Peize Liu
St. Peter's College
University of Oxford

Problem Sheet 1

B1.2: Set Theory

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For this sheet assume the Empty Set Axiom, the Axioms of Extensionality, Pairs, Unions and the Comprehension Scheme. In question 1 **only** assume also the Power Set Axiom: if X is a set there is a set $\mathcal{P}(X)$ whose elements are precisely the subsets of X (this set is called the power set of X).

I will try to write proofs as formal as possible. In particular I shall adopt the axioms and rules of inference of Deductive System K, together with the available ZF axioms.

We use the first-order language $\mathcal{L} := \{\in, \subseteq; P, \bigcup, \mathcal{P}; \varnothing\}$, where $\in$ and $\subseteq$ are binary predicates, P is a binary function, $\bigcup$ and $\mathcal{P}$ are unary functions, and $\varnothing$ is a constant.

The equality symbol $\doteq$ is used in $\mathcal{L}$ which indicates that two terms have the same value under any model and assignment. The equality symbol $=$ is used in metalanguage which indicates that two strings are equal.

The ZF axioms we shall use in this sheet are listed below:

1. *Extensionality:* $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \leftrightarrow x \doteq y)$;
2. *Pairs:* $\forall x \forall y \forall z(z \in P(x, y) \leftrightarrow (x \doteq z \vee y \doteq z))$;
3. *Unions:* $\forall x \forall y(y \in \bigcup x \leftrightarrow \exists z(y \in z \wedge z \in x))$;
4. *Empty Set:* $\forall x \neg x \in \varnothing$;
5. *Power Sets:* $\forall x \forall y(y \in \mathcal{P}(x) \leftrightarrow y \subseteq x)$;
6. *Comprehension Scheme: Let* $\varphi \in \mathrm{Form}(\mathcal{L})$ *and* $z, w_1, \dots, w_k \in \mathrm{Free}(\varphi)$. *Then* $\forall x \forall w_1 \cdots \forall w_k \exists y \forall z(z \in y \leftrightarrow (z \in x \wedge \varphi))$.

The predicate $\subseteq$ is introduced for convenience. It satisfies $\forall x \forall y(x \subseteq y \leftrightarrow \forall z(z \in x \to z \in y))$.

Question 1 α-

For each statement give either a proof or a counterexample.

(i) $\mathcal{P}(\bigcup X) = X$

(ii) $\bigcup \mathcal{P}(X) = X$

(iii) If $\mathcal{P}(a) \subseteq \mathcal{P}(b)$ then $a \subseteq b$.

Proof. We claim the following rule of inference of K: Let $\varphi, \chi \in \mathrm{Form}(\mathcal{L})$ and let ψ be a sub-formula of φ. Then from $\psi \leftrightarrow \chi$ and φ infer $\varphi[\chi/\psi]$. **[Substitution]**

It is trivially true by Gödel's Completeness Theorem.

(i) The sentence is inconsistent with ZF set theory.

Consider $X := \varnothing$. Let $\{\varnothing\} := P(\varnothing, \varnothing)$.

1 :	$\forall x \forall y \forall z(z \in P(x, y) \leftrightarrow (x \doteq z \vee y \doteq z))$	[Pairs]
2 :	$(z \in \{\varnothing\} \leftrightarrow (z \doteq \varnothing \vee z \doteq \varnothing))$	[A4+MP 3 Times]
3 :	$(z \doteq \varnothing \vee z \doteq \varnothing \leftrightarrow z \doteq \varnothing)$	[Tautology]
4 :	$z \in \{\varnothing\} \leftrightarrow z \doteq \varnothing$	[Substitution 2,3]
5 :	$\forall x \forall y(y \in \bigcup x \leftrightarrow \exists z(y \in z \wedge z \in x))$	[Unions]
6 :	$(y \in \bigcup \varnothing \leftrightarrow \exists z(y \in z \wedge z \in \varnothing))$	[A4+MP Twice]
7 :	$\forall z \neg z \in \varnothing$	[Empty Set]
8 :	$(\forall z \neg z \in \varnothing \to \neg \exists z(y \in z \wedge z \in \varnothing))$	[Theorem in K]
9 :	$\neg \exists z(y \in z \wedge z \in \varnothing)$	[MP 7,8]
10 :	$\neg y \in \bigcup \varnothing$	[Tautology + MP 6,9]
11 :	$\forall x(\forall y \neg y \in x \leftrightarrow x \doteq \varnothing)$	[Uniqueness of Empty Set]
12 :	$(\neg y \in \bigcup \varnothing \leftrightarrow \bigcup \varnothing \doteq \varnothing)$	[A4+MP Twice]
13 :	$\bigcup \varnothing \doteq \varnothing$	[MP]
14 :	$\forall x \forall y(y \in \mathcal{P}(x) \leftrightarrow y \subseteq x)$	[Power Sets]
15 :	$y \subseteq \varnothing \leftrightarrow y \in \mathcal{P}(\varnothing)$	[A4+MP Twice]
16 :	$\forall y \forall x(y \subseteq x \leftrightarrow \forall z(z \in y \to z \in x))$	[Def of $\subseteq$]
17 :	$y \subseteq \varnothing \leftrightarrow \forall z(z \in y \to z \in \varnothing)$	[A4+MP Twice]
18 :	$\neg z \in \varnothing$	[A4+MP 7]
19 :	$(\neg z \in \varnothing \to ((z \in y \to z \in \varnothing) \to \neg z \in y))$	[Tautology]

20 : $(((z \in y \to z \in \varnothing) \to \neg z \in y) \to (\forall z(z \in y \to z \in \varnothing) \to \forall z\, \neg z \in y))$ [Theorem in K]
21 : $(\forall z(z \in y \to z \in \varnothing) \to \forall z\, \neg z \in y))$ [MP 18,19,20]
22 : $(y \subseteq \varnothing \to \forall z\, \neg z \in y)$ [Substitution 21,24]
23 : $(\forall z\, \neg z \in y \leftrightarrow y \doteq \varnothing)$ [A4+MP 11]
24 : $(y \subseteq \varnothing \to y \doteq \varnothing)$ [Substitution 22,23]
25 : $\varnothing \subseteq \varnothing$ [Def of $\subseteq$ + Tautology + MP]
26 : $(y \doteq \varnothing \to (\varnothing \subseteq \varnothing \to y \subseteq \varnothing))$ [Substitution]
27 : $(y \doteq \varnothing \to y \subseteq \varnothing)$ [DT+MP 25 26]
28 : $(y \doteq \varnothing \leftrightarrow y \subseteq \varnothing)$ [Def of $\leftrightarrow$ + MP 24, 27]
29 : $(y \in \mathcal{P}(\varnothing) \leftrightarrow y \doteq \varnothing)$ [Substitution 15,28]
30 : $(y \in \mathcal{P}(\varnothing) \leftrightarrow y \in \{\varnothing\})$ [Substitution 4,29]
31 : $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \leftrightarrow x \doteq y)$ [Extensionality]
32 : $((y \in \mathcal{P}(\varnothing) \leftrightarrow y \in \{\varnothing\}) \to \mathcal{P}(\varnothing) \doteq \{\varnothing\})$ [A4+MP]
33 : $\mathcal{P}(\varnothing) \doteq \{\varnothing\}$ [MP 31,32]
34 : $\mathcal{P}(\varnothing) \doteq \mathcal{P}(\bigcup \varnothing)$ [Substitution 13,33]
35 : $\mathcal{P}(\bigcup \varnothing) \doteq \{\varnothing\}$ [Transitivity of $\doteq$]
36 : $\varnothing \in \{\varnothing\}$ [Substitution 4]
37 : $\forall y\, \neg y \in \{\varnothing\} \leftrightarrow \varnothing \doteq \{\varnothing\}$ [A4+MP 11]
38 : $(\varnothing \in \{\varnothing\} \to \exists y\, y \in \{\varnothing\})$ [Theorem in K]
39 : $\neg \varnothing \doteq \{\varnothing\}$ [Tautology+MP 36,37,38]
40 : $\neg \mathcal{P}(\bigcup \varnothing) \doteq \varnothing$ [Substitution]
41 : $\forall x(\mathcal{P}(\bigcup x) \doteq x)$ [Premise]
42 : $\mathcal{P}(\bigcup \varnothing) \doteq \varnothing$ [A4+MP]

