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A+

Problem Sheet 4

C3.11: Riemannian Geometry

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Section A: Introductory

Question 1

Let $f : (M, g) \to (N, h)$ be a surjective local isometry between connected Riemannian manifolds.

(a) Show that if (M, g) is complete then (N, h) is complete.

(b) If (N, h) is complete, is (M, g) complete? Give a proof or a counterexample.

Let $(\widetilde{M}, \widetilde{g})$ be the universal cover of (M, g) with the covering metric.

(c) Show that $(\widetilde{M}, \widetilde{g})$ is complete if and only if (M, g) is complete.

Question 2

Let B^n be the unit ball in $\mathbb{R}^n$ and let

$$g = \frac{4 \sum_{i=1}^n \mathrm{d}x_i^2}{\left(1 - \sum_{i=1}^n x_i^2\right)^2}$$

By considering normalized geodesics in (B^n, g) through 0, show that (B^n, g) is complete.

Question 3

Let (N, g) be an oriented $(n+1)$-dimensional Riemannian manifold. Let $f : N \to \mathbb{R}$ be a smooth function and let $h = e^{2f} g$.

(a) Let ∇^g and ∇^h be the Levi-Civita connections of g and h. Show that

$$\nabla^h_X Y = \nabla^g_X Y + X(f)Y + Y(f)X - g(X, Y)\nabla^g f$$

for all vector fields X, Y on N.

(b) Let M be a connected oriented hypersurface in (N, g) with unit normal vector field ν so that the shape operator satisfies

$$S_\nu = \lambda \text{ id}$$

for a smooth function $\lambda : M \to \mathbb{R}$. Show that the shape operator of M in (N, h) satisfies

$$S_{e^{-f}\nu} = \mu \text{id}$$

for a smooth function $\mu : M \to \mathbb{R}$ which should be identified in terms of λ and f.

Now let $R > 0$, let

$$M = \left\{ (x_1, \dots, x_{n+1}) \in H^{n+1} : \sum_{i=1}^{n+1} x_i^2 = R^2 \right\}$$

with its standard orientation and let h be the hyperbolic metric on H^{n+1}.

(c) Calculate the mean curvature and sectional curvatures of M in $\left(H^{n+1}, h\right)$ with its induced metric.

Section B: Core

Question 4

(a) Let $(M.g)$ be a complete Riemannian manifold with non-positive sectional curvature, let p, q be points in M and let α be a curve in M from p to q.

Show that there is a unique geodesic γ in (M, g) from p to q which is homotopic to α.

(b) Let (M, g) be an oriented even-dimensional manifold with positive sectional curvature and let $\gamma : \mathcal{S}^1 \to (M, g)$ be a closed geodesic.

Show that there is a closed curve $\alpha : \mathcal{S}^1 \to (M, g)$ homotopic to γ such that $L(\alpha) < L(\gamma)$.

Proof. (a) By taking the path component of p, we may assume that M is connected. Let $\widetilde{M}$ be the universal cover of M. Then $\widetilde{M}$ is connected, simply-connected, complete, with non-positive sectional curvature. By Cartan-Hadamard Theorem, $\widetilde{M}$ is diffeomorphic to $\mathbb{R}^n$. As the universal cover is defined up to homeomorphism, we can actually take $\widetilde{M} = \mathbb{R}^n$. Moreover, the covering map is $\exp_p : \mathrm{T}_pM \cong \mathbb{R}^n \to M$. Hence the lift of geodesics in M are straight lines in $\mathbb{R}^n$.

Let $\pi : (\mathbb{R}^n, \widetilde{p}) \to (M, p)$ be the covering map. By path lifting property, $\alpha : [0, 1] \to M$ uniquely lifts to $\widetilde{\alpha} : [0, 1] \to \mathbb{R}^n$ such that $\widetilde{\alpha}(0) = \widetilde{p}$. Let $\widetilde{q} := \widetilde{\alpha}(1)$. Now let $\widetilde{\gamma} : [0, 1] \to \mathbb{R}^n$ be the unique geodesic (i.e. straight line) connecting $\widetilde{p}$ and $\widetilde{q}$. Since $\mathbb{R}^n$ is simply-connected, $\widetilde{\alpha} \simeq \widetilde{\gamma}$ and hence $\alpha \simeq \gamma := \pi \circ \widetilde{\gamma}$. To show that γ is unique, suppose γ' is another geodesic connecting p and q such that $\gamma \simeq \gamma'$. By homotopy lifting property, the homotopy H from γ to γ' lifts to a homotopy $\widetilde{H}$ from $\widetilde{\gamma}$ to $\widetilde{\gamma}'$. But then we must have $\widetilde{\gamma} = \widetilde{\gamma}'$ by uniqueness of geodesic in $\mathbb{R}^n$. Hence γ is unique. ✓

(b) First we need to prove the following lemma (*hint from Exercise 9.4 of do Carmo*):

There exists a parallel vector field V along γ such that $V(1) = V(0)$.

