then $u[t/x_i] = x_k = u$, and $\tilde{v'}(u) = v'(x_k) = v(x_k) = \tilde{v}(u[t/x_i])$.

Induction case: Suppose that the result holds for $u$ with length less than $n$. Now assume that $u$ has length $n$. Then $u = f_k(u_1, ..., u_m)$ for some terms $u_1, ..., u_m$. By induction hypothesis, we have $\tilde{v'}(u_\ell) = \tilde{v}(u_\ell[t/x_i])$ for $\ell \in \{1, ..., m\}$. We have

$$\begin{aligned}\tilde{v'}(u) &= \tilde{v'}(f_k(u_1, ..., u_m)) = \overline{f_k}(\tilde{v'}(u_1), ..., \tilde{v'}(u_m)) = \overline{f_k}(\tilde{v}(u_1[t/x_i]), ..., \tilde{v}(u_m[t/x_i])) \\ &= \tilde{v}(f_k(u_1[t/x_i], ..., u_m[t/x_i])) = \tilde{v}(u[t/x_i]).\end{aligned}$$

✓

(ii) Proof of $\vdash (t_1 \doteq t_2 \to t_2 \doteq t_1)$: By Deduction Theorem, it suffices to prove that $t_1 \doteq t_2 \vdash t_2 \doteq t_1$. Suppose that $x_k$ is a variable that does not occur free in $t_1$ and $t_2$.

| | | |
|---|---|---|
| $\alpha_1$: | $\forall x_k(x_k \doteq x_k)$ | [A6] |
| $\alpha_2$: | $(\forall x_k(x_k \doteq x_k) \to t_1 \doteq t_1)$ | [A4] |
| $\alpha_3$: | $t_1 \doteq t_1$ | [MP $\alpha_1, \alpha_2$] |
| $\alpha_4$: | $(t_1 \doteq t_2 \to (t_1 \doteq t_1 \to t_2 \doteq t_1))$ | [A7] ← |
| $\alpha_5$: | $t_1 \doteq t_2$ | [Premise] |
| $\alpha_6$: | $(t_1 \doteq t_1 \to t_2 \doteq t_1)$ | [MP $\alpha_4, \alpha_5$] |
| $\alpha_7$: | $t_2 \doteq t_1$ | [MP $\alpha_3, \alpha_6$] |

No, you cannot use A7 like this! Here t1 and t2 are terms, but K(L) as defined in lecture only has "change of variables" as A7
You should first prove
xi = xj → xj = xi
and then use A4 twice

Proof of $\{t_1 \doteq t_2, t_2 \doteq t_3\} \vdash t_1 \doteq t_3$: Suppose that $x_k$ is a variable that does not occur free in $t_1$, $t_2$ and $t_3$.

| | | |
|---|---|---|
| $\alpha_1$: | $\forall x_k(x_k \doteq x_k)$ | [A6] |
| $\alpha_2$: | $(\forall x_k(x_k \doteq x_k) \to t_1 \doteq t_1)$ | [A4] |
| $\alpha_3$: | $t_1 \doteq t_1$ | [MP $\alpha_1, \alpha_2$] |
| $\alpha_4$: | $(t_1 \doteq t_2 \to (t_1 \doteq t_1 \to t_2 \doteq t_1))$ | [A7] |
| $\alpha_5$: | $t_1 \doteq t_2$ | [Premise] |
| $\alpha_6$: | $(t_1 \doteq t_1 \to t_2 \doteq t_1)$ | [MP $\alpha_4, \alpha_5$] |
| $\alpha_7$: | $t_2 \doteq t_1$ | [MP $\alpha_3, \alpha_6$] |
| $\alpha_8$: | $(t_2 \doteq t_1 \to (t_2 \doteq t_3 \to t_1 \doteq t_3))$ | [A7] |
| $\alpha_9$: | $(t_2 \doteq t_3 \to t_1 \doteq t_3)$ | [MP $\alpha_7, \alpha_8$] |
| $\alpha_{10}$: | $t_2 \doteq t_3$ | [Premise] |
| $\alpha_{11}$: | $t_1 \doteq t_3$ | [MP $\alpha_9, \alpha_{10}$] |

Similar problems with A7

□

**Question 2** α

(i) Prove that $\vdash (\forall x_i(A \to B) \to (\exists x_i A \to B))$ for any formulae $A, B$ provided that the variable $x_i$ does not occur free in $B$.