You do not need this — you only need 25: ∅ ⊆ ∅, so that ∅ ∈ P(∅), hence by extensionality ¬ ∅ = P(∅)

It is mentioned in question, so you can stop at a counterexample (and assume it is a trivial FOPC truth that counterexamples prove the negation of the corresponding universal statements)

Note that Line 40 and 42 imply that $\forall x(\mathcal{P}(\bigcup x) \doteq x)$ is inconsistent with ZF axioms.

(ii) The statement is provable from the ZF axioms:

1 : $\forall x \forall y(y \in \bigcup x \leftrightarrow \exists z(y \in z \wedge z \in x))$ [Unions]
2 : $(y \in \bigcup \mathcal{P}(x) \leftrightarrow \exists z(y \in z \wedge z \in \mathcal{P}(x)))$ [A4+MP Twice]
3 : $\forall x \forall y(y \in \mathcal{P}(x) \leftrightarrow y \subseteq x)$ [Power Sets]
4 : $(z \in \mathcal{P}(x) \leftrightarrow z \subseteq x)$ [A4+MP]
5 : $(y \in \bigcup \mathcal{P}(x) \leftrightarrow \exists z(y \in z \wedge z \subseteq x))$ [Substitution 2,4]
6 : $\forall x \forall y(y \subseteq x \leftrightarrow \forall z(z \in y \to z \in x))$ [Def of $\subseteq$]
7 : $(x \subseteq x \leftrightarrow \forall z(z \in x \to z \in x))$ [A4+MP Twice]
8 : $\forall z(z \in x \to z \in x)$ [Theorem in K]
9 : $x \subseteq x$ [MP 7,8]
10 : $(y \in x \to (y \in x \wedge x \subseteq x))$ [Tautology + 9]
11 : $((y \in x \wedge x \subseteq x) \to \exists z(y \in z \wedge z \subseteq x))$ [Theorem in K]
12 : $(y \in x \to \exists z(y \in z \wedge z \subseteq x))$ [HS 10,11]
13 : $(z \subseteq x \leftrightarrow \forall w(w \in z \to w \in x))$ [A4+MP 6]
14 : $((y \in z \wedge z \subseteq x) \leftrightarrow (y \in z \wedge \forall w(w \in z \to w \in x)))$ [Substitution 13]
15 : $((y \in z \wedge \forall w(w \in z \to w \in x)) \to y \in x)$ [Theorem in K]
16 : $((y \in z \wedge z \subseteq x) \to y \in x)$ [Substitution 14,15]
17 : $(((y \in z \wedge z \subseteq x) \to y \in x) \to (\exists z(y \in z \wedge z \subseteq x) \to y \in x))$ [Theorem in K]
18 : $(\exists z(y \in z \wedge z \subseteq x) \to y \in x)$ [MP 16,17]
19 : $(y \in x \leftrightarrow \exists z(y \in z \wedge z \subseteq x))$ [Def of $\leftrightarrow$ + MP 12,18]
20 : $(y \in x \leftrightarrow y \in \bigcup \mathcal{P}(x))$ [Substitution 5,19]
21 : $\forall y \forall z(y \doteq z \leftrightarrow \forall w(w \in y \leftrightarrow w \in z))$ [Extensionality]
22 : $((y \in x \leftrightarrow y \in \bigcup \mathcal{P}(x)) \to x \doteq \bigcup \mathcal{P}(x))$ [A4+MP 3 Times]
23 : $x \doteq \bigcup \mathcal{P}(x)$ [MP 21,22]

You can first simplify to (∃z (y ∈ z ∧ ∀w (w ∈ z → w ∈ x)) ↔ y ∈ x) and prove both sides (backward is trivial with assignment z := x), so you do not need lines 7-12

(iii) The statement is provable from the ZF axioms. By Deduction Theorem, it suffices to prove $\mathcal{P}(a) \subseteq \mathcal{P}(b) \vdash a \subseteq b$:

1 : $\forall x \forall y(y \in \mathcal{P}(x) \leftrightarrow y \subseteq x)$ [Power Sets]
2 : $(y \in \mathcal{P}(a) \leftrightarrow y \subseteq a)$ [A4 + MP Twice]
3 : $(y \in \mathcal{P}(b) \leftrightarrow y \subseteq b)$ [A4 + MP Twice]
4 : $\mathcal{P}(a) \subseteq \mathcal{P}(b)$ [Premise]
5 : $(y \in P(a) \to y \in P(b))$ [Def of $\subseteq$+A4+MP 4]

Apply A4 by assigning {x := a, y := a} and {x := a, y := b}, and you'll find this surprisingly simple

$$
\begin{array}{lll}
6: & (y \subseteq a \to y \subseteq b) & \text{[Substitution 2,3,5]} \\
7: & (y \subseteq a \leftrightarrow \forall z(z \in y \to z \in a)) & \text{[Def of } \subseteq\text{+A4+MP]} \\
8: & (y \subseteq b \leftrightarrow \forall z(z \in y \to z \in b)) & \text{[Def of } \subseteq\text{+A4+MP]} \\
9: & (\forall z(z \in y \to z \in a) \to \forall z(z \in y \to z \in b)) & \text{[Substitution 6,7,8]} \\
10: & ((\forall z(z \in y \to z \in a) \to \forall z(z \in y \to z \in b)) \to \forall z(z \in a \to z \in b)) & \text{[Theorem in } K\text{]} \\
11: & \forall z(z \in a \to z \in b) & \text{[MP 9,10]} \\
12: & a \subseteq b & \text{[Def of } \subseteq\text{+A4+MP 11]}
\end{array}
$$

✓

□

Question 2 α-

(a) Prove that the unordered pair $\{x, y\}$ of x and y is the unique set whose elements are precisely x and y.