For $t \in [0, 1]$, let $(\mathrm{T}_{\gamma(t)}\gamma)^\perp$ be the orthonormal complement of $\dot{\gamma}$ in $\mathrm{T}_{\gamma(t)}M$. Let $\tau_t : (\mathrm{T}_{\gamma(0)}\gamma)^\perp \to (\mathrm{T}_{\gamma(t)}\gamma)^\perp$ be the parallel transport of vector fields from $\gamma(0)$ to $\gamma(t)$ along γ. It is clear that $\{\tau_t\}_{t\in[0,1]}$ is a one-parameter family of orientation preserving isometries of $\mathbb{R}^n$. Since M is even-dimensional, by Synge–Weinstein Theorem, τ_t has a fixed vector for each t. In particular, when we take $t = 1$, we obtain a parallel vector field $V(t)$ along γ such that $V(0) = V(1)$.

Suppose that for all closed curve α with $\alpha \simeq \gamma$, we have $L(\alpha) \geqslant L(\gamma)$. We construct a variation of γ as follows. Let V be a normalised parallel vector field along γ constructed as above. So V is orthogonal to $\dot{\gamma}$ and $V(0) = V(1)$. Let

$$f(s, t) := \exp_{\gamma(t)}(sV(t))$$

Then $V_f(t) = \dfrac{\partial f_i}{\partial s}(0, t) = V(t)$. In particular $\dot{V}(t) = 0$. Substituting into the second variation formula, we obtain

$$\frac{1}{2}\ddot{E}_f(0) = -L(\gamma)^2 \int_0^1 K(V, \dot{\gamma})\,\mathrm{d}t - g\left(\frac{\mathrm{D}}{\mathrm{D}s}\frac{\partial f_i}{\partial s}(0, t), \dot{\gamma}(t)\right)\Bigg|_{t=0}^{t=1}$$

Since $V(0) = V(1)$ and $\dot{\gamma}(0) = \dot{\gamma}(1)$, then we have

$$\frac{1}{2}\ddot{E}_f(0) = -L(\gamma)^2 \int_0^1 K(V, \dot{\gamma})\,\mathrm{d}t < 0$$

On the other hand, $\alpha_s(t) := f(s, t)$ defines a family of closed curves homotopy to $\alpha_0 = \gamma$. By assumption, $L(\alpha_s) \geqslant L(\gamma)$ for all s. Then $E_f(s) \geqslant E_f(0)$ for all s. It means that $\dot{E}_f(0) = 0$ and $\ddot{E}_f(0) \geqslant 0$. This is a contradiction. We conclude that γ has a homotopic curve with smaller length. □

Very well done!

Question 5

(a) Let $n, m \in \mathbb{N}$. Show that $\mathcal{S}^n \times \mathcal{S}^m$ admits a Riemannian metric of positive Ricci curvature if and only if $n \geqslant 2$ and $m \geqslant 2$.

(b) Let G be a connected Lie group with identity e which admits a bi-invariant Riemannian metric. Suppose that the centre of the Lie algebra $\mathfrak{g} = T_e G$ is trivial.

Show that G and its universal cover are compact, and hence that $\mathrm{SL}(n, \mathbb{R})$ does not admit a bi-invariant metric for $n \geqslant 2$.

[*You may assume that the results of Problem sheet 3 Question 4 extend to any Lie group with a bi-invariant Riemannian metric.*]

(c) Show that $\mathbb{RP}^2 \times \mathbb{RP}^2$ does not admit a Riemannian metric of positive sectional curvature.

[*Hint: You may want to think about the orientable double cover.*]

Proof. (a) First we show that for $(n, m) \in \{(1,1), (1,2), (2,1)\}$, $S^n \times S^m$ does not admit positive Ricci curvature. Note that $S^n \times S^m$ is compact and hence geodesically complete by Hopf–Rinow Theorem. If it admits positive Ricci curvature, then by a Corollary Bonnet–Myers Theorem, $\pi_1(S^n \times S^m)$ is finite. But we know that

$$\pi_1(S^n \times S^m) \cong \pi_1(S^n) \times \pi_1(S^m) \cong \begin{cases} \mathbb{Z}^2, & (n,m) = (1,1) \\ \mathbb{Z}, & (n,m) = (1,2), (2,1) \\ 0, & n, m \geqslant 2 \end{cases}$$

So we have a contradiction.

Now suppose that $n, m \geqslant 2$. We equip S^n and S^m with the round metric, and equip $S^n \times S^m$ with the product metric. We know that S^n has constant sectional curvature $K = 1$. So $S^n \times S^m$ has non-negative sectional curvature. Let $(X_1, X_2) \in \mathrm{T}_{(p_1,p_2)}(S^n \times S^m)$. We may assume that $X_1 \neq 0$. Since $n \geqslant 2$, we can find normalised $Y_1 \in \mathrm{T}_{p_1} S^n$ orthogonal to X_1 in S^n. So

$$\begin{aligned} \mathrm{Ric}((X_1, X_2), (X_1, X_2)) &\geqslant R((Y_1, 0), (X_1, X_2), (X_1, X_2), (Y_1, 0)) \\ &= R^{S^n}(Y_1, X_1, X_1, Y_1) = g^{S^n}(X_1, X_1) K^{S^n}(X_1, Y_1) \\ &= g^{S^n}(X_1, X_1) > 0 \end{aligned}$$

Hence the Ricci curvature is positive. ✓

(b) By Question 4 of Sheet 3, the Riemann curvature for $X, Y, Z \in \mathfrak{g}$ is given by

$$R(X,Y)Z = -\frac{1}{4}[[X,Y],Z] = \frac{1}{4}\,\mathrm{ad}_Z \circ \mathrm{ad}_X(Y)$$

For $X, Y \in \mathfrak{g}$, the Ricci curvature is given by

$$\begin{aligned} \mathrm{Ric}(X,Y) &= \sum_{i=1}^{n} g(R(E_i, X, Y), E_i) = -\frac{1}{4}\sum_{i=1}^{n} g([[E_i, X], Y], E_i) \\ &= -\frac{1}{4}\sum_{i=1}^{n} g(\mathrm{ad}_X \circ \mathrm{ad}_Y(E_i), E_i) = -\frac{1}{4}\mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) \\ &= -\frac{1}{4}\kappa(X,Y) \end{aligned}$$

where κ is the Killing form on $\mathfrak{g}$. Since $\mathfrak{z}(\mathfrak{g}) = \{0\}$, κ is non-degenerate. The fact that the sectional curvature is non-negative implies that $\mathrm{Ric}(X,X) > 0$ for normalised $X \in \mathfrak{g}$. Since the unit ball in $\mathfrak{g}$

is compact, $\text{Ric}(X, X) > c > 0$ for some constant c. Using the bi-invariance of the metric we have $\text{Ric}(X, X) > c > 0$ for all normalised vector field X in G.

Since G is connected, $\exp_e : \mathfrak{g} \to G$ is surjective, and hence G is geodesically complete by Hpof–Rinow Theorem. Now by Bonnet–Myers Theorem, G is compact. The universal cover $\widetilde{G}$ of G is also complete with positive Ricci curvature. So $\widetilde{G}$ is also compact.

Note that $\text{SL}(2, \mathbb{R})$ is unbounded, because for $M_\alpha := \text{diag}(\alpha, \alpha^{-1}) \in \text{SL}(2, \mathbb{R})$,

$$\|M_\alpha\|^2 = \alpha^2 + \alpha^{-2} \to \infty \qquad \text{as } \alpha \to \infty$$

So $\text{SL}(2, \mathbb{R})$ is not compact. By the above proof we deduce that $\text{SL}(2, \mathbb{R})$ does not admit a bi-invariant metric. Since $\text{SL}(2, \mathbb{R})$ embeds into $\text{SL}(n, \mathbb{R})$ for $n \geqslant 2$, we conclude that $\text{SL}(n, \mathbb{R})$ does not admit a bi-invariant metric. ✓

(c) Suppose that $\mathbb{RP}^2 \times \mathbb{RP}^2$ admits positive sectional curvature. By Künneth's Theorem, the highest homology group

$$H_4(\mathbb{RP}^2 \times \mathbb{RP}^2) \cong H_2(\mathbb{RP}^2) \otimes H_2(\mathbb{RP}^2) = 0$$

Then $\mathbb{RP}^2 \times \mathbb{RP}^2$ is non-orientable. It has a connected oriented double cover M. Since $\mathbb{RP}^2 \times \mathbb{RP}^2$ is compact with positive sectional curvature, so is M. In particular M is geodesically complete. By Synge's Theorem, M is simply-connected.

On the other hand, note that $\pi_1(\mathbb{RP}^2 \times \mathbb{RP}^2) \cong \pi_1(\mathbb{RP}^2) \times \pi_1(\mathbb{RP}^2) \cong (\mathbb{Z}/2)^2$. $\pi_1(M)$ is an index two subgroup of $\pi_1(\mathbb{RP}^2 \times \mathbb{RP}^2)$. So $\pi_1(M) \cong \mathbb{Z}/2$. In particular M is not simply-connected. Contradiction. Therefore $\mathbb{RP}^2 \times \mathbb{RP}^2$ does not admit positive sectional curvature. Very good! □

Section C: Optional

Question 6

Determine whether each of the following statements is true or false, and give a proof or counterexample as appropriate.

(a) The unitary group $U(m)$ admits a Riemannian metric with strictly positive Ricci curvature for some $m > 1$.

(b) The manifold $\mathcal{S}^n \times \mathcal{S}^m$ admits a Riemannian metric with non-positive sectional curvature if and only if $n = m = 1$.

(c) Euclidean space $\mathbb{R}^n$ admits a constant curvature 1 Riemannian metric for any $n > 1$.

(d) If K is the Klein bottle then $K \times \mathcal{S}^n$ admits a Riemannian metric with positive sectional curvature for any $n > 1$.

(e) Complex projective space $\mathbb{CP}^n$ admits a constant curvature 1 Riemannian metric if and only if $n = 1$.

[*Hint: You may assume that* $\pi_1(\mathbb{CP}^n) = \{1\}$ *and* $H^2(\mathbb{CP}^n) \neq 0$ *for all* n.]