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**Problem Sheet 1**

# String Theory I

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# 1 Veneziano and Virasoro-Shapiro amplitudes

Before they were recognized as describing relativistic quantum-mechanical strings, a great was made by studying the properties of scattering amplitudes as a part of the dual model p. problem, you will perform some analysis on the two most famous dual model amplitudes.

**Question 1.1**

Consider elastic scattering of identical scalar particles with masses $\alpha' m^2 = -\alpha(0) = -1$. In terms of the Mandelstam variables

$$s = -\left(p_1 + p_2\right)^2, \quad t = -\left(p_1 + p_4\right)^2, \quad u = -\left(p_1 + p_3\right)^2$$

find the expression for the scattering angle $\theta_s$ in center-of-momentum frame. Charac of high energy fixed-angle scattering in terms of Mandelstam variables. What does ($s \gg 1, t < 0$ fixed) look like in center-of-momentum frame?

**Question 1.2**

The Veneziano amplitude for the scattering of open-string tachyons of mass $\alpha' m^2 = -1$ is given by

$$\mathcal{A}_V(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}, \quad \alpha(x) := 1 + \alpha' x$$

Show that the Veneziano amplitude can also be defined by the series expansion

$$\mathcal{A}_V(s,t) = -\sum_{n=0}^{\infty} \frac{(\alpha(s)+1)(\alpha(s)+2)\cdots(\alpha(s)+n)}{n!} \frac{1}{\alpha(t)-n}.$$

Deduce that the Veneziano amplitude displays Dolen-Horn-Schmid duality.

**Question 1.3**

Show that in the Regge limit, the Veneziano amplitude behaves according to [a]

$$\mathcal{A}(s,t) \sim \Gamma(-\alpha(t))(-\alpha(s))^{\alpha(t)}$$

This is subtle: you must actually define the Regge limit so that $s$ has a small imaginar $s$-channel poles on the positive $s$ axis. Can you justify this trick?

[a] Stirling's formula for the Gamma function should prove useful for this.

**Question 1.4**

Show that in high-energy, fixed-angle scattering, the Veneziano amplitude behaves according to

$$\mathcal{A}(s,t) \sim F\left(\theta_s\right)^{-\alpha(s)}$$

Find the function $F\left(\theta_s\right)$ and show that this behavior is exponentially soft. As in the previous part, you should give $s$ a small imaginary part to avoid $s$-channel poles.

**Question 1.5**

The Virasoro-Shapiro amplitude for the scattering of identical scalar tachyons of mass $\alpha' m^2 = -4$ is given by [a]

$$\mathcal{A}_{VS}(s,t,u) = \frac{\Gamma\left(-\alpha_c(s)\right)\Gamma\left(-\alpha_c(t)\right)\Gamma\left(-\alpha_c(u)\right)}{\Gamma\left(-\alpha_c(s)-\alpha_c(t)\right)\Gamma\left(-\alpha_c(t)-\alpha_c(u)\right)\Gamma\left(-\alpha_c(u)-\alpha_c(s)\right)}, \quad \alpha_c(x) := 1 + \frac{\alpha' x}{4}.$$

Find and argue the validity of an expression for $\mathcal{A}_{VS}(s,t,u)$ as a sum of, say, $t$ - and $u$-channel exchange contributions (i.e., as a sum of terms which have simple poles in $t$ or $u$ with residues that are polynomials in $s$, as arise from tree-level exchange diagrams in the $t$ or $u$ channels in field theory).

[a]Recall that $s+t+u = 4m^2 = -16/\alpha'$ for this amplitude. We write the $u$ dependence to make crossing symmetry manifest.

**Question 1.6**

Find expressions for the Regge and high-energy, fixed-angle limits of the Virasoro-Shapiro amplitude.

## 2 Classical string dynamics

Although the analysis of the quantum string is quite a bit more involved than that of the classical string, it is important to remember that there are some points of contact between classical and quantum theories. In this exercise, you will analyze the classical bosonic string via the Nambu-Goto action, which is less suitable for quantization than the Polyakov action, but is a perfectly good (and equivalent) classical theory.