(b) Let $\phi(z, w_1, \ldots, w_k)$ be a formula of $\mathcal{L}$ and $w_1, \ldots, w_k, x$ sets. Prove that the subset y of x afforded by the Comprehension Scheme is unique with the stated property.

Proof. (a) The formal formula to prove is $\forall w(\forall z(z \in w \leftrightarrow (x \doteq z \vee y \doteq z)) \to w \doteq \{x, y\})$, where $\{x, y\}$ should be interpreted as $P(x, y)$ in our language $\mathcal{L}$. By Deduction Theorem and A4 Axiom, it suffices to prove that

$$\forall z(z \in w \leftrightarrow (x \doteq z \vee y \doteq z)) \vdash w \doteq \{x, y\}$$

Otherwise, your claim is "for any specific assignment to z & w, LHS implies RHS" — this is false

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$$
\begin{array}{lll}
1: & \forall x \forall y \forall z(z \in \{x, y\} \leftrightarrow (x \doteq z \vee y \doteq z)) & \text{[Pairs]} \\
2: & (z \in \{x, y\} \leftrightarrow (x \doteq z \vee y \doteq z)) & \text{[A4+MP 3 Times]} \\
3: & (z \in w \leftrightarrow (x \doteq z \vee y \doteq z)) & \text{[Premise]} \\
4: & (z \in w \leftrightarrow z \in \{x, y\}) & \text{[Substitution 2,3]} \\
5: & \forall z(z \in w \leftrightarrow z \in \{x, y\}) & [\forall\ 4] \\
6: & \forall a \forall b(a \doteq b \leftrightarrow \forall z(z \in a \leftrightarrow z \in b)) & \text{[Extensionality]} \\
7: & (w \doteq \{x, y\} \leftrightarrow \forall z(z \in a \leftrightarrow z \in b)) & \text{[A4+MP Twice]} \\
8: & w \doteq \{x, y\} & \text{[MP 6,7]}
\end{array}
$$

Did you notice that this generalisation is invalid given your claim with free variable z?

(b) The formal formula to prove is $\forall y \forall y'(\forall z(z \in y \leftrightarrow (z \in x \wedge \phi)) \wedge \forall z'(z' \in y \leftrightarrow (z' \in x \wedge \phi))) \to y \doteq y')$. By Deduction Theorem and A4 Axiom it suffices to prove that

$$\{\forall z(z \in y \leftrightarrow (z \in x \wedge \phi)), \forall z'(z' \in y \leftrightarrow (z' \in x \wedge \phi))\} \vdash y \doteq y'$$

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$$
\begin{array}{lll}
1: & \forall z(z \in y \leftrightarrow (z \in x \wedge \phi)) & \text{[Premise]} \\
2: & (z \in y \leftrightarrow (z \in x \wedge \phi)) & \text{[A4+MP]} \\
3: & \forall z'(z' \in y' \leftrightarrow (z' \in x \wedge \phi)) & \text{[Premise]} \\
4: & (z \in y' \leftrightarrow (z \in x \wedge \phi)) & \text{[A4+MP]} \\
5: & (z \in y \leftrightarrow z \in y') & \text{[Substitution 2,4]} \\
6: & \forall z(z \in y \leftrightarrow z \in y') & [\forall\ 5] \\
7: & \forall a \forall b(a \doteq b \leftrightarrow \forall z(z \in a \leftrightarrow z \in b)) & \text{[Extensionality]} \\
8: & (y \doteq y' \leftrightarrow \forall z(z \in y \leftrightarrow z \in y')) & \text{[A4+MP Twice]} \\
9: & y \doteq y' & \text{[MP 6,8]}
\end{array}
$$

✓

□

Question 3 α

Let a be a set. Prove that $\{a\} \times \{a\} = \{\{\{a\}\}\}$.

Proof. In the metalanguage, the Cartesian product of two sets X and Y is the set of all ordered pairs $\langle x, y\rangle$ where $x \in X$ and $y \in Y$. The formal definition and existence of the Cartesian product relies on the Axiom of Power Sets, without which we can only define the Cartesian product of finite sets (in the sense of metalanguage) by enumerating all its elements.

You can define any Cartesian product. You just cannot guarantee its existence without power sets; and actually in this specific finite case, the existence itself can be proven formally

In this way, the Cartesian product of $\{a\}$ and $\{a\}$ is the unique singleton $\{\langle a, a\rangle\}$.

By definition, $\langle a, a\rangle = \{\{a\}, \{a, a\}\} = \{\{a\}, \{a\}\} = \{\{a\}\}$. Hence $\{a\} \times \{a\} = \{\{\{a\}\}\}$. ✓ □

Question 4 α

(a) Show that if we define an ordered triple (a, b, c) of sets to be $\langle\langle a, b\rangle, c\rangle$ then this definition "works": i.e. if $(a, b, c) = (a', b', c')$ then $a = a'$, $b = b'$ and $c = c'$.

(b) For each of the following alternative possible definitions of an ordered triple, prove that the definition "works" or give a counterexample.

(i) $(a, b, c)_1 = \{\{a\}, \{a, b\}, \{a, b, c\}\}$

(ii) $(a, b, c)_2 = \{\langle 0, a\rangle, \langle 1, b\rangle, \langle 2, c\rangle\}$ (where $0 = \varnothing$, $1 = \{0\}$, $2 = \{0, 1\}$)

(iii) $(a, b, c)_3 = (\{0, a\}, \{1, b\}, \{2, c\})$ where $(., ., .)$ is as in part (a)

(iv) $(a, b, c)_4 = \{\{0, a\}, \{1, b\}, \{2, c\}\}$.

Sketch of Proof. (a) $\langle x, y\rangle$ is defined to be $\{\{x\}, \{x, y\}\} := P(P(x, x), P(x, y))$. It has been proven in the lectures that

$$\forall x\forall x'\forall y\forall y'(\langle x, y\rangle \doteq \langle x', y'\rangle \leftrightarrow (x \doteq x' \wedge y \doteq y'))$$

We shall prove that

$$(\langle\langle a, b\rangle, c\rangle \doteq \langle\langle a', b'\rangle, c'\rangle \leftrightarrow (a \doteq a' \wedge b \doteq b' \wedge c \doteq c'))$$

1 :	$\forall x\forall x'\forall y\forall y'(\langle x, y\rangle \doteq \langle x', y'\rangle \leftrightarrow (x \doteq x' \wedge y \doteq y'))$	[Ordered Pairs]
2 :	$(\langle\langle a, b\rangle, c\rangle \doteq \langle\langle a', b'\rangle, c'\rangle \leftrightarrow (\langle a, b\rangle \doteq \langle a', b'\rangle \wedge c \doteq c'))$	[A4+MP]
3 :	$(\langle a, b\rangle \doteq \langle a', b'\rangle \leftrightarrow (a \doteq a' \wedge b \doteq b'))$	[A4+MP]
4 :	$(\langle\langle a, b\rangle, c\rangle \doteq \langle\langle a', b'\rangle, c'\rangle \leftrightarrow (a \doteq a' \wedge b \doteq b' \wedge c \doteq c'))$	[Substitution 2,3] ✓

(b) (i) The definition does not work. Before proceeding the proof we must first prove that for any sets x, y, z there exists a unique set $\{x, y, z\}$ which contains exactly x, y, z as elements. The formal statement is

$$\forall x\forall y\forall z\forall w\forall A((w \in A \leftrightarrow (w \doteq x \vee w \doteq y \vee w \doteq z)) \leftrightarrow A \doteq \{x, y, z\})$$

Actually, instead of defining a new function "ordered triple", we sometimes define finite extensional set expressions in general as: {a1, a2, a3, a4, …} = … ∪{∪{{a1, a2}, {a3}}, {a4}} … whose existence is guaranteed by pair and union axioms

We defer the proof to Question 7.(i).

