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Problem Sheet 5

Working with the FRW metric

B5: General Relativity

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RECALL:

Dynamical evolution equation:

$$\dot{R}^2 - \frac{8\pi G\rho R^2}{3} = 2E \text{ (Energy Form)} = -\frac{c^2}{a^2} \text{ (Curvature Form)} = -kc^2 \text{ (FRW Form)}$$

FRW metric, R dimensions of length, $k = 0, \pm 1$:

$$-c^2 d\tau^2 = -c^2 dt^2 + \frac{R^2 dr^2}{1-kr^2} + R^2 r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Curvature form, with $R_0 = 1$ and R dimensionless, a^2 positive or negative:

$$-c^2 d\tau^2 = -c^2 dt^2 + \frac{R^2 dr^2}{1-r^2/a^2} + R^2 r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Remark. I will try to use the Penrose abstract index notation throughout this problem sheet. In this convention, the latin letters $a, b, c, d, e...$ are abstract indices; the latin letters $i, j, k, \ell, m...$ are specific indices which range from 1 to 3; the greek letters $\mu, \nu, \rho, \sigma, \tau...$ are specific indices which range from 0 to 3.

Question 1. A big bang, but in empty space you say. Really?

a.) Show that the dynamical field equation for the scale factor $R(t)$ for an empty space $\rho = 0$ leads to an FRW metric of the form

$$-d\tau^2 = -dt^2 + \frac{t^2 dr^2}{1+r^2} + r^2 t^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Use $c = 1$ for this problem!

b.) Wait...Surely empty space must be Minkowski spacetime. Though this metric does not look static, there *must* be a coordinate transformation that turns this metric into a static Minkowski form. In other words, we ought to be able to find two functions, s and T,

$$s = s(r,t), \quad T = T(r,t) \quad \text{or equivalently} \quad r = r(s,T), \quad t = t(s,T)$$

that transform the metric of part (1a) into an old friend:

$$-d\tau^2 = -dT^2 + ds^2 + s^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

By inspection, we must have

$$s(r,t) = rt$$

Why "by inspection?" Explain convincingly why it is as simple as this, in just one to two sentences.

c.) Using $s = rt$, and by then demanding that the coefficient of dT^2 be -1 after the coordinate change, show that $T = \sqrt{s^2+t^2}$ (up to an additive function of s which you may safely discard), and thereby derive the second coordinate transformation:

$$T = t\sqrt{1+r^2}$$

Give the explicit functional forms for $r(s,T)$ and $t(s,T)$.

d.) Complete the full coordinate transformation for $d\tau^2$ and verify in detail that the Minkowski metric emerges. You may find it to your advantage to express $\partial t/\partial s$ and $\partial r/\partial s$ in terms of r and t, and $\partial r/\partial T$ in terms of $\partial t/\partial T$, before you begin. This is a valuable lesson: it is easy to be fooled by coordinates.

Proof. a.) The Friedmann equation with $\rho = 0$ in the form of FRW metric is given by

$$\dot{R}^2 = -k$$

where $R(t)$ is the scale factor. It integrates to

$$R(t) = \sqrt{-k}t + \text{const}$$

Since R is real-valued, we must have $k = 0$ or -1.

For $k = 0$, we can set $R(0) = 1$ so that the FRW metric is simply the Minkowski metric:

$$g_{ab} = -\mathrm{d}t_{ab}^2 + R(t)^2\left(\frac{\mathrm{d}r_{ab}^2}{1-kr^2} + r^2\mathrm{d}\theta_{ab}^2 + r^2\sin^2\theta\mathrm{d}\varphi_{ab}^2\right) = -\mathrm{d}t_{ab}^2 + \mathrm{d}r_{ab}^2 + r^2\mathrm{d}\theta_{ab}^2 + r^2\sin^2\theta\mathrm{d}\varphi_{ab}^2$$