**Question 2.1**

Recall the Nambu-Goto Lagrangian for the relativistic string,

$$\mathcal{L}_{\text{NG}} = -T\sqrt{-h},$$

where $h$ is the determinant of the induced metric on the worldsheet. Derive the equations of motion implied by this Lagrangian for both open and closed strings, and show that in both cases they imply

$$\partial_i K^i_\mu = 0, \quad K^i_\mu := \frac{\delta \mathcal{L}}{\delta\left(\frac{\partial x^\mu(\xi)}{\partial \xi^i}\right)},$$

where $\xi^i$ are worldsheet coordinates and $x^\mu(\xi)$ are space-time coordinates. Using the equations of motion, show that the following quantity is conserved by the string dynamics,

$$P^\mu(\tau) = \int_0^\pi d\sigma K^{\mu\tau}(\sigma,\tau),$$

where $\sigma$ and $\tau$ are spatial and temporal coordinates on the string worldsheet. What is the interpretation of this quantity?

*Proof.* The induced metric on the world sheet is given by

indices?

$$h = \frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j}\eta_{\mu\nu}\cancel{\mathrm{d}\xi^i \mathrm{d}\xi^j}$$

$$h_{ij} = \frac{\partial x^\mu}{\partial \xi^i}\frac{\partial x^\nu}{\partial \xi^j}\eta_{\mu\nu}$$

$$ds^2 = h_{ij}(\xi)\, d\xi^i d\xi^j$$

So the Nambu-Goto Lagrangian is given by

$$\mathcal{L}_{\mathrm{NG}} = -T\sqrt{\det h} = -T\sqrt{-\left(\frac{\partial x}{\partial \tau}\right)^2\left(\frac{\partial x}{\partial \sigma}\right)^2 + \left(\frac{\partial x}{\partial \tau}\cdot\frac{\partial x}{\partial \sigma}\right)^2}$$

For closed strings, the variation $\delta\mathcal{L}$ has the natural boundary conditions. For open strings, the variation $\delta\mathcal{L}$ has the fixed boundary conditions on $\sigma$. By calculus of variations, both cases produce the Euler-Lagrange equation:

$$\frac{\partial}{\partial \xi^i}\frac{\partial \mathcal{L}_{\mathrm{NG}}}{\partial(\partial_i x^\mu)} - \frac{\partial \mathcal{L}_{\mathrm{NG}}}{\partial x^\mu} = 0$$

$$\frac{\partial \mathcal{L}_{NG}}{\partial X^\mu} = 0$$

If $K^i_\mu := \dfrac{\partial \mathcal{L}_{\mathrm{NG}}}{\partial(\partial_i x^\mu)}$, then we have the equations of motion

$$\partial_i K^i_\mu = 0 \iff \frac{\partial K^\tau_\mu}{\partial \tau} = -\frac{\partial K^\sigma_\mu}{\partial \sigma}$$

For

$$P^\mu(\tau) = \int_0^\pi \mathrm{d}\sigma\, K^{\mu\tau}(\sigma,\tau) = \eta^{\mu\nu}\int_0^\pi \mathrm{d}\sigma\, K^\tau_\nu(\sigma,\tau)$$

we have

$$\frac{\partial P^\mu}{\partial \tau} = \eta^{\mu\nu}\int_0^\pi \mathrm{d}\sigma\,\frac{\partial K^\tau_\nu}{\partial \tau} = -\eta^{\mu\nu}\int_0^\pi \mathrm{d}\sigma\,\frac{\partial K^\sigma_\nu}{\partial \sigma} = -\eta^{\mu\nu}(K^\sigma_\nu(\tau,\pi) - K^\sigma_\nu(\tau,0))$$

For closed strings, the periodic boundary condition suggests that $K^\sigma_\nu(\tau,\pi) = K^\sigma_\nu(\tau,0)$. For open strings, the boundary conditions force $K^\sigma_\nu(\tau,\pi) \overset{(\bullet)}{=} K^\sigma_\nu(\tau,0) = 0$. In both cases, we have $\partial_\tau P^\mu = 0$. Hence $P^\mu$ is conserved by the string dynamics. This should be understood as the momentum of the string. □

from Neumann boundary conditions on the endpoints

FREE ENDPOINTS: $\partial_\sigma X^\mu(\tau,\sigma^*) = 0$, $\sigma^* = \{0,\pi\} \Rightarrow (\bullet)$

**Question 2.2**

Show that in conformal gauge, [a] the following quantity is also conserved for both open and closed strings,

$$M^{\mu\nu} = \int_0^\pi d\sigma\,(x^\mu(\sigma,\tau)K^{\nu\tau}(\sigma,\tau) - x^\nu(\sigma,\tau)K^{\mu\tau}(\sigma,\tau))\,.$$

[a] In the lectures and most textbooks, conformal gauge is introduced only for the Polyakov action. Convince yourself that there should be an analogous gauge choice for the Nambu-Goto string.

*Proof.* In the conformal gauge for the Nambu-Goto action, the induced metric on the worldsheet is chosen such that $h_{\tau\sigma} = 0$ and $\operatorname{tr} h = 0$. Then

$$\mathcal{L}_{\mathrm{NG}} = -\frac{T}{2}\left(-\frac{\partial x}{\partial \tau}\cdot\frac{\partial x}{\partial \tau} + \frac{\partial x}{\partial \sigma}\cdot\frac{\partial x}{\partial \sigma}\right)$$

So $K^\tau_\mu = T\partial_\tau x_\mu$ and $K^\sigma_\mu = -T\partial_\sigma x_\mu$. $\partial_i K^{\mu i} = 0$ impplies that $\partial^2_\tau x^\mu = \partial^2_\sigma x^\mu$. We have

$$\begin{aligned}
\frac{\partial M^{\mu\nu}}{\partial \tau} &= T\frac{\partial}{\partial \tau}\int_0^\pi \mathrm{d}\sigma\,(x^\mu\partial_\tau x^\nu - x^\nu\partial_\tau x^\mu) \\
&= T\int_0^\pi \mathrm{d}\sigma\,(x^\mu\partial^2_\tau x^\nu - x^\nu\partial^2_\tau x^\mu) \\
&= T\int_0^\pi \mathrm{d}\sigma\,(x^\mu\partial^2_\sigma x^\nu - x^\nu\partial^2_\sigma x^\mu) \\
&= T\,(x^\mu\partial_\sigma x^\nu - x^\nu\partial_\sigma x^\mu)\Big|_{\sigma=0}^{\sigma=\pi}
\end{aligned}$$

As in the previous question,the quantity above vanishes for both closed strings and open strings. Hence

$M^{\mu\nu}$ is conserved. This is the anuglar momentum of the string. ☐

## Question 2.3

Consider an open string in conformal gauge. Show that the end-points of the string move through spacetime at the speed of light.

*Proof.* In the conformal gauge, we have $\mathrm{tr}\, h \overset{?}{=} -\partial_\tau x \cdot \partial_\tau x + \partial_\sigma x \cdot \partial_\sigma x = 0$. At the end-points of an open string, we have $\partial_\sigma x = 0$. Therefore $\partial_\tau x \cdot \partial_\tau x = 0$. This shows that the vector $\partial_\tau x$ is light-like in Minkowski spacetime. So physically we can say that it moves through spacetime at the speed of light. ☐

## Question 2.4

Verify that the following is a solution of the equations of motion for the Nambu-Goto string in conformal gauge:

$$\begin{aligned}
x^0(\sigma,\tau) &= \frac{1}{2}\left(p + \frac{a^2}{p}\right) n\tau \\
x^1(\sigma,\tau) &= \frac{1}{2}\left(p - \frac{a^2}{p}\right) n\tau \\
x^2(\sigma,\tau) &= a\cos(n\sigma)\cos(n\tau) \\
x^3(\sigma,\tau) &= a\cos(n\sigma)\sin(n\tau) \\
x^\mu(\sigma,\tau) &= 0 \qquad\qquad \mu \geqslant 4.
\end{aligned}$$

Describe the motion of this string through spacetime. Find an analogous solution where the center of mass of the string is stationary. Find the relationship between spacetime energy and angular momentum for this family of solutions.

*Proof.* First we need to verify that the solution is in conformal gauge. The induced metric on the worldsheet:

$$\begin{aligned}
h_{\tau\tau} &= \frac{\partial x}{\partial \tau} \cdot \frac{\partial x}{\partial \tau} \\
&= -\frac{1}{4}\left(p + \frac{a^2}{p}\right)^2 n^2 + \frac{1}{4}\left(p - \frac{a^2}{p}\right)^2 n^2 + a^2n^2\cos^2(n\sigma)\sin^2(n\tau) + a^2n^2\cos^2(n\sigma)\cos^2(n\tau) \\
&= a^2(-\cancel{1}\,n^2 + n^2\cos^2(n\sigma)) = -a^2 n^2 \mathrm{sen}^2(n\sigma) \\
h_{\tau\sigma} &= \frac{\partial x}{\partial \tau} \cdot \frac{\partial x}{\partial \sigma} \\
&= a^2n^2\cos(n\sigma)\sin(n\sigma)\cos(n\tau)\sin(n\tau) - a^2n^2\cos(n\sigma)\sin(n\sigma)\cos(n\tau)\sin(n\tau) \\
&= 0 \\
h_{\sigma\sigma} &= \frac{\partial x}{\partial \sigma} \cdot \frac{\partial x}{\partial \sigma} \\
&= a^2n^2\sin^2(n\sigma)\cos^2(n\tau) + a^2n^2\sin^2(n\sigma)\sin^2(n\tau) \\
&= a^2n^2\sin^2(n\sigma)
\end{aligned}$$

So $h_{\tau\tau} + h_{\sigma\sigma} = 0$. In the conformal gauge, the equations of motion are simple: $\partial_\tau^2 x^\mu = \partial_\sigma^2 x^\mu$. We have:

$$\partial_\tau^2 x^0 = \partial_\sigma^2 x^0 = 0, \qquad \partial_\tau^2 x^1 = \partial_\sigma^2 x^1 = 0, \qquad \partial_\tau^2 x^2 = \partial_\sigma^2 x^2 = -n^2x^2, \qquad \partial_\tau^2 x^3 = \partial_\sigma^2 x^3 = -n^2x^3$$

So this is a solution for the Nambu-Goto string in conformal gauge.

In the $x^1$-direction, the string moves at constant speed. On the $x^2x^3$-plane, the string rotates along its centre at constant angular velocity.

For a similar solution where the centre of mass of the string is stationary, we can simply neglect the motion in the $x^1$-direction by setting $p = a$. So we have

$$x^0(\sigma,\tau) = an\tau, \quad x^1(\sigma,\tau) = 0, \quad x^2(\sigma,\tau) = a\cos(n\sigma)\cos(n\tau), \quad x^3(\sigma,\tau) = a\cos(n\sigma)\sin(n\tau)$$

Let $\boldsymbol{K}^\tau = (K_1^\tau, ..., K_3^\tau)$. The classical (spatial) angular momentum is given by

(T) missing

$$\boldsymbol{M} = \int_0^\pi \mathrm{d}\sigma\, \boldsymbol{x} \wedge \boldsymbol{K}^\tau = \int_0^\pi \mathrm{d}\sigma\, \left(x^2\partial_\tau x^3 - x^3\partial_\tau x^2\right) \boldsymbol{e}_1 = na^2 \int_0^\pi \mathrm{d}\sigma\, \cos^2(n\sigma)\boldsymbol{e}_1 = \frac{\pi n a^2}{2}\boldsymbol{e}_1$$

The spacetime energy

$$E = \frac{1}{2}\int_0^\pi \mathrm{d}\sigma\, \boldsymbol{K}^\tau \cdot \boldsymbol{K}^\tau = \frac{1}{2}a^2n^2 \int_0^\pi \mathrm{d}\sigma\, \cos^2(n\sigma) = \cancel{\frac{\pi n^2 a^2}{4}} \qquad \square$$

- $E = \dots = Tan\pi$
- $M^{23} = \dots = \frac{\pi n a^2 T}{2}$

$\Rightarrow$ relation?

$E^2 \sim M^{23} \rightarrow$ leading Regge traj.

Apart from a few missing factors around and the computation of the spacetime energy, you did a good job!

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