Second, we claim the following proposition:

$$(x \doteq y \to \{x, y, z\} \doteq \{x, z\})$$

Third, consider the assignment $a \mapsto 0$, $b \mapsto 0$, $c \mapsto 1$, $a' \mapsto 0$, $b' \mapsto 1$ and $c' \mapsto 1$. Note that

$$(0, 0, 1)_1 := \{\{0\}, \{0, 0\}, \{0, 0, 1\}\} \doteq \{\{0\}, \{0\}, \{0, 1\}\} \doteq \{\{0\}, \{0, 1\}\} \doteq \{1, 2\}$$
$$(0, 1, 1)_1 := \{\{0\}, \{0, 1\}, \{0, 1, 1\}\} \doteq \{\{0\}, \{0, 1\}, \{0, 1\}\} \doteq \{\{0\}, \{0, 1\}\} \doteq \{1, 2\}$$

Hence $(0, 0, 1)_1 \doteq (0, 1, 1)_1$. But it is clear that $0 \not\doteq 1$. ✓

(ii) The definition works. Suppose that $\{\langle 0, a\rangle, \langle 1, b\rangle, \langle 2, c\rangle\} \doteq \{\langle 0, a'\rangle, \langle 1, b'\rangle, \langle 2, c'\rangle\}$. Then $\langle 0, a\rangle \in \{\langle 0, a'\rangle, \langle 1, b'\rangle, \langle 2, c'\rangle\}$. In particular, $\langle 0, a\rangle \doteq \langle 0, a'\rangle \vee \langle 0, a\rangle \doteq \langle 1, b'\rangle \vee \langle 0, a\rangle \doteq \langle 2, c'\rangle$. But it is clear that $0 \not\doteq 1$ and $1 \not\doteq 2$ (0). So $\langle 0, a\rangle \not\doteq \langle 1, b'\rangle \wedge \langle 0, a\rangle \not\doteq \langle 2, c'\rangle$. Therefore $\langle 0, a\rangle \doteq \langle 0, a\rangle$ and $a \doteq a'$. Similarly $b \doteq b'$ and $c \doteq c'$. ✓

(iii) The definition works. Suppose that $(\{0, a\}, \{1, b\}, \{2, c\}) \doteq (\{0, a'\}, \{1, b'\}, \{2, c'\})$. Then $\{0, a\} \doteq \{0, a'\} \wedge \{1, b\} \doteq \{1, b'\} \wedge \{2, c\} \doteq \{2, c'\}$. $\{0, a\} \doteq \{0, a'\}$ implies that $a \doteq a'$. Similarly $b \doteq b'$ and $c \doteq c'$. ✓

(iv) The definition does not work. Consider the assignment $a \mapsto 2$, $b \mapsto 0$, $c \mapsto 1$, $a' \mapsto 1$, $b' \mapsto 2$ and $c' \mapsto 0$.

$$(2, 0, 1)_4 := \{\{0, 2\}, \{1, 0\}, \{2, 1\}\} \doteq \{\{0, 1\}, \{1, 2\}, \{2, 0\}\}$$
$$(1, 2, 0)_4 := \{\{0, 1\}, \{1, 2\}, \{2, 0\}\}$$

Hence $(2, 0, 1)_4 \doteq (1, 2, 0)_4$. But it is clear that $0 \not\doteq 1$. ✓ □

Question 5 β

A set a is called *transitive* if $\bigcup a \subseteq a$, i.e. if, for all sets x, if $x \in a$ then $x \subseteq a$. Prove that

(i) $\varnothing$ is transitive

(ii) if a is transitive then so is $a \cup \{a\}$ (this set is denoted a^+)

(iii) a is transitive if and only if $\bigcup(a \cup \{a\}) = a$

(iv) a is transitive if and only if, for all sets x, y, if $x \in y \in a$ then $x \in a$

(v) the intersection of any (non-empty) set of transitive sets is transitive

(vi) the union of any set of transitive sets is transitive

Write a formula in $\mathcal{L}$ with a free variable x expressing "x is transitive".

Proof. The formal formula of x being transitive is

$$\forall y(y \in x \to \forall z(z \in y \to z \in x))$$ ✓

Let T be a unary predicate such that $T(x)$ if and only if the set x is transitive. Hence we have

$$\forall x(T(x) \leftrightarrow \forall y(y \in x \to y \subseteq x))$$

(i) We shall prove that $T(\varnothing)$:

1 :	$\forall y\, \neg y \in \varnothing$	[Empty Set]
2 :	$(\forall y\, \neg y \in \varnothing \to \neg y \in \varnothing)$	[A4]
3 :	$\neg y \in \varnothing$	[MP 1,2]
4 :	$(\neg y \in \varnothing \to (y \in \varnothing \to y \subseteq \varnothing))$	[Tautology]
5 :	$(y \in \varnothing \to y \subseteq \varnothing)$	[MP 3,4]
6 :	$\forall y(y \in \varnothing \to y \subseteq \varnothing)$	[$\forall$ 5]
7 :	$\forall x(T(x) \leftrightarrow \forall y(y \in x \to y \subseteq x))$	[Def of T]
8 :	$(\forall x(T(x) \leftrightarrow \forall y(y \in x \to y \subseteq x)) \to (T(\varnothing) \leftrightarrow \forall y(y \in \varnothing \to y \subseteq \varnothing)))$	[A4]
9 :	$T(\varnothing) \leftrightarrow \forall y(y \in \varnothing \to y \subseteq \varnothing)$	[MP 7,8]
10 :	$T(\varnothing)$ ✓	[MP 6,10]

Hence $\varnothing$ is transitive.

(ii) We define a^+ or $a \cup \{a\}$ to be $\bigcup P(a, P(a,a))$. First we shall prove the following lemma: **[Lemma 1]** $\forall y(y \in a^+ \leftrightarrow (y \in a \vee y \doteq a))$.