For $k = -1$, we can set $R(0) = 0$, so $R(t) = t$. Substituting into the FRW metric:

$$g_{ab} = -\mathrm{d}t_{ab}^2 + R^2(t)\left(\frac{\mathrm{d}r_{ab}^2}{1-kr^2} + r^2\mathrm{d}\theta_{ab}^2 + r^2\sin^2\theta\mathrm{d}\varphi_{ab}^2\right) = -\mathrm{d}t_{ab}^2 + \frac{t^2\mathrm{d}r_{ab}^2}{1+r^2} + r^2t^2\mathrm{d}\theta_{ab}^2 + r^2t^2\sin^2\theta\mathrm{d}\varphi_{ab}^2$$

b.) The metric tensor in two coordinate systems:

$$g_{ab} = -\mathrm{d}t_{ab}^2 + \frac{t^2\mathrm{d}r_{ab}^2}{1+r^2} + r^2t^2(\mathrm{d}\theta_{ab}^2 + \sin^2\theta\mathrm{d}\varphi_{ab}^2) = -\mathrm{d}T_{ab}^2 + \mathrm{d}s_{ab}^2 + s^2\left(\mathrm{d}\theta_{ab}^2 + \sin^2\theta\mathrm{d}\phi_{ab}^2\right)$$

We can equate the coefficient of $(\mathrm{d}\theta_{ab}^2 + \sin^2\theta\mathrm{d}\varphi_{ab}^2)$ and obtain $s^2 = r^2t^2$. We can set $s = rt$.

c.) The transition matrix from $\{t, r, \theta, \varphi\}$ to $\{T, s, \theta, \varphi\}$ is given by

$$J = \begin{pmatrix} \frac{\partial T}{\partial t} & \frac{\partial T}{\partial r} & 0 & 0 \\ \frac{\partial s}{\partial t} & \frac{\partial s}{\partial r} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial t} & \frac{\partial T}{\partial r} & 0 & 0 \\ r & t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

With respect to the bases $\{t, r, \theta, \varphi\}$ and $\{T, s, \theta, \varphi\}$, the metric tensor in matrix form is given by

$$\begin{pmatrix} -1 & & & \\ & \frac{t^2}{1+r^2} & & \\ & & r^2t^2 & \\ & & & r^2t^2\sin^2\theta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & s^2 & \\ & & & s^2\sin^2\theta \end{pmatrix}$$

Therefore

$$\begin{pmatrix} -1 & & & \\ & \frac{t^2}{1+r^2} & & \\ & & r^2t^2 & \\ & & & r^2t^2\sin^2\theta \end{pmatrix} = \begin{pmatrix} \frac{\partial T}{\partial t} & \frac{\partial T}{\partial r} & 0 & 0 \\ r & t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & s^2 & \\ & & & s^2\sin^2\theta \end{pmatrix} \begin{pmatrix} \frac{\partial T}{\partial t} & \frac{\partial T}{\partial r} & 0 & 0 \\ r & t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Computing the (1,1) entry on the RHS:

$$-1 = -\left(\frac{\partial T}{\partial t}\right)^2 + r^2 \implies \frac{\partial T}{\partial t} = \sqrt{1+r^2} \implies T = t\sqrt{1+r^2} + f(t)$$

where f is some arbitrary function of t. We have the freedom to set $f = 0$. We obtain that $T = t\sqrt{1+r^2}$.

Inverting the functions we have

$$t(s, T) = \sqrt{T^2 - s^2}, \qquad r(s, T) = \frac{s}{\sqrt{T^2 - s^2}}$$

d.) We don't need to invert the transition matrix J. From (c) we know that

$$J = \begin{pmatrix} \sqrt{1+r^2} & \frac{tr}{\sqrt{1+r^2}} & 0 & 0 \\ r & t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

And then we complete the matrix multiplication in (c). Since all matrices involved are block-diagnoal, it suffices to consider the 2×2 block on the upperleft.

$$\begin{pmatrix} \sqrt{1+r^2} & \dfrac{tr}{\sqrt{1+r^2}} \\ r & t \end{pmatrix}^\intercal \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{1+r^2} & \dfrac{tr}{\sqrt{1+r^2}} \\ r & t \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \dfrac{t^2}{1+r^2} \end{pmatrix}$$

This verifies that our choice of coordinate transformation is valid. The metric is indeed flat Minkowski. □

Question 2. A radiation/matter universe.

Repent now, or face a calculation for an Eternal, Infinite Universe of Fire and Brimstone! Well...radiation and matter, actually. Much the same. Anyway, it's too late to repent, the calculation begins. Solve the dynamical cosmological equation (Energy form) for $R(t)$ for the case of an arbitrary mixture of radiation and nonrelativistic matter in a spatially flat universe ($E = 0$). Assume a current energy density of $\rho_{\gamma_0}c^2$, and a matter density ρ_{m0}. In terms of the "inferno ratio" $I = \rho_{\gamma_0}/\rho_{m0}$, you should find

$$(R+I)^{3/2} - 3I(R+I)^{1/2} + 2I^{3/2} = \frac{3\Omega_{m0}^{1/2}H_0 t}{2}$$

(Note: This cubic equation is simple enough that the analytic solution is useful. Here it is [no need to prove]:

$$R = 4I\cos^2\left[\frac{1}{3}\cos^{-1}Q\right] - I, \quad Q = \frac{3H_0 t\Omega_{m0}^{1/2}}{4I^{3/2}} - 1$$

This holds as long as $-1 \leqslant Q < 1$. When $Q \geqslant 1$, replace cos and $\cos^{-1}$ with cosh and $\cosh^{-1}$.)

Proof. The Friedmann equation in flat space:

$$H^2 = \frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}(\rho_\gamma + \rho_m)$$

We recall from the notes the definition of the Ω_0 parameter:

$$\Omega_{m_0} := \frac{8\pi G}{3H_0^2}\rho_{m_0}$$

We know that $\rho_m \propto R^{-3}$ and $\rho_\gamma \propto R^{-4}$. For $R_0 = 1$, we have $\rho_m = \rho_{m_0}R^{-3}$ and $\rho_\gamma = \rho_{\gamma_0}R^{-4}$. Substituting into the Friedmann equation:

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}\left(\rho_{m_0}R^{-3} + \rho_{\gamma_0}R^{-4}\right) = \frac{1}{R^4}\frac{8\pi G}{3}\rho_{m_0}(R+I) = \frac{R+I}{R^4}H_0^2\Omega_{m_0}$$

Rearranging and integrating the expression:

$$\frac{R}{\sqrt{R+I}}\dot{R} = H_0\Omega_{m_0}^{1/2} \implies \int_1^R \frac{R}{\sqrt{R+I}}\,\mathrm{d}R = \int_0^t H_0\Omega_{m_0}^{1/2}\,\mathrm{d}t \implies \frac{2}{3}(R+I)^{3/2} - 2I(R+I)^{1/2} + \frac{4}{3}I^{3/2} = H_0\Omega_{m_0}^{1/2}t$$

We deduce that

$$(R+I)^{3/2} - 3I(R+I)^{1/2} + 2I^{3/2} = \frac{3}{2}H_0\Omega_{m_0}^{1/2}t$$

In addition, it is worth noting that, the Friedmann equation at current time is given by

$$H_0^2 = \frac{8\pi G}{3}(\rho_{m_0} + \rho_{\gamma_0}) = H_0^2\Omega_{m_0}(I+1)$$

which implies that $\Omega_{m_0} = 1/(I+1)$. □

Question 3. A bullet in an E-dS universe.

Shoot a bullet into an Einstein-de Sitter universe at start of time. Nothing is actually pushing or pulling the bullet, but each comoving observer will see the bullet fly by at a different velocity as it passes. The question is, how far does the bullet get? More precisely, what is the largest comoving coordinate distance r the bullet attains if it starts at $r = 0, R = 0$? The metric is standard E-dS:

$$-c^2 d\tau^2 = -c^2 dt^2 + R^2 dr^2 + R^2 r^2 d\Omega^2$$

$R(t)$ is the usual scale factor. We will use $d\varpi = R dr$ for the proper physical distance. Other standