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Problem Sheet 3

B2.1: Introduction to Representation Theory

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Question 1

Let V be a finite dimensional $\mathbb{C}G$-module and let $g \in G$. Prove that

$$\chi_{S^2V}(g) = \frac{1}{2}\left(\chi_V(g)^2 + \chi_V\left(g^2\right)\right) \quad \text{and} \quad \chi_{\Lambda^2V}(g) = \frac{1}{2}\left(\chi_V(g)^2 - \chi_V\left(g^2\right)\right)$$

Proof. Since V is a $\mathbb{C}[G]$-module, for each $g \in G$, let $g_V \in \mathrm{GL}(V)$ given by the left multiplication by g. By Question 1 in Sheet 1, g_V is diagonalisable. V has a basis $\{v_1, .., v_n\}$ consisting of the eigenvectors of g_V with corresponding eigenvalues $\lambda_1, ..., \lambda_n$. That is,

$$g_V(v_i) = \lambda_i v_i, \qquad i \in \{1, ..., n\}$$

$\left\{\frac{1}{2}(v_i \otimes v_j + v_j \otimes v_i) : 1 \leqslant j \leqslant i \leqslant n\right\}$ is a basis of S^2V by Lemma 4.13. g induces $g_{\mathrm{S}^2V} \in \mathrm{GL}(\mathrm{S}^2V)$ such that

$$g_{\mathrm{S}^2V}\left(\frac{1}{2}(v_i \otimes v_j + v_j \otimes v_i)\right) = \frac{1}{2}\left(g_V(v_i) \otimes g_V(v_j) + g_V(v_j) \otimes g_V(v_i)\right) = \frac{1}{2}\lambda_i\lambda_j(v_i \otimes v_j + v_j \otimes v_i), \qquad 1 \leqslant j \leqslant i \leqslant n$$

Hence $\frac{1}{2}(v_i \otimes v_j + v_j \otimes v_i)$ is an eigenvector of g_{S^2V} with eigenvalue $\lambda_i\lambda_j$. Since the eigenvectors span a basis of S^2V, we have:

$$\chi_{\mathrm{S}^2V}(g) = \operatorname{tr} g_{\mathrm{S}^2V} = \sum_{i=1}^{n}\sum_{j=1}^{i}\lambda_i\lambda_j = \frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i\lambda_j + \sum_{i=1}^{n}\lambda_i^2\right) = \frac{1}{2}\left(\operatorname{tr} g_V\right)^2 + \operatorname{tr} g_V^2 = \frac{1}{2}\left(\chi_V(g)^2 + \chi_V\left(g^2\right)\right)$$

$\{v_i \wedge v_j : 1 \leqslant j < i \leqslant n\}$ is a basis of $\bigwedge^2 V$ by Lemma 4.13. g induces $g_{\bigwedge^2 V} \in \mathrm{GL}(\bigwedge^2 V)$ such that

$$g_{\bigwedge^2 V}(v_i \wedge v_j) = \frac{1}{2}\left(g_V(v_i) \otimes g_V(v_j) - g_V(v_j) \otimes g_V(v_i)\right) = \frac{1}{2}\lambda_i\lambda_j(v_i \otimes v_j - v_j \otimes v_i) = \lambda_i\lambda_j v_i \wedge v_j, \qquad 1 \leqslant j < i \leqslant n$$

Hence $v_i \wedge v_j$ is an eigenvector of $g_{\bigwedge^2 V}$ with eigenvalue $\lambda_i\lambda_j$. Since the eigenvectors span a basis of $\bigwedge^2 V$, we have:

$$\chi_{\bigwedge^2 V}(g) = \sum_{i=2}^{n}\sum_{j=1}^{i-1}\lambda_i\lambda_j = \frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i\lambda_j - \sum_{i=1}^{n}\lambda_i^2\right) = \frac{1}{2}\left(\chi_V(g)^2 - \chi_V\left(g^2\right)\right)$$

□

Question 2

Show that every group homomorphism from G to an abelian group A is trivial on the commutator subgroup G' and hence factors through G/G'. Show that if N is a normal subgroup of G with G/N abelian, then $G' \leqslant N$.

Proof. For $g, h \in N$, since G/N is Abelian, we have

$$(gN)(hN) = (hN)(gN) \implies h^{-1}g^{-1}hgN = N \implies [h, g] = h^{-1}g^{-1}hg \in N$$

Hence $G' \leqslant N$.

Suppose that $\varphi : G \to A$ is a group homomorphism. Then by First Isomorphism Theorem we have

$$G/\ker\varphi \cong \operatorname{im}\varphi \leqslant A$$

Then $G/\ker\varphi$ is Abelian. We have $G' \leqslant \ker\varphi$. That is, $\varphi|_{G'} = 0$. Hence $\varphi : G \to A$ induces $\widetilde{\varphi} : G/G' \to A$ via $\widetilde{\varphi}(gG') := \varphi(g)$ for $g \in G$. □

Question 3

Let k be an algebraicaly closed field.

(a) Suppose that G is abelian. Prove that every simple kG-module is one-dimensional.

(b) Prove that the converse holds provided that $|G| \neq 0$ in k.

(c) Deduce from (a) that G has precisely $\left|G : G'\right|$ complex linear characters.

Proof. (a) Suppose that V is a simple $k[G]$-module. For $g \in G$, g_V is a k-linear map. Since G is Abelian, for $h \in G$ and $v \in V$,

$$g_V(h \cdot v) = gh \cdot v = hg \cdot v = h \cdot g_V(v)$$

By extending the equation linearly, g_V is a $k[G]$-module endomorphism. By Schur's Lemma, there exists $\lambda \in k$ such that $g_V = \lambda 1_V$. In particular, $\langle v \rangle$ is a sub-$k[G]$-module of V for any $v \in V \setminus \{0\}$, If $\dim_k V > 1$ then this contradicts that V is simple. Hence $\dim_k V = 1$.

(b) For $\operatorname{char} k \nmid |G|$, by Corollary 3.20 of Artin-Wedderburn Theorem, we have the $k[G]$-module isomorphism:

$$k[G] \cong V_1^{\dim_k V_1} \oplus \cdots \oplus V_r^{\dim_k V_r}$$

where $V_1, \ldots, V_r$ is a complete list of simple $k[G]$-modules up to isomorphism. By assumption we have $\dim_k V_1 = \cdots = \dim_k V_r = 1$. Then $|G| = \dim_k k[G] = r$. By Corollary 3.16, r is exactly the number of conjugacy classes of G. Therefore every conjugacy class of G is a singleton. Hence G is Abelian.

(c) *$k = \mathbb{C}$ is assumed for this part.*

Note that G/G' is Abelian (by Question 2 or Part A Group Theory). By (a) every simple $\mathbb{C}[G/G']$-module is one-dimensional over $\mathbb{C}$. Let $V_1, \ldots, V_r$ be a list of such modules. By Artin-Wedderburn Theorem we have

$$\mathbb{C}[G/G'] \cong V_1 \oplus \cdots \oplus V_r$$

Each irreducible representation $\rho_i : G/G' \to \mathrm{GL}(V_i)$ lifts to a representation $\dot{\rho}_i : G \to \mathrm{GL}(V_i)$ via $\dot{\rho}_i(g) := \rho_i(gG')$. Since $\rho_1, \ldots, \rho_n$ are non-isomorphic, neither are $\dot{\rho}_1, \ldots, \dot{\rho}_n$. Then G has at least $\dim_k \mathbb{C}[G/G'] = [G : G']$ non-isomorphic one-dimensional irreducible representations.

On the other hand, suppose that $\varphi : G \to \mathrm{GL}(V)$ is a one-dimensional irreducible representation of G over $\mathbb{C}$. Note that $\dim_{\mathbb{C}} V = 1$ implies that $\mathrm{GL}(V) \cong \mathbb{C}^\times$, which is Abelian. Then by Question 2, φ induces a group homomorphism $\widetilde{\varphi} : G/G' \to \mathrm{GL}(V)$. Then $V \cong V_i$ as $\mathbb{C}[G/G']$-modules for some $i \in \{1, \ldots, r\}$.

We conclude that G has exactly $[G : G']$ non-isomorphic one-dimensional irreducible representations. By row orthogonality theorem, the characters and the isomorphism classes of finite-dimensional $\mathbb{C}[G]$-modules are in bijective correspondence. Hence G has $[G : G']$ complex linear characters. □

Question 4

Calculate the character of the representation $\rho : S_4 \to \mathrm{GL}_3(\mathbb{R})$ from Example 1.3(d). Let $V = \mathbb{C}^3$ be the $\mathbb{C}S_4$-module obtained by viewing ρ as a complex representation $S_4 \to \mathrm{GL}_3(\mathbb{C})$. Decompose $V \otimes V$ as a direct sum of irreducible representations.

Proof. Note that in a permutation group, two elements are conjugate if and only if they have the same cycle type. S_4 has 5 conjugacy classes with representatives:

$$e,\ (12),\ (123),\ (12)(34),\ (1234)$$

For $\rho(g) \in \mathrm{SO}_3(\mathbb{R}) \leqslant \mathrm{GL}_3(\mathbb{R})$, ρ_g is a planar rotation, so with a suitable choice of basis it has matrix

$$M_g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_g & -\sin\theta_g \\ 0 & \sin\theta_g & \cos\theta_g \end{pmatrix}$$

In particular, $\theta_g = \dfrac{2\pi}{n}$ if g has order n. Hence $\chi_\rho(g) = \operatorname{tr} M_g = 2\cos\theta_g = 1 + 2\cos\dfrac{2\pi}{\mathrm{o}(g)}$.

- $\chi_\rho(e) = \dim_{\mathbb{R}} \mathbb{R}^3 = 3$.
- Since (12) has order 2, $\chi_\rho(12) = 1 + \cos\dfrac{2\pi}{2} = -1$.
- Since $(12)(34)$ has order 2, $\chi_\rho((12)(34)) = 1 + \cos\dfrac{2\pi}{2} = -1$.
- Since (123) has order 3, $\chi_\rho(123) = 1 + \cos\dfrac{2\pi}{3} = 1 + 2 \cdot \left(-\dfrac{1}{2}\right) = 0$.

- Since (1234) has order 4, $\chi_\rho(1234) = 1 + \cos\dfrac{2\pi}{4} = 1$.

For the $\mathbb{C}[S_4]$-module $V \otimes V$, by Lemma 4.13 we have the initial decomposition $V \otimes V = \mathrm{S}^2 V \oplus \bigwedge^2 V$, where $\dim_{\mathbb{C}} \mathrm{S}^2 V = 6$ and $\dim_{\mathbb{C}} \bigwedge^2 V = 3$. For the decomposition of $\mathrm{S}^2 V$ and $\bigwedge^2 V$, we use the character table of S_4 from Example 5.24:

	e	(12)	(12)(34)	(123)	(1234)
$\lvert g^G \rvert$	1	3	8	6	6
$\lvert C_G(g) \rvert$	24	8	3	4	4
$\widetilde{\mathbf{1}}$	1	1	1	1	1
$\widetilde{\epsilon}$	1	-1	1	1	-1
$\widetilde{\chi_W}$	2	0	2	-1	0
χ_4	3	1	-1	0	-1
χ_V	3	-1	-1	0	1

By row orthogonality theorem we have

$$\chi_{\mathrm{S}^2 V} = \langle \chi_{\mathrm{S}^2 V}, \widetilde{\mathbf{1}} \rangle \widetilde{\mathbf{1}} + \langle \chi_{\mathrm{S}^2 V}, \widetilde{\epsilon} \rangle \widetilde{\epsilon} + \langle \chi_{\mathrm{S}^2 V}, \widetilde{\chi_W} \rangle \widetilde{\chi_W} + \langle \chi_{\mathrm{S}^2 V}, \chi_4 \rangle \chi_4 + \langle \chi_{\mathrm{S}^2 V}, \chi_V \rangle \chi_V$$
$$\chi_{\bigwedge^2 V} = \langle \chi_{\bigwedge^2 V}, \widetilde{\mathbf{1}} \rangle \widetilde{\mathbf{1}} + \langle \chi_{\bigwedge^2 V}, \widetilde{\epsilon} \rangle \widetilde{\epsilon} + \langle \chi_{\bigwedge^2 V}, \widetilde{\chi_W} \rangle \widetilde{\chi_W} + \langle \chi_{\bigwedge^2 V}, \chi_4 \rangle \chi_4 + \langle \chi_{\bigwedge^2 V}, \chi_V \rangle \chi_V$$

with the inner product defined in Definition 5.12. Using the formula proven in Question 1, we have

	e	(12)	(12)(34)	(123)	(1234)
$\chi_{\mathrm{S}^2 V}$	6	2	2	0	0
$\chi_{\bigwedge^2 V}$	3	-1	-1	0	1

By performing explicit calculations we obtain

$$\chi_{\mathrm{S}^2 V} = \widetilde{\mathbf{1}} + \widetilde{\chi_W} + \chi_4$$
$$\chi_{\bigwedge^2 V} = \chi_V$$

We still need to find the $\mathbb{C}[S_4]$-module corresponding to χ_4, which is a 3-dimensional subspace of $\mathbb{C}^4$ invariant under the action S_4 which is not isomorphic to $\mathbb{C}^3$ as $\mathbb{C}[S_4]$-modules. The only such subspace is

$$U := \left\{ (x^1, x^2, x^3, x^4) \in \mathbb{C}^4 : \sum_{i=1}^{4} x^i = 0 \right\}$$

The row orthogonality theorem implies that the character uniquely determines the representation. We have

$$\mathrm{S}^2 V \cong \mathbb{C} \oplus W \oplus U, \qquad \textstyle\bigwedge^2 V \cong V$$

Hence

$$V \otimes V \cong \mathbb{C} \oplus W \oplus U \oplus V$$

as $\mathbb{C}[S_4]$-modules. The corresponding decomposition of representation is

$$\rho_{V \otimes V} = \rho_{\mathbb{C}} \oplus \rho_W \oplus \rho_U \oplus \rho_V$$

□

Question 5

(a) Let χ be a character of G. Show that $\{g \in G : \chi(g) = \chi(1)\}$ is a normal subgroup of G.

(b) Prove that G is simple if and only if $\chi(g) \neq \chi(1)$ for every $g \neq 1$ and $\chi \neq \mathbb{1}$.

Proof. The notation is confusing. I shall use e to denote the identity in G. The underlying field $k = \mathbb{C}$.

(a) Let $H := \{g \in G : \chi(g) = \chi(e)\}$. Let $\rho : G \to \mathrm{GL}(V)$ be the representation associated with the character χ. We claim that $H = \ker \rho$. For $g \in \ker \rho$,

$$\chi(g) = \operatorname{tr} \rho(g) = \operatorname{tr} 1_V = \operatorname{tr} \rho(e) = \chi(e)$$

Hence $g \in H$ and $\ker \rho \subseteq H$.

On the other hand, suppose that $g \in H$. Suppose that $\rho(g)$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ (counting multiplicities). Then

$$\chi(g) = \sum_{i=1}^{n} \lambda_i = n = \chi(e)$$

Since $|G| < \infty$, by Lagrange's Theorem $\mathrm{o}(g) < \infty$. $\rho(g)^{\mathrm{o}(g)} = \rho(e) = 1_V$ has the unique eigenvalue 1. Then we have

$$\lambda_1^{\mathrm{o}(g)} = \cdots = \lambda_n^{\mathrm{o}(g)} = 1$$

So the eigenvalues of $\rho(g)$ are roots of unity. Moreover,

$$\left| \sum_{i=1}^{n} \lambda_i \right| \leqslant \sum_{i=1}^{n} |\lambda_i| = \sum_{i=1}^{n} 1 = n$$

with equality holds if and only if $\lambda_1 = \cdots = \lambda_n$. Hence

$$\lambda_1 = \cdots = \lambda_n \wedge \sum_{i=1}^{n} \lambda_i = n \implies \lambda_1 = \cdots = \lambda_n = 1 \implies \rho(g) = 1_V \implies g \in \ker \rho$$

Hence $\ker\rho \subseteq H$. We deduce that $H = \ker\rho \lhd G$.

(b) Suppose that G is simple. Then $H \lhd G$ is either $\{e\}$ or G. If $H = G$, then by (a) $\ker\rho = G$. The representation is trivial and $\chi = 1$. If $H = \{e\}$, then $\chi(g) \neq \chi(1)$ for $g \neq e$.

Conversely, suppose that G is not simple. G has a proper non-trivial normal subgroup N. Then G/N is non-trivial and has at least one irreducible representation $\varphi : G/N \to \mathrm{GL}(V)$. φ can be lifted to a non-trivial representation $\dot{\varphi} : G \to \mathrm{GL}(V)$ via $\dot{\varphi}(g) := \varphi(gN)$. Then the inflated character $\widetilde{\chi}(g) = \chi(gN)$. We have $\widetilde{\chi}(g) = \widetilde{\chi}(e)$ for $g \in N \setminus \{e\}$. □

Question 6

Let G act on a finite set X and consider the permutation module $V := \mathbb{C}X$.

(a) Let $g \in G$. Prove that $\chi_V(g) = \left|\mathrm{Fix}_X(g)\right|$ where $\mathrm{Fix}_X(g) := \{x \in X : g \cdot x = x\}$

(b) Prove that $\sum_{g \in G} \chi_V(g) = r|G|$, where r is the number of G-orbits on X.

(c) Suppose now that the action of G on X is 2-transitive, that is G has two orbits acting on $X \times X$ in the action defined by $g \cdot (x_1, x_2) = \left(g \cdot x_1, g \cdot x_2\right)$. Show that $\sum_{g \in G} \chi_V(g)^2 = 2|G|$ and deduce that $V = \mathbb{1} \oplus W$ for some simple submodule W of V.

Proof. *$\mathbb{C}^X$ or $\mathbb{C}^{\oplus X}$ are better notions for free $\mathbb{C}$-vector spaces on X.*

(a) Suppose that $X = \{x_1, \ldots, x_n\}$. Let G acts on X by $\rho : G \to \mathrm{Sym}(X)$, which extends linearly to $\rho : G \to \mathrm{GL}(V)$. With respect to the basis $\{x_1, \ldots, x_n\}$, the matrix M_g of $\rho(g)$ is given by

$$\left(M_g\right)_{ij} = \begin{cases} 1, & g \cdot x_i = x_j \\ 0, & \text{otherwise} \end{cases}$$

Hence

$$\chi_V(g) = \operatorname{tr} M_g = \sum_{i=1}^{n} \mathbf{1}_{\{g \cdot x_i = x_i\}} = |\mathrm{Fix}_X(g)|$$

(b) This is the orbit counting formula in Prelim Group Theory:

$$\sum_{g \in G} \chi_V(g) = \sum_{g \in G} |\mathrm{Fix}_X(g)| = r|G|$$

The proof is to count $|G \times X|$ in two ways and apply the orbit-stabliser theorem. I omit the details here.

(c) The group action $\widetilde{\rho} : G \to X \times X$ extends linearly to the representation $\widetilde{\rho} : G \to \mathrm{GL}(\mathbb{C}^{\oplus(X \times X)})$, where $\mathbb{C}^{\oplus(X \times X)} \cong V \otimes V$ canonically as $\mathbb{C}[G]$-modules. Then by Proposition 5.21.(c), we have

$$\chi_{V \otimes V}(g) = \chi_V(g)^2$$

for each $g \in G$. By (b) we have

$$\sum_{g\in G} \chi_V(g)^2 = \sum_{g\in G} \chi_{V\otimes V}(g) = 2|G|$$

The action $\widetilde{\rho}: G \to \mathrm{Sym}(X\times X)$ is 2-transitive. Suppose that $\rho: G \to \mathrm{Sym}(X)$ is not transitive. Then for any two distinct orbits A and B of X, $A\times A$, $A\times B$, $B\times A$ and $B\times B$ are distinct orbits of $X\times X$, which is a contradiction. Hence ρ is transitive. We have $\sum_{g\in G} \chi_V(g) = |G|$.

Consider the subspace

$$U := \left\langle \sum_{i=1}^{n} x_i \right\rangle \leqslant V$$

The subspace is G-stable and hence is a sub-$\mathbb{C}[G]$-module of V. By Maschke's Theorem, there exists another sub-$\mathbb{C}[G]$-module W of V such that $V = U \oplus W$. Passing to characters we have $\chi_V = \chi_U + \chi_W$.

Note that for $g \in G$,

$$g \cdot \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} g\cdot x_i = \sum_{i=1}^{n} x_i$$

Hence the sub-representation of ρ on U is trivial. $\chi_U = 1$.

Now we consider the decomposition of W into irreducible simple $\mathbb{C}[G]$-modules:

$$W \cong \bigoplus_{i=1}^{r} V_i^{\alpha_i}, \qquad \chi_W = \sum_{i=1}^{r} \alpha_i \chi_{V_i}$$

Then by row orthogonality theorem we have $\langle \chi_W, \chi_W\rangle = \sum_{i=1}^{r} \alpha_i^2$, where

$$\langle \chi_W, \chi_W\rangle = \langle \chi_V, \chi_V\rangle - 2\langle \chi_V, \chi_U\rangle + \langle \chi_U, \chi_U\rangle = \frac{1}{|G|}\left(\sum_{g\in G} \chi_V(g)^2 - 2\sum_{g\in G}\chi_V(g) + |G|\right) = \frac{1}{|G|}(2|G| - 2|G| + |G|) = 1$$

Since $\alpha_1, \ldots, \alpha_r$ are non-negative integers, we must have $\alpha_i = \delta_{ij}$ for some $j \in \{1, \ldots, r\}$. Hence $W \cong V_j$ is a simple $\mathbb{C}[G]$-module. We have decomposed the representation

$$\rho_V = \mathbf{1} \oplus \rho_W$$

into two irreducible sub-representations. □

Question 7

Find the character tables of the quaternion group Q_8 and the dihedral group D_8 of order 8. Does the character table determine the group up to isomorphism?

Proof. The quaternion group $Q_8 = \{\pm 1, \pm \mathrm{i}, \pm \mathrm{j}, \pm \mathrm{k}\}$ has 5 conjugacy classes:

$$\{1\},\ \{-1\},\ \{\mathrm{i}, -\mathrm{i}\},\ \{\mathrm{j}, -\mathrm{j}\},\ \{\mathrm{k}, -\mathrm{k}\}$$

Note that $\{1,-1\} \lhd Q_8$ and $Q_8/\{1,-1\} \cong V_4$. We first find the character table of V_4.

Note that V_4 has a trivial centre so $V_4' = \{e\}$. By Question 3.(c), all 4 characters of V_4 are linear. $\chi(g)$ is a fourth root of unity for each character χ and $g \in V_4$. With row and column orthogonality theorem, we can easily write down the whole table:

V_4	1	i	j	k
$\mathbf{1}$	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

The characters of V_4 inflate to characters of Q_8. We have the table

Q_8	$\{1\}$	$\{-1\}$	$\{\mathrm{i},-\mathrm{i}\}$	$\{\mathrm{j},-\mathrm{j}\}$	$\{\mathrm{k},-\mathrm{k}\}$
$\widetilde{\mathbf{1}}$	1	1	1	1	1
$\widetilde{\chi}_2$	1	1	1	-1	-1
$\widetilde{\chi}_3$	1	1	-1	1	-1
$\widetilde{\chi}_4$	1	1	-1	-1	1
χ_5	a	b	c	d	e

where $a \in \mathbb{N}$ and $b, c, d \in \mathbb{C}$ are to be determined. By column orthogonality theorem we have

$$\begin{aligned}
1^2+1^2+1^2+1^2+|a|^2 &= |C_G(1)| = 8\\
1^2+1^2+1^2+1^2+|b|^2 &= |C_G(-1)| = 8\\
1^2+1^2+(-1)^2+(-1)^2+|c|^2 &= |C_G(\mathrm{i})| = 4\\
1^2+(-1)^2+1^2+(-1)^2+|d|^2 &= |C_G(\mathrm{j})| = 4\\
1^2+(-1)^2+(-1)^2+1^2+|e|^2 &= |C_G(\mathrm{k})| = 4
\end{aligned}$$

Hence $a = 2$, $b = \pm 2$, $c = d = e = 0$. By row orthogonality theorem,

$$8\langle \widetilde{\mathbf{1}}, \chi_5\rangle = a+b+c+d+e = 0 \implies b = -2$$

Hence the character table of Q_8 is given by

Q_8	$\{1\}$	$\{-1\}$	$\{\mathrm{i},-\mathrm{i}\}$	$\{\mathrm{j},-\mathrm{j}\}$	$\{\mathrm{k},-\mathrm{k}\}$
$\widetilde{\mathbf{1}}$	1	1	1	1	1
$\widetilde{\chi}_2$	1	1	1	-1	-1
$\widetilde{\chi}_3$	1	1	-1	1	-1
$\widetilde{\chi}_4$	1	1	-1	-1	1
χ_5	2	-2	0	0	0

Next, we consider the dihedral group $D_8 = \langle r, s \mid r^2, s^4, rsrs\rangle$. It has 5 conjugacy classes:

$$\{e\},\ \{s^2\},\ \{s, s^3\},\ \{r, rs^2\},\ \{rs, rs^3\}$$

D_8 and Q_8 have the same number of conjugacy classes and the same sizes for each class. Note that $\{e, s^2\} \lhd D_8$ and $D_8/\{e, s^2\} \cong V_4$. Then $D_8 \cong C_2 \rtimes V_4$.

Note that when we solve the character table of Q_8, we use no information of Q_8 more than the size of conjugacy classes and that $Q_8 \cong C_2 \rtimes V_4$. This forces D_8 to have the same character table as Q_8. But $Q_8 \not\cong D_8$. Hence the character table cannot determine the group up to isomorphism. □

Question 8

Let H be another finite group whose character table is equal to the character table of G. Prove that $|G'| = |H'|$ and that $|Z(G)| = |Z(H)|$.

Proof. Let $\chi_1, \ldots, \chi_r$ be a complete list of characters of G and $\chi'_1, \ldots, \chi'_s$ be a complete list of characters of H. Let $C_1, \ldots, C_r$ be the conjugacy classes of G and $C'_1, \ldots, C'_s$ be the conjugacy classes of H. G and H has the same character table implies that $r = s$, and $\chi_i(g) = \chi'_i(h)$ for $g \in C_j$, $h \in C'_j$, $i, j \in \{1, \ldots, r\}$.

Take $g \in C_j$ and $h \in C'_j$. By column orthogonality theorem, we have

$$\frac{|G|}{|C_j|} = |C_G(g)| = \sum_{i=1}^{r} |\chi_i(g)|^2 = \sum_{i=1}^{r} |\chi'_i(h)|^2 = |C_H(h)| = \frac{|H|}{|C'_j|}$$

Without loss of generality we assume that $|G| \geqslant |H|$. Take $C_j = \{e_G\}$. Then we have $|G| = |H|/|C'_j| \leqslant |H|$. Hence we must have $|C'_j| = 1$ and $|G| = |H|$. Therefore $|C_j| = |C'_j|$ for all conjugacy classes.

Note that $g \in Z(G)$ if and only if the conjugacy class of g is the singleton $\{g\}$. Therefore $Z(G) = Z(H)$ are the number of singleton conjugacy classes in G (and in H). □

Question 9

(a) Let χ be a character of G. Show that $\chi\left(g^{-1}\right) = \overline{\chi(g)}$ for all $g \in G$.

(b) Show that $g \in G$ is conjugate to g^{-1} if and only if $\chi(g) \in \mathbb{R}$ for every character χ of G.

Proof. The underlying field $k = \mathbb{C}$.

(a) Let $\rho : G \to \mathrm{GL}(V)$ be the representation associated with χ. Suppose that $\rho(g)$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ (counting multiplicities). Then $\rho(g^{-1}) = \rho(g)^{-1}$ has eigenvalues $\lambda_1^{-1}, \ldots, \lambda_n^{-1}$. In Question 6.(a) we have proven that the eigenvalues are roots of unity. Then

$$\lambda_i \lambda_i^{-1} = 1 = |\lambda_i|^2 = \lambda_i \overline{\lambda_i} \implies \overline{\lambda_i} = \lambda_i^{-1}$$

for $i \in \{1, \ldots, n\}$. Hence

$$\chi(g^{-1}) = \operatorname{tr} \rho(g^{-1}) = \sum_{i=1}^{n} \lambda_i^{-1} = \sum_{i=1}^{n} \overline{\lambda_i} = \overline{\sum_{i=1}^{n} \lambda_i} = \overline{\chi(g)}$$

(b) Suppose that g is conjugate to g^{-1} in G. Since χ is a class function, we have

$$\chi(g) = \chi(g^{-1}) = \overline{\chi(g)} \implies \chi(g) \in \mathbb{R}$$

Conversely, suppose that $\chi(g) \in \mathbb{R}$ for every character χ. Suppose that g is not conjugate to g^{-1}. Let $\chi_1, \ldots, \chi_r$ be a complete list of irreducible characters of G. By column orthogonality theorem, we have

$$0 = \sum_{i=1}^{r} \overline{\chi_i(g^{-1})} \chi_i(g) = \sum_{i=1}^{r} \chi_i(g)^2$$

Then $\chi_1(g) = \cdots = \chi_r(g) = 0$. But for the trivial representation $\mathbf{1}$, we have $\chi_{\mathbf{1}}(g) = 1$, which is a contradiction. Hence g is conjugate to g^{-1}. □