1 :	$\forall y(y \in \bigcup P(a, P(a,a)) \leftrightarrow \exists z(y \in z \vee z \in P(a, P(a,a))))$	[Unions]
2 :	$(y \in \bigcup P(a, P(a,a)) \leftrightarrow \exists z(y \in z \wedge z \in P(a, P(a,a))))$	[A4+MP]
4 :	$\forall z(z \in P(a, P(a,a)) \leftrightarrow (z \doteq a \vee z \doteq \{a\}))$	[Pairs]
5 :	$(z \in P(a, P(a,a)) \leftrightarrow (z \doteq a \vee z \doteq \{a\}))$	[A4+MP]
6 :	$(y \in \bigcup P(a, P(a,a)) \leftrightarrow \exists z(y \in z \wedge (z \doteq a \vee z \doteq \{a\})))$	[Substitution 3,5]
7 :	∃z $((y \in z \wedge (z \doteq a \vee z \doteq \{a\})) \leftrightarrow (y \in a \vee y \in \{a\}))$	[Theorem in K] ← Without ∃z it's not a theorem
8 :	$(y \in \bigcup P(a, P(a,a)) \leftrightarrow \exists z(y \in a \vee y \in \{a\}))$	[Substitution 6,7]
9 :	$(y \in \{a\} \leftrightarrow y \doteq a)$ ← Above is why you can drop ∃z here	[Theorem]
10 :	$(y \in \bigcup P(a, P(a,a)) \leftrightarrow \exists z(y \in a \vee y \doteq a))$	[Substitution 8,9]
11 :	$\forall y(y \in \bigcup P(a, P(a,a)) \leftrightarrow \exists z(y \in a \vee y \doteq a))$	[$\forall$ 10]

Next, we shall prove that $(T(a) \to T(a^+))$. By Deduction Theorem it suffices to prove $T(a) \vdash T(a^+)$, which is equivalent to $T(a) \vdash (y \in a^+ \to y \subseteq a^+)$. Again by Deduction Theorem it suffices to prove $\{T(a), y \in a^+\} \vdash y \subseteq a^+$.

1 :	$\forall x(T(x) \leftrightarrow \forall y(y \in x \rightarrow y \subseteq x))$	[Def of T]
2 :	$(T(a) \leftrightarrow \forall y(y \in a \rightarrow y \subseteq a))$	[A4+MP]
3 :	$T(a)$	[Premise]
4 :	$\forall y(y \in a \rightarrow y \subseteq a)$	[MP 2,3]
5 :	$(y \in a \rightarrow y \subseteq a)$	[A4+MP]
6 :	$\forall y(y \in a^+ \leftrightarrow (y \in a \vee y \doteq a))$	[Lemma 1]
7 :	$(y \in a^+ \leftrightarrow (y \in a \vee y \doteq a))$	[A4+MP]
8 :	$y \in a^+$	[Premise]
9 :	$(y \in a \vee y \doteq a)$	[MP 7,8]
10 :	$a \subseteq a^+$	[Corollary of Lemma 1]
11 :	$(a \subseteq a^+ \rightarrow (y \doteq a \rightarrow y \subseteq a^+))$	[Theorem in K]
12 :	$(y \doteq a \rightarrow y \subseteq a^+)$	[MP 10,11]
13 :	$\forall x \forall y \forall z(x \subseteq y \rightarrow (y \subseteq z \rightarrow x \subseteq z))$	[Lemma 2]
14 :	$(y \subseteq a \rightarrow (a \subseteq a^+ \rightarrow y \subseteq a^+))$	[A4+MP]
15 :	$(y \in a \rightarrow (a \subseteq a^+ \rightarrow y \subseteq a^+))$	[HS 5,14]
16 :	$(a \subseteq a^+ \rightarrow (y \in a \rightarrow y \subseteq a^+))$	[DT+MP 15]
17 :	$(y \in a \rightarrow y \subseteq a^+)$	[MP 10,16]
18 :	$(((y \in a \rightarrow y \subseteq a^+) \wedge (y \doteq a \rightarrow y \subseteq a^+)) \rightarrow ((y \in a \vee y \doteq a) \rightarrow y \subseteq a^+))$	[Tautology]
19 :	$((y \in a \vee y \doteq a) \rightarrow y \subseteq a^+)$	[MP 12,17,18]
20 :	$y \subseteq a^+$	[MP 12,19]

In Line 13 of the proof of (ii), we claimed the following lemma: **[Lemma 2]** $\forall x \forall y \forall z(x \subseteq y \rightarrow (y \subseteq z \rightarrow x \subseteq z))$. We only need to prove that $\{x \subseteq y, y \subseteq z\} \vdash x \subseteq z$. The proof is as follows:

It is simpler if you also expand x ⊆ z and prove: {x ⊆ y, y ⊆ z, w ∈ x} ⊢ w ∈ z

1 :	$\forall x \forall y(x \subseteq y \leftrightarrow \forall w(w \in x \rightarrow w \in y))$	[Def of $\subseteq$]
2 :	$(x \subseteq y \leftrightarrow \forall w(w \in x \rightarrow w \in y))$	[A4+MP]
3 :	$(y \subseteq z \leftrightarrow \forall w(w \in y \rightarrow w \in z))$	[A4+MP]
4 :	$(x \subseteq z \leftrightarrow \forall w(w \in x \rightarrow w \in z))$	[A4+MP]
5 :	$x \subseteq y$	[Premise]
6 :	$y \subseteq z$	[Premise]
7 :	$\forall w(w \in x \rightarrow w \in y)$	[MP]
8 :	$\forall w(w \in y \rightarrow w \in z)$	[MP]
9 :	$(w \in x \rightarrow w \in y)$	[A4+MP]
10 :	$(w \in y \rightarrow w \in z)$	[A4+MP]
11 :	$(w \in x \rightarrow w \in z)$	[HS]
12 :	$\forall w(w \in x \rightarrow w \in z)$	[$\forall$]
13 :	$x \subseteq z$	[MP 4,12]

(iii) We shall prove that $(T(a) \leftrightarrow a \doteq \bigcup a^+)$. First we prove $a \doteq \bigcup a^+ \vdash T(a)$:

1 :	$(x \in a \rightarrow (y \in x \rightarrow (\exists z(y \in z \wedge z \in a))))$	[Theorem in K]
2 :	$(x \in a \rightarrow (y \in x \rightarrow (y \in a \vee \exists z(y \in z \wedge z \in a))))$	[Tautology+MP]
3 :	$\forall y(x \in a \rightarrow (y \in x \rightarrow (y \in a \vee \exists z(y \in z \wedge z \in a))))$	[$\forall$]
4 :	$(\exists z(y \in z \wedge (z \in a \vee z \doteq a)) \leftrightarrow (y \in a \vee \exists z(y \in z \wedge z \in a)))$	[Theorem in K]
5 :	$\forall y(x \in a \rightarrow (y \in x \rightarrow \exists z(y \in z \wedge (z \in a \vee z \doteq a))))$	[Substitution]
6 :	$(z \in a^+ \leftrightarrow (z \in a \vee z \doteq a))$	[Lemma 1]
7 :	$\forall y(x \in a \rightarrow (y \in x \rightarrow \exists z(y \in z \wedge z \in a^+)))$	[Substitution]
8 :	$\forall y(x \in a \rightarrow (y \in x \rightarrow y \in \bigcup a^+))$	[Unions]
9 :	$(x \in a \rightarrow x \subseteq \bigcup a^+)$	[Def of $\subseteq$]
10 :	$\forall x(x \in a \rightarrow x \subseteq \bigcup a^+)$	[$\forall$]
11 :	$a \doteq \bigcup a^+$	[Premise]
12 :	$\forall x(x \in a \rightarrow x \subseteq a)$	[Substitution]
13 :	$T(a)$	[Def of$T$]

To use Def. of ⊆, you need ∀y to be inside (the first arrow) instead; so you should not use generalisation at step 3: you take x ∈ a as a promise, MP with 8, use generalisation, and then DT to get the appropriate claim: (x ∈ a → ∀y (y ∈ x → y ∈ a+))

Next we prove $T(a) \vdash a \doteq \bigcup a^+$:

1 :	$T(a)$	[Premise]
2 :	$\forall y(y \in a \to y \subseteq a)$	[Def of T]
3 :	$\forall y(y \in a \to y \subseteq a) \to (\exists y(x \in y \land y \in a) \to x \in a))$	[Theorem in K]
4 :	$(\exists y(x \in y \land y \in a) \to x \in a)$	[MP]
5 :	$((x \in a \lor \exists y(x \in y \land y \in a)) \to x \in a)$	[Tautology+MP]
6 :	$(x \in a \to (x \in a \lor \exists y(x \in y \land y \in a)))$	[Theorem in K]
7 :	$(x \in a \leftrightarrow (x \in a \lor \exists y(x \in y \land y \in a)))$	[Def of $\leftrightarrow$]
8 :	$(\exists y(x \in y \land (y \in a \lor y \doteq a)) \leftrightarrow (x \in a \lor \exists y(x \in y \land y \in a)))$	[Theorem in K]
9 :	$(x \in a \leftrightarrow \exists y(x \in y \land (y \in a \lor y \doteq a)))$	[Substitution]
10 :	$(y \in a^+ \leftrightarrow (y \in a \lor y \doteq a))$	[Lemma 1]
11 :	$(x \in a \leftrightarrow \exists y(x \in y \land y \in a^+))$	[Substitution]
12 :	$(x \in \bigcup a^+ \leftrightarrow \exists y(x \in y \land y \in a^+))$	[Unions]
13 :	$(x \in a \leftrightarrow x \in \bigcup a^+)$	[Substitution]
14 :	$\forall x(x \in a \leftrightarrow x \in \bigcup a^+)$	[$\forall$]
15 :	$a \doteq \bigcup a^+$	[Extensionality]

Then by Deduction Theorem and definition of $\leftrightarrow$ we deduce that $\vdash (T(a) \leftrightarrow a \doteq \bigcup a^+)$.

(iv) We shall prove that $(T(a) \leftrightarrow \forall x \forall y((x \in y \land y \in a) \to x \in a))$. We first prove $T(a) \vdash \forall x \forall y((x \in y \land y \in a) \to x \in a)$.

1 :	$T(a)$	[Premise]
2 :	$(y \in a \to y \subseteq a)$	[Def of T]
3 :	$((x \in y \land y \in a) \to (x \in y \land y \subseteq a))$	[Substitution]
4 :	$((x \in y \land y \subseteq a) \to x \in a)$	[Theorem]
5 :	$((x \in y \land y \in a) \to x \in a)$	[HS]
6 :	$\forall x \forall y((x \in y \land y \in a) \to x \in a)$	[$\forall$]

More directly, this is just tautology + MP 2

Next we prove $\forall x \forall y((x \in y \land y \in a) \to x \in a) \vdash T(a)$.

1 :	$\forall x \forall y((x \in y \land y \in a) \to x \in a)$	[Premise]
2 :	$(\forall x \forall y((x \in y \land y \in a) \to x \in a) \leftrightarrow \forall y(y \in a \to \forall x(x \in y \to x \in a)))$	[Theorem in K]
3 :	$\forall y(y \in a \to \forall x(x \in y \to x \in a))$	[MP]
4 :	$\forall y(y \in a \to y \subseteq a)$	[Def of $\subseteq$]
5 :	$T(a)$	[Def of T]

Then by Deduction Theorem and definition of $\leftrightarrow$ we deduce that $\vdash (T(a) \leftrightarrow \forall x \forall y((x \in y \land y \in a) \to x \in a))$.

∀y should be inside: … ∀y (y ∈ x → T(y)) …

(v) We shall prove that $\forall x \forall y((\neg x \doteq \varnothing \land (y \in x \to T(y))) \to T(\bigcap x))$, where the intersection (unary) function is defined via Axiom Scheme of Comprehension:

$$\forall z \left(z \in \bigcap x \leftrightarrow \left(z \in \bigcup x \land \forall y\,(y \in x \to z \in y)\right)\right)$$

or, informally,

$$\bigcap x := \left\{z \in \bigcup x :\ \forall y\,(y \in x \to z \in y)\right\}$$

By Deduction Theorem, it suffices to prove that $\{\neg x \doteq \varnothing,\ \forall y\,(y \in x \to T(y))\} \vdash T(\bigcap x)$. As $T(\bigcap x)$ if and only if $\forall z(z \in \bigcap x \to z \subseteq \bigcap x)$, by Deduction Theorem, it suffices to prove that

$$\left\{\neg x \doteq \varnothing,\ \forall y\,(y \in x \to T(y))\,,\ z \in \bigcap x\right\} \vdash z \subseteq \bigcap x$$

Note that $z \subseteq \bigcap x$ if and only if $\forall w(w \in z \to w \in \bigcap x)$. By Deduction Theorem, it suffices to prove that

$$\left\{\neg x \doteq \varnothing,\ \forall y\,(y \in x \to T(y))\,,\ z \in \bigcap x,\ w \in z\right\} \vdash w \in \bigcap x$$

1 :	$\forall y(y \in x \to T(y))$	[Premise]
2 :	$(y \in x \to T(y))$	[A4+MP]
3 :	$T(y) \leftrightarrow \forall z(z \in y \to z \subseteq y)$	[Def of T]
4 :	$((T(y) \leftrightarrow \forall z(z \in y \to z \subseteq y)) \to (T(y) \to (z \in y \to z \subseteq y)))$	[Theorem in K]
5 :	$(T(y) \to (z \in y \to z \subseteq y))$	[MP]
6 :	$(y \in x \to (z \in y \to z \subseteq y))$	[HS]
7 :	$z \in \bigcap x$	[Premise]
8 :	$(y \in x \to z \in y)$	[Def of $\bigcap$]
9 :	$(y \in x \to z \subseteq y)$	[DT+MP] ← More directly, this is just A2 + MP x2
10 :	$w \in z$	[Premise]
11 :	$(w \in z \to (z \subseteq y \to w \in y))$	[Theorem]
12 :	$(z \subseteq y \to w \in y)$	[MP]
13 :	$(y \in x \to w \in y)$	[HS]
14 :	$\forall y(y \in x \to w \in y)$	[$\forall$]
15 :	$w \in \bigcap x$ ←	[Def of $\bigcap$]

You also need w ∈ ∪x to infer this. Did you notice that you never used the condition ¬x = ∅? This is where you use it (to get ∃y y ∈ x and hence ∃y (y ∈ x ∧ w ∈ y)

(vi) We shall prove that $\forall x \forall y((y \in x \to T(y)) \to T(\bigcup x))$. Similar to (v), it suffices to prove

Again, ∀y should be inside: … ∀y (y ∈ x → T(y)) …

$$\left\{\forall y\,(y \in x \to T(y))\,,\ z \in \bigcup x,\ w \in z\right\} \vdash w \in \bigcup x$$

1 :	$\forall y(y \in x \to T(y))$	[Premise]
2 :	$(y \in x \to T(y))$	[A4+MP]
3 :	$T(y) \leftrightarrow \forall z(z \in y \to z \subseteq y)$	[Def of$T$]
4 :	$((T(y) \leftrightarrow \forall z(z \in y \to z \subseteq y)) \to (T(y) \to (z \in y \to z \subseteq y)))$	[Theorem in K]
5 :	$(T(y) \to (z \in y \to z \subseteq y))$	[MP]
6 :	$(y \in x \to (z \in y \to z \subseteq y))$	[HS]
7 :	$((y \in x \to (z \in y \to z \subseteq y)) \to ((y \in x \wedge z \in y) \to (y \in x \wedge z \subseteq y)))$	[Theorem in K]
8 :	$((y \in x \wedge z \in y) \to (y \in x \wedge z \subseteq y))$	[MP]
9 :	$\exists y(y \in x \wedge z \in y)$	[Unions]
10 :	$\exists y(y \in x \wedge z \subseteq y)$	[MP]
11 :	$w \in z$	[Premise]
12 :	$(w \in z \to (z \subseteq y \to w \in y))$	[Theorem]
13 :	$(z \subseteq y \to w \in y)$	[MP]
14 :	$\exists y(y \in x \wedge w \in y)$	[Substitution]
15 :	$w \in \bigcup x$ ✓	[Def of$\bigcup$]

□

Question 6 α-

Prove the following

(i) If x is a set, there is no set whose elements are all the sets y with $y \notin x$.

(ii) There is no set of all one-element sets.

(iii) There is no set of all two-element sets.

Sketch of Proof. (i) We shall prove that $\neg\exists z\forall y(y \in z \leftrightarrow \neg y \in x)$. It suffices to prove that $\exists z\forall y(y \in z \leftrightarrow \neg y \in x)$ is inconsistent with ZF axioms. Fix z as a constant symbol in $\mathcal{L}$ that satisfies the existential statement. Then $x \cup z := \bigcup P(x, z)$ satisfies that $\forall y(y \in x \cup z)$. In other words, $x \cup z$ is the set of all sets. We will have Russell's Paradox as $(x \cup z \in x \cup z \leftrightarrow x \cup z \notin x \cup z)$. ✓

(ii) A set x is a singleton if and only if $\exists z\forall y(y \in x \to y \doteq z)$. We shall prove that

No, empty set also satisfies this statement

$$\varphi := \neg\exists w\forall x\underline{\exists z\forall y}((y \in x \to y \doteq z) \to x \in w)$$

Inside next bracket — Double-sided arrow? Otherwise w can contain other sets

It suffices to prove that $\neg\varphi$ is inconsistent with ZF axioms. Fix w as a constant symbol in $\mathcal{L}$ that satisfies the existential statement in $\neg\varphi$. Consider $\bigcup w$. Note that for all sets x, $\{x\}$ is a singleton. And we have $(\{x\} \in w \to x \in \bigcup w)$. By *modus ponens* and generalization we deduce that $\forall x(x \in\in \bigcup w)$. In other words, $\bigcup w$ is the set of all sets. This is impossible by Russell's Paradox. ✓

(iii) A set x is a doubleton if and only if $\exists z \exists w(\neg z \doteq w \land \forall y(y \in x \rightarrow (y \doteq z \lor y \doteq w)))$. We shall prove that

Again, this applies to any set with no more than two elements

$$\psi := \neg\exists a \forall x \exists z \exists w((\neg z \doteq w \land \forall y(y \in x \rightarrow (y \doteq z \lor y \doteq w))) \rightarrow x \in a)$$

Inside next bracket — Again, double-sided arrow?

It suffices to prove that $\neg\psi$ is inconsistent with ZF axioms. Fix a as a constant symbol in $\mathcal{L}$ that satisfies the existential statement in $\neg\psi$. Consider $\bigcup a$. For any set x, there exists a set y such that $x \neq y$. This is because there are at least two different sets:

$$0 := \varnothing, \ 1 := \{\varnothing\}$$

Then $\{x, y\}$ is a doubleton. Hence $\{x, y\} \in a$ and $x \in \bigcup a$. We deduce that $\forall x(x \in \bigcup a)$. In other words, $\bigcup a$ is the set of all sets. This is impossible by Russell's Paradox. ✓ □

Question 7 β

(a) Prove that

(i) if a, b, c are sets then $\{a, b, c\}$ is a set.

(ii) if $x_1, ..., x_n$ are sets then $\{x_1, ..., x_n\}$ is a set (here $n \in \mathbb{N}$).

(iii) if X is a finite set then $\mathcal{P}(X)$ is a set.

(iv) if X is a finite set then the collection of all two-element subsets of X is a set.

(b) Suppose X is a set all of whose elements are finite sets. Prove that there is a set Y consisting of all the elements of X that have an *even* number of elements. (Note that it is not sufficient that Y "is" a subset of X.)

Sketch of Proof. (a) (i) We define $\{a, b, c\}$ to be $\bigcup P(P(a, b), P(c, c))$. We shall prove that this is the unique set that satisfies $\forall x(x \in \{a, b, c\} \leftrightarrow (x \doteq a \lor x \doteq b \lor x \doteq c))$. The uniqueness is trivial by the Axiom of Extensionality.

1 :	$(x \in \bigcup P(P(a, b), P(c, c)) \leftrightarrow \exists y(x \in y \land y \in P(P(a, b), P(c, c))))$	[Unions]
2 :	$(y \in P(P(a, b), P(c, c)) \leftrightarrow (y \doteq P(a, b) \lor y \doteq P(c, c)))$	[Pairs]
3 :	$(x \in \bigcup P(P(a, b), P(c, c)) \leftrightarrow \exists y(x \in y \land (y \doteq P(a, b) \lor y \doteq P(c, c))))$	[Substitution]
4 :	$(\exists y(x \in y \land (y \doteq P(a, b) \lor y \doteq P(c, c))) \leftrightarrow (x \in P(a, b) \lor x \in P(c, c)))$	[Theorem in K]
5 :	$(x \in \bigcup P(P(a, b), P(c, c)) \leftrightarrow (x \in P(a, b) \lor x \in P(c, c)))$	[Substitution]
6 :	$(x \in P(a, b) \leftrightarrow (x \doteq a \lor x \doteq b))$	[Pairs]
7 :	$(x \in P(c, c) \leftrightarrow (x \doteq c \lor x \doteq c))$	[Pairs]
8 :	$(x \in \bigcup P(P(a, b), P(c, c)) \leftrightarrow (x \doteq a \lor x \doteq b \lor x \doteq c \lor x \doteq c))$	[Substitution]
9 :	$((x \doteq c \lor x \doteq c) \leftrightarrow x \doteq c)$	[Tautology]
10 :	$(x \in \bigcup P(P(a, b), P(c, c)) \leftrightarrow (x \doteq a \lor x \doteq b \lor x \doteq c))$ ✓	[Substitution]

(ii) We use induction on n. The case $n = 0, 1, 2, 3$ is proven. Suppose that $\{x_1, ..., x_n\}$ is a set. Formally, suppose that there exists a unique set $\{x_1, ..., x_n\}$ such that

$$\forall x \left(x \in \{x_1, ..., x_n\} \leftrightarrow \bigvee_{i=1}^{n} x \doteq x_i \right)$$

We shall prove that there exists a unique set $\{x_1, ..., x_{n+1}\}$ such that

$$\forall x \left(x \in \{x_1, ..., x_{n+1}\} \leftrightarrow \bigvee_{i=1}^{n+1} x \doteq x_i \right)$$

We define $\{x_1, ..., x_{n+1}\}$ to be $\bigcup P(\{x_1, ..., x_n\}, P(x_{n+1}, x_{n+1}))$. The uniqueness is trivial by the Axiom of

Extensionality.

$$\begin{array}{lll}
1: & (x \in \bigcup P(\{x_1,...,x_n\}, P(x_{n+1}, x_{n+1})) & \\
 & \leftrightarrow \exists y(x \in y \wedge y \in P(\{x_1,...,x_n\}, P(x_{n+1}, x_{n+1})))) & \text{[Unions]} \\
2: & (y \in P(\{x_1,...,x_n\}, P(x_{n+1}, x_{n+1})) \leftrightarrow (y \doteq \{x_1,...,x_n\} \vee y \doteq P(x_{n+1}, x_{n+1}))) & \text{[Pairs]} \\
3: & (x \in \bigcup P(\{x_1,...,x_n\}, P(x_{n+1}, x_{n+1})) & \\
 & \leftrightarrow \exists y(x \in y \wedge (y \doteq \{x_1,...,x_n\} \vee y \doteq P(x_{n+1}, x_{n+1})))) & \text{[Substitution]} \\
4: & (\exists y(x \in y \wedge (y \doteq \{x_1,...,x_n\} \vee y \doteq P(x_{n+1}, x_{n+1}))) & \\
 & \leftrightarrow (x \in \{x_1,...,x_n\} \vee x \in P(x_{n+1}, x_{n+1}))) & \text{[Theorem in } K\text{]} \\
5: & (x \in \bigcup P(\{x_1,...,x_n\}, P(x_{n+1}, x_{n+1})) \leftrightarrow (x \in \{x_1,...,x_n\} \vee x \in P(x_{n+1}, x_{n+1}))) & \text{[Substitution]} \\
6: & (x \in \{x_1,...,x_n\} \leftrightarrow \bigvee_{i=1}^{n} x \doteq x_i) & \text{[IH]} \\
7: & (x \in P(x_{n+1}, x_{n+1}) \leftrightarrow (x \doteq x_{n+1} \vee x \doteq x_{n+1})) & \text{[Pairs]} \\
8: & ((x \doteq x_{n+1} \vee x \doteq x_{n+1}) \leftrightarrow x \doteq x_{n+1}) & \text{[Tautology]} \\
9: & (x \in \bigcup P(\{x_1,...,x_n\}, P(x_{n+1}, x_{n+1})) \leftrightarrow (\bigvee_{i=1}^{n} x \doteq x_i \vee x = x_{n+1})) & \text{[Substitution]} \\
10: & \left((\bigvee_{i=1}^{n} x \doteq x_i \vee x = x_{n+1}) \leftrightarrow \bigvee_{i=1}^{n+1} x \doteq x_i\right) & \text{[Tautology]} \\
11: & \left(x \in \bigcup P(\{x_1,...,x_n\}, P(x_{n+1}, x_{n+1})) \leftrightarrow \bigvee_{i=1}^{n+1} x \doteq x_i\right) & \text{[Substitution]}
\end{array}$$

✓

(iii) Suppose that X is a finite set in the sense that there exists sets $x_1, ..., x_n$ such that $X = \{x_1, ..., x_n\}$. Let $\varphi_i \in \{x = x_i, \varnothing\}$. Then if $Y \subseteq X$, then we have $\forall x(x \in Y \leftrightarrow \bigvee_{i=1}^{n} \varphi_i)$. In particular, by the Axiom of Extensionality there are 2^n distinct subsets of X. The union of these subsets of X is the power set of X. ✓

Better if you give a way to explicitly list these sets (by induction, for example)

(iv) Suppose that X is a finite set in the sense that there exists sets $x_1, ..., x_n$ such that $X = \{x_1, ..., x_n\}$. Then X has $\binom{n}{2} = \frac{n(n-1)}{2}$ distinct doubleton subsets. The union of these sets is a set.

Again, better if you give a way to explicitly list these sets, or directly use (iii) and comprehension

(b) First we formalize the property that "a set has an even number of elements." Let

$$\chi_{i,j} := \begin{cases} \neg x_i \doteq x_j & i \neq j \\ x_i \doteq x_j & i = j \end{cases}$$

For a set a, a has $2n$ elements if and only if it satisfies

$$\varphi_n := \exists x_1 \cdots \exists x_{2n} \left(\bigwedge_{i=1}^{2n} \bigwedge_{j=1}^{2n} \chi_{i,j} \wedge \forall x \left(x \in a \to \bigvee_{i=1}^{2n} x \doteq x_i \right) \right)$$

a has an even number of elements if and only if it satisfies $\varphi := \bigvee_{n \in \mathbb{N}} \varphi_n$. However, $\varphi \notin \text{Form}(\mathcal{L})$ so the Axiom Scheme of Comprehension is not applicable. □

Given finite A, In (a)(iv) you already you already know that there exists a unique set of all doubletons in A (expressible in FOPC). Define this as P2(A). Then we know that A is even iff a subset of P2(A) (i.e. a set in P(P2(A))) partitions A. So we can formulate:
Y = {x ∈ X : ∃y (y ∈ P(P2(x)) ∧ ∪y = x ∧ ∀z∀z' (z ∈ y ∧ z' ∈ y ∧ ¬z = z' → ∩{z, z'} = ∅))}

Question 8 α

Prove that there exist infinitely many sets.

Sketch of Proof. Suppose for contradiction that there are only finitely many sets in the sense that we can list them as $x_1, ..., x_n$. Then by Question 7.(a).(ii), $\{x_1, ..., x_n\}$ is a set. In particular it is the set of all sets. This is impossible by Russell's Paradox. ✓ □