(ii) Let $\phi$ be a formula with just one variable $x_i$, occurring free and let $\Delta$ be a set of sentences. Assume that the constant symbol $c_j$ does not occur in $\phi$ nor in any sentence in $\Delta$, and that $\Delta \vdash \phi[c_j/x_i]$. Sketch a proof that

$\Delta \vdash \phi$.

[*Hint: First reduce to the case that $\Delta$ is finite and choose $m$ so large that the variable $x_m$ does not occur in $\Delta$ nor in any formula in a derivation of $\phi[c_j/x_i]$ from hypotheses $\Delta$. Then change every occurrence of $c_j$ to $x_m$.*]

*Proof.* (i) By Deduction Theorem, it suffices to prove that $\forall x_i(A \to B) \vdash (\exists x_i A \to B)$.

| | | |
|---|---|---|
| $\alpha_1$: | $\forall x_i(A \to B)$ | [Premise] |
| $\alpha_2$: | $(\forall x_i(A \to B) \to (A \to B))$ | [A4] |
| $\alpha_3$: | $(A \to B)$ | [MP $\alpha_1$, $\alpha_2$] |
| $\alpha_4$: | $((A \to B) \to (\neg B \to \neg A))$ | [Tautology] |
| $\alpha_5$: | $(\neg B \to \neg A)$ | [MP $\alpha_3$, $\alpha_4$] |
| $\alpha_6$: | $\forall x_i(\neg B \to \neg A)$ | [$\forall$ $\alpha_5$] |
| $\alpha_7$: | $(\forall x_i(\neg B \to \neg A) \to (\neg B \to \forall x_i \neg A))$ | [A5, $x_i \notin \text{Free}(\neg B)$] |
| $\alpha_8$: | $(\neg B \to \forall x_i \neg A)$ | [MP $\alpha_6$, $\alpha_7$] |
| $\alpha_9$: | $((\neg B \to \forall x_i \neg A) \to (\neg \forall x_i \neg A \to B))$ | [Tautology] |
| $\alpha_{10}$: | $(\neg \forall x_i \neg A \to B)$ | [MP $\alpha_8$, $\alpha_9$] |
| $\alpha_{11}$: | $(\exists x_i A \to B)$ | [$\alpha_{10}$, Definition of $\exists$] |

✓

(ii) Suppose that $\phi_1, \ldots, \phi_n$ is a proof of $\Delta \vdash \phi[c_j/x_i]$. Let $\Delta' \subseteq \Delta$ be the set of sentences involved in the proof. Thus $\phi_1, \ldots, \phi_n$ is also a proof of $\Delta' \vdash \phi[c_j/x_i]$, where $\Delta'$ is finite.

Suppose that $x_m$ is a variable that does not occur in any formula of $\phi_1, \ldots, \phi_n$. Next we use induction to prove that $\Delta' \vdash \phi_k[x_m/c_j]$ for all $k \leqslant n$. Assume that $\Delta' \vdash \phi_\ell[x_m/c_j]$ for all $\ell < k$.

(a) If $\phi_k$ is an axiom, then it is clear that $\phi_k[x_m/c_j]$ is also an axiom. Thus we have $\Delta' \vdash \phi_k[x_m/c_j]$.

Could elaborate why free conditions of A4/A5 still hold

(b) If $\phi_k \in \Delta'$, then by assumption $c_j$ does not occur in $\phi_k$. Thus $\Delta' \vdash \phi_k = \phi_k[x_m/c_j]$.

(c) If $\phi_k$ follows from *modus ponens* of $\phi_s$ and $\phi_t$ for some $s, t < k$, then by induction hypothesis, $\Delta' \vdash \phi_s[x_m/c_j]$ and $\Delta' \vdash \phi_t[x_m/c_j]$. It is clear that $\phi_k[x_m/c_j]$ follows from *modus ponens* of $\phi_s[x_m/c_j]$ and $\phi_t[x_m/c_j]$. Hence $\Delta' \vdash \phi_k[x_m/c_j]$.

(d) If If $\phi_k$ follows from generalization of $\phi_s$ for some $s < k$, then by induction hypothesis, $\Delta' \vdash \phi_s[x_m/c_j]$. It is clear that $\phi_k[x_m/c_j]$ follows from generalization of $\phi_s[x_m/c_j]$. Hence $\Delta' \vdash \phi_k[x_m/c_j]$. ✓

Therefore we have $\Delta' \vdash \phi_n[x_m/c_j] = \phi[c_j/x_i][x_m/c_j] = \phi[x_m/x_i]$.

| | |
|---|---|
| $\Delta' \vdash \phi[x_m/x_i]$ | |
| $\Delta' \vdash \forall x_m \phi[x_m/x_i]$ | [$\forall$] |
| $\Delta' \vdash (\forall x_m \phi[x_m/x_i] \to \phi[x_m/x_i][x_i/x_m])$ | [A4, $x_i \notin \text{Free}(\phi[x_m/x_i])$] |
| $\Delta' \vdash \phi[x_m/x_i][x_i/x_m] = \phi$ | [MP] |
| $\Delta \vdash \phi[x_m/x_i][x_i/x_m] = \phi$ | [Thinning] |

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**Question 3** α

Derive the following theorems:

(i) If $x_i$ does not occur free in $A$ then

(a) $\vdash (\exists x_i(A \to B) \to (A \to \exists x_i B))$,

(b) $\vdash ((A \to \exists x_i B) \to \exists x_i(A \to B))$.

(ii) If the only variables occurring free in $\phi$ are $x_i$ and $x_j$ $(i \neq j)$ then

(a) $\vdash (\forall x_i \neg \phi \to \neg \forall x_j \phi)$,

(b) $\vdash (\exists x_i \forall x_j \phi \to \forall x_j \exists x_i \phi)$.

*Proof.* (i) (a) The theorem is equivalent to $(\neg \forall x_i \neg (A \to B) \to (A \to \neg x_i \neg B))$. First we prove $\neg (A \to \neg \forall x_i \neg B) \vdash \forall x_i \neg (A \to B)$:

$$
\begin{array}{lll}
\alpha_1: & \neg(A \to \neg\forall x_i \neg B) & \text{[Premise]} \\
\alpha_2: & (\neg(A \to \neg\forall x_i \neg B) \to A) & \text{[Tautology]} \\
\alpha_3: & (\neg(A \to \neg\forall x_i \neg B) \to \forall x_i \neg B) & \text{[Tautology]} \\
\alpha_4: & A & \text{[MP } \alpha_1, \alpha_2\text{]} \\
\alpha_5: & \forall x_i \neg B & \text{[MP } \alpha_1, \alpha_3\text{]} \\
\alpha_6: & (\forall x_i \neg B \to \neg B) & \text{[A4]} \\
\alpha_7: & \neg B & \text{[MP } \alpha_5, \alpha_6\text{]} \\
\alpha_8: & (A \to (\neg B \to \neg(A \to B))) & \text{[Tautology]} \\
\alpha_9: & (\neg B \to \neg(A \to B)) & \text{[MP } \alpha_4, \alpha_8\text{]} \\
\alpha_{10}: & \neg(A \to B) & \text{[MP } \alpha_7, \alpha_9\text{]} \\
\alpha_{11}: & \forall x_i \neg(A \to B) & \text{[}\forall\ \alpha_{10}\text{]}
\end{array}
$$

Do note that generalisation depends on xi not occurring free in premise (i.e. in A)

Next, by Deduction Theorem, $\vdash (\neg(A \to \neg\forall x_i \neg B) \to \forall x_i \neg(A \to B))$. Note that

$$((\neg(A \to \neg\forall x_i \neg B) \to \forall x_i \neg(A \to B)) \to (\neg\forall x_i \neg(A \to B) \to (A \to \neg\forall x_i \neg B)))$$

is a tautology. By *modus ponens* we have $\vdash (\neg\forall x_i \neg(A \to B) \to (A \to \neg\forall x_i \neg B))$.

(b) The theorem is equivalent to $((A \to \neg\forall x_i \neg B) \to \neg\forall x_i \neg(A \to B))$. First we prove $\forall x_i \neg(A \to B) \vdash \neg(A \to \neg\forall x_i \neg B)$:

$$
\begin{array}{lll}
\alpha_1: & \forall x_i \neg(A \to B) & \text{[Premise]} \\
\alpha_2: & (\forall x_i \neg(A \to B) \to \neg(A \to B)) & \text{[A4]} \\
\alpha_3: & \neg(A \to B) & \text{[MP } \alpha_1, \alpha_2\text{]} \\
\alpha_4: & (\neg(A \to B) \to A) & \text{[Tautology]} \\
\alpha_5: & (\neg(A \to B) \to \neg B) & \text{[Tautology]} \\
\alpha_6: & A & \text{[MP } \alpha_3, \alpha_4\text{]} \\
\alpha_7: & \neg B & \text{[MP } \alpha_3, \alpha_5\text{]} \\
\alpha_8: & \forall x_i \neg B & \text{[}\forall\ \alpha_7\text{]} \\
\alpha_9: & (A \to (\forall x_i \neg B \to \neg(A \to \neg\forall x_i \neg B))) & \text{[Tautology]} \\
\alpha_{10}: & (\forall x_i \neg B \to \neg(A \to \neg\forall x_i \neg B)) & \text{[MP } \alpha_6, \alpha_9\text{]} \\
\alpha_{11}: & \neg(A \to \neg\forall x_i \neg B) & \text{[MP } \alpha_8, \alpha_{10}\text{]}
\end{array}
$$

Next, by Deduction Theorem, $\vdash (\neg(A \to \neg\forall x_i \neg B) \to \forall x_i \neg(A \to B))$. Note that

$$((\neg(A \to \neg\forall x_i \neg B) \to \forall x_i \neg(A \to B)) \to (\neg\forall x_i \neg(A \to B) \to (A \to \neg\forall x_i \neg B)))$$

is a tautology. By *modus ponens* we have $\vdash (\neg\forall x_i \neg(A \to B) \to (A \to \neg\forall x_i \neg B))$.

(ii) (a) It is clear that $\{\forall x_i \neg\phi,\ \forall x_j \phi\} \vdash \forall x_j \phi$. We can prove that $\{\forall x_i \neg\phi,\ \forall x_j \phi\} \vdash \neg\forall x_j \phi$:

$$
\begin{array}{lll}
\alpha_1: & \forall x_i \neg\phi & \text{[Premise]} \\
\alpha_2: & (\forall x_i \neg\phi \to \neg\phi) & \text{[A4]} \\
\alpha_3: & \neg\phi & \text{[MP } \alpha_1, \alpha_2\text{]} \\
\alpha_4: & \forall x_j \phi & \text{[Premise]} \\
\alpha_5: & (\forall x_j \phi \to \phi) & \text{[A4]} \\
\alpha_6: & \phi & \text{[MP } \alpha_4, \alpha_5\text{]} \\
\alpha_7: & (\phi \to (\neg\phi \to \neg\forall x_j \phi)) & \text{[Tautology]} \\
\alpha_8: & (\neg\phi \to \neg\forall x_j \phi) & \text{[MP } \alpha_6, \alpha_7\text{]} \\
\alpha_9: & \neg\forall x_j \phi & \text{[MP } \alpha_3, \alpha_8\text{]}
\end{array}
$$

By proof by contradiction, we have $\forall x_i \neg\phi \vdash \neg\forall x_j \phi$. By Deduction Theorem, we have $\vdash (\forall x_i \neg\phi \to \neg\forall x_j \phi)$.

(b) Informal proof as follows:

| | | |
|---|---|---|
| 1: | $\vdash (\forall x_j \phi \to \phi)$ | [A4] |
| 2: | $\vdash ((\forall x_j \phi \to \phi) \to (\neg\phi \to \neg\forall x_j \phi))$ | [Tautology] |
| 3: | $\vdash (\neg\phi \to \neg\forall x_j \phi)$ | [MP 1,2] |
| 4: | $\vdash (\forall x_i \neg\phi \to \neg\phi)$ | [A4] |
| 5: | $\forall x_i \neg\phi \vdash \neg\phi$ | [MP 4] |
| 6: | $\forall x_i \neg\phi \vdash \neg\forall x_j \phi$ | [MP 3,5] |
| 7: | $\forall x_i \neg\phi \vdash \forall x_i \neg\forall x_j \phi$ | [$\forall$ 6] |
| 8: | $\vdash (\forall x_i \neg\phi \to \forall x_i \neg\forall x_j \phi)$ | [DT] |
| 9: | $\vdash ((\forall x_i \neg\phi \to \forall x_i \neg\forall x_j \phi) \to (\neg\forall x_i \neg\forall x_j \phi \to \neg\forall x_i \neg\phi))$ | [Tautology] |
| 10: | $\vdash (\neg\forall x_i \neg\forall x_j \phi \to \neg\forall x_i \neg\phi)$ | [MP 8,9] |
| 11: | $\vdash (\exists x_i \forall x_j \phi \to \exists x_i \phi)$ | [Definition of $\exists$] |
| 12: | $\exists x_i \forall x_j \phi \vdash \exists x_i \phi$ | [MP 11] |
| 13: | $\exists x_i \forall x_j \phi \vdash \forall x_j \exists x_i \phi$ | [$\forall$ 12] |
| 14: | $\vdash (\exists x_i \forall x_j \phi \to \forall x_j \exists x_i \phi)$ | [DT] |

✓ □

**Question 4** β

Let $f, g$ be binary function symbols, $P$ a binary predicate symbol, $c, d$ constant symbols and let $\mathcal{L} := \{f, g; P; c, d\}$.

Consider $\mathcal{R} := \langle \mathbb{R}; +, \cdot; <; 0, 1\rangle$ as $\mathcal{L}$-structure. Let $h$ be a unary function symbol, let $\mathcal{L}' := \mathcal{L} \cup \{h\}$ and let $\mathcal{R}'$ be $\mathcal{R}$ together with an interpretation $h_{\mathcal{R}'}$ of $h$ in $\mathcal{R}$.

Find $\mathcal{L}'$-formulae $\phi$ and $\psi$ such that

(i) $\mathcal{R}' \vDash \phi$ if and only if $h_{\mathcal{R}'}$ is continuous.

(ii) $\mathcal{R}' \vDash \psi$ if and only if $h_{\mathcal{R}'}$ is differentiable.

*Solution.* (i) Informally, the definition of the continuity of $h_{\mathcal{R}'}$ is

$$\forall x_0 \in \mathbb{R}\, \forall \varepsilon > 0\, \exists \delta > 0\, \forall x \in \mathbb{R}\, (|x_0 - x| < \delta \to |h_{\mathcal{R}'}(x_0) - h_{\mathcal{R}'}(x)| < \varepsilon)$$

Corrections are directly marked to the formulas. Please make sense of them yourself.

which is the abbreviation of

$$\forall x_0 \in \mathbb{R} \forall \varepsilon \in \mathbb{R}(0 < \varepsilon \to \exists \delta \in \mathbb{R}(0 < \delta \to \forall x \in \mathbb{R}((x - \delta < x_0 \wedge x_0 < x + \delta) \to (h_{\mathcal{R}'}(x) - \varepsilon < h_{\mathcal{R}'}(x_0) \wedge h_{\mathcal{R}'}(x_0) < h_{\mathcal{R}'}(x) + \varepsilon))))$$

In these problems you could have moved the subtract parts to the other side of the inequality (not for division though, because of sign problems). This do not always give simpler expressions, but it reduces the amount of variables & connectives you need to deal with

Since the subtraction is not an original function in the model $\mathcal{R}$, the sentence can also be expressed without the subtraction:

$$\forall x_0 \in \mathbb{R} \forall \varepsilon \in \mathbb{R}(0 < \varepsilon \to \exists \delta \in \mathbb{R}(0 < \delta \to \forall x \in \mathbb{R}((\forall \delta' \in \mathbb{R}(\delta + \delta' = 0 \wedge x + \delta' < x_0) \wedge x_0 < x + \delta) \to (\forall \varepsilon' \in \mathbb{R}(\varepsilon + \varepsilon' = 0 \wedge h_{\mathcal{R}'}(x) + \varepsilon' < h_{\mathcal{R}'}(x_0)) \wedge h_{\mathcal{R}'}(x_0) < h_{\mathcal{R}'}(x) + \varepsilon))))$$

In the language of $\mathcal{L}'$, the sentence $\phi$ is

$$\forall x_0 \forall x_1 (P(c, x_1) \to \exists x_2 (P(c, x_2) \to \forall x_3 ((\forall x_4 (f(x_2, x_4) \doteq c \wedge P(f(x_3, x_4), x_0)) \wedge P(x_0, f(x_3, x_2))) \to (\forall x_5 (f(x_1, x_5) \doteq c \wedge P(f(h(x_3), x_5), h(x_0))) \wedge P(h(x_0), f(h(x_3), x_1))))))$$

(ii) Informally, the definition of the differentiability of $h_{\mathcal{R}'}$ is

$$\forall x_0 \in \mathbb{R}\, \exists L \in \mathbb{R}\, \forall \varepsilon > 0\, \exists \delta > 0\, \forall x \in \mathbb{R}\, \left(0 < |x - x_0| < \delta \to \left|\frac{h_{\mathcal{R}'}(x) - h_{\mathcal{R}'}(x_0)}{x - x_0} - L\right| < \varepsilon\right)$$

which is equivalent to

$$\forall x_0 \in \mathbb{R} \exists L \in \mathbb{R} \forall \varepsilon \in \mathbb{R}(0 < \varepsilon \to \exists \delta \in \mathbb{R}(0 < \delta \to \forall x \in \mathbb{R}((\forall \delta' \in \mathbb{R}(\delta + \delta' = 0 \wedge x_0 + \delta' < x) \wedge x < x_0 + \delta \wedge x \neq x_0) \to (\forall x_0^- \in \mathbb{R}(x_0 + x_0^- = 0 \wedge \forall D \in \mathbb{R}((x + x_0^-) \cdot D = 1 \wedge \forall a \in \mathbb{R}(h_{\mathcal{R}'}(x_0) + a = 0 \wedge \forall \varepsilon' \in \mathbb{R}(\varepsilon + \varepsilon' = 0 \wedge L + \varepsilon' < (h_{\mathcal{R}'}(x) + a) \cdot D \wedge (h_{\mathcal{R}'}(x) + a) \cdot D < L + \varepsilon)))))))$$

In the language $\mathcal{L}'$, the sentence $\psi$ is

Here there is no need for x ≠ x0, because you are using universal quantifiers below, and "for any inverse of 0, ….." is trivially true, so you needn't exclude that case

$$\forall x_0 \exists L \forall \varepsilon (P(c,\varepsilon) \to \exists \delta (P(c,\delta) \to \forall x((\forall \delta' (f(\delta,\delta') \doteq c \wedge P(f(x_0,\delta'),x)) \wedge P(x, f(x_0,\delta)) \wedge \neg x \doteq x_0) \to$$
$$(\forall x_0^- (f(x_0, x_0^-) \doteq c \wedge \forall D(g(f(x, x_0^-), D) \doteq d \wedge \forall a (f(h(x_0), a) \doteq c \wedge \forall \varepsilon' (f(\varepsilon, \varepsilon') \doteq$$
$$c \wedge P(f(L,\varepsilon'), g(f(h(x),a),D)) \wedge P(g(f(h(x),a),D), f(L,\varepsilon))))))))))$$

□

**Question 5** α-

(i) Let $\mathcal{L} := \{+, \cdot; 0, 1\}$ be the language of rings (i.e. $+$ a binary function symbol etc.). Write down sets of formulae $\Phi_p$ (for $p$ a prime or $p = 0$) whose models are exactly all fields of characteristic $p$.

(ii) State the Compactness Theorem for sets of sentences and show how it follows from the Soundness and Completeness Theorems.

(iii) Prove that $\Phi_0$ in (i) cannot be chosen finite.

*Proof.* (i) The formulae as axioms of fields:

$$\begin{aligned}
\alpha_1 &= \forall x \forall y \; x + y \doteq y + x \\
\alpha_2 &= \forall x \forall y \forall z \; (x + y) + z \doteq x + (y + z) \\
\alpha_3 &= \forall x \; x + 0 \doteq x \\
\alpha_4 &= \forall x \exists y \; x + y \doteq 0 \\
\alpha_5 &= \forall x \forall y \; x \cdot y \doteq y \cdot x \\
\alpha_6 &= \forall x \forall y \forall z \; (x \cdot y) \cdot z \doteq x \cdot (y \cdot z) \\
\alpha_7 &= \forall x \; x \cdot 1 \doteq x \\
\alpha_8 &= \forall x (\neg x \doteq 0 \to \exists y \; x \cdot y \doteq 1) \\
\alpha_9 &= \forall x \forall y \forall z \; (x + y) \cdot z \doteq x \cdot z + y \cdot z
\end{aligned}$$

You also want ¬0 = 1, otherwise the trivial ring will satisfy your Φp for each prime p below

Inductively, we define the terms $\psi_p := (\psi_{p-1} + 1)$ where $\psi_1 := 1$. We define the formulae $\phi_p := \psi_p \doteq 0$.

If $p$ is a prime, we have $\Phi_p = \{\alpha_1, \ldots, \alpha_9, \phi_p\}$.

If $p = 0$, we have $\Phi_0 = \{\alpha_1, \ldots, \alpha_9\} \cup \{\neg \phi_p : p \in \mathbb{Z}_+\}$. ✓

(ii) Compactness Theorem:

*Let $\mathcal{L}$ be a first-order language and $\Gamma \subseteq \mathrm{Sent}(\mathcal{L})$. Then $\Gamma$ has a model if and only if every finite subset of it has a model.* ✓

The forward direction is trivial. For the backword direction, suppose that $\Gamma$ does not have a model. In particular, for any $\alpha \in \mathrm{Sent}(\mathcal{L})$, we have $\Gamma \vDash \alpha$ and $\Gamma \vDash \neg\alpha$. By Completeness Theorem, we have $\Gamma \vdash \alpha$ and $\Gamma \vdash \neg\alpha$. But both proofs involve a finite subset $\Gamma' \subseteq \Gamma$. Then $\Gamma' \vdash \alpha$ and $\Gamma' \vdash \neg\alpha$. By Soundness Theorem, $\Gamma' \vDash \alpha$ and $\Gamma' \vDash \neg\alpha$. Hence $\Gamma'$ does not have a model. ✓

(iii) Suppose that $\Phi_0$ can be finitely axiomatised. Assume that $\Phi_0 = \{\gamma_1, \ldots, \gamma_n\}$. Let $\Delta' := \{\neg(\gamma_1 \wedge \cdots \wedge \gamma_n)\}$. Then any field with positive characteristic is a model of $\Delta'$.

It should be ¬φ_p

Let $\Delta := \Delta' \cup \{\phi_p : p \in \mathbb{Z}_+\}$. For each finite subset $\tilde{\Delta} \subseteq \Delta$, let $q = \max\{p \in \mathbb{Z}_+ : \phi_p \in \tilde{\Delta}\}$. There exists a prime $r > q$ such that fields of characteristic $r$ are models of $\tilde{\Delta}$. By Compactness Theorem, $\Delta$ has a model. But fields that satisfy $\phi_p$ for all $p \in \mathbb{Z}_+$ are exactly fields of characteristic 0. This is a contradiction. □

No, a model that is not a field can satisfy your Δ. You should included the field axioms in Δ', so that its models are "fields that are not 0-characteristic fields"

**Question 6** α-

(i) Axiomatise the first-order theory $\Sigma$ of ordered fields in the language $\mathcal{L} := \{+, \cdot; <; 0, 1\}$.

(ii) Which of the following is a model of $\Sigma$:

($\alpha$) $\mathbb{Q}$ with the usual interpretations,

($\beta$) $\mathbb{R}$ with the usual interpretations,

($\gamma$) $\mathbb{C}$ with $a + bi < c + di$ if and only if $a^2 + b^2 < c^2 + d^2$,

($\delta$) $\mathbb{F}_p$ with $0 < 1 < 2 < \cdots < p-1$.

(iii) Is $\Sigma$ consistent? Is it maximally consistent?

(iv) Recall that the ordering on $\mathbb{Q}$ (*resp.* on $\mathbb{R}$) is *Archimedean*, i.e. for every $x \in \mathbb{Q}$ (*resp.* $x \in \mathbb{R}$) there is some $n \in \mathbb{N}$ with $-n < x < n$. Use the Compactness Theorem to prove that Archimedeanity is not a first-order property. [*Hint: introduce a new constant symbol* $c$.]

*Proof.* (i) The following are axioms of ordered fields:

$$\begin{aligned}
\alpha_1 &= \forall x \forall y\, x + y \doteq y + x \\
\alpha_2 &= \forall x \forall y \forall z\, (x+y)+z \doteq x+(y+z) \\
\alpha_3 &= \forall x\, x + 0 \doteq x \\
\alpha_4 &= \forall x \exists y\, x + y \doteq 0 \\
\alpha_5 &= \forall x \forall y\, x \cdot y \doteq y \cdot x \\
\alpha_6 &= \forall x \forall y \forall z\, (x \cdot y) \cdot z \doteq x \cdot (y \cdot z) \\
\alpha_7 &= \forall x\, x \cdot 1 \doteq x \\
\alpha_8 &= \forall x (\neg x \doteq 0 \to \exists y\, x \cdot y \doteq 1) \\
\alpha_9 &= \forall x \forall y \forall z\, (x+y) \cdot z \doteq x \cdot z + y \cdot z \\
\alpha_{10} &= \forall x \forall y ((x < y \wedge \neg y < x \wedge \neg x \doteq y) \vee (\neg x < y \wedge y < x \wedge \neg x \doteq y) \vee (\neg x < y \wedge \neg y < x \wedge x \doteq y)) \\
\alpha_{11} &= \forall x \forall y \forall z (x < y \to (y < z \to x < z)) \\
\alpha_{12} &= \forall x \forall y \forall z (x < y \to x + z < y + z) \\
\alpha_{13} &= \forall x \forall y (0 < a \to (0 < b \to 0 < a \cdot b))
\end{aligned}$$

$\Sigma = \{\alpha_1, ..., \alpha_{13}\}$.

(ii) It is clear from analysis that ($\alpha$) and ($\beta$) are models of ordered fields.

($\gamma$) is not an ordered field. Note that we have $\neg 1 < \mathrm{i}$ and $\neg \mathrm{i} < 1$ in $\mathbb{C}$ by definition. But also $\neg 1 \doteq \mathrm{i}$ in $\mathbb{C}$. This violates the trichotomy of order.

($\delta$) is not an ordered field. From $\alpha_{12}$ we infer that $p-2 < p-1 \to p-1 < p = 0$. But $0 < p-1$ in $\mathbb{F}_p$. Contradiction.

You are using Soundness here, because you need "if $\Sigma$ proves contradictions, then its models satisfy contradictions"

(iii) Since $\Sigma$ has a model, by Completeness Theorem, it is consistent.

$\Sigma$ is not maximally consistent. Since $\sqrt{2} \notin \mathbb{Q}$, the sentence

$$\phi := \exists x\, x \cdot x \doteq 1 + 1$$

satisfies that $\mathbb{Q} \models \neg\phi$ and $\mathbb{R} \models \phi$. But both $\mathbb{Q}$ and $\mathbb{R}$ are models of $\Sigma$. Therefore $\Sigma \models \phi$ and $\Sigma \models \neg\phi$. By Completeness Theorem, $\Sigma \vdash \phi$ and $\Sigma \vdash \neg\phi$. Hence $\Sigma$ is not maximally consistent.

Doesn't this make $\Sigma$ inconsistent? You need that $\Sigma$ doesn't prove either (again by Soundness)

(iv) Suppose for contradiction that $\chi \in \mathrm{Sent}(\mathcal{L}^{\mathrm{FOPC}})$ describes the Archimedean property. Consider the sentences defined inductively by $\phi_n := 1 + \phi_{n-1}$ where $\phi_0 := 0 < c$ and $c$ is a constant symbol. Consider the set $\Gamma := \chi \cup \{\phi_n : n \in \mathbb{N}\}$. It is clear that $\mathbb{Q}$ is a model of any finite subset of $\Gamma$. However, any field that satisfies $\{\phi_n : n \in \mathbb{N}\}$ is non-Archimedean. Therefore $\Gamma$ has no models. This contradicts the Compactness Theorem. ∎

Definitions like this mess up the operator priorities (namely, if you consider the full expression without abbreviation of brackets, then such definitions are not possible). It's better to define the terms first, like you did in Q5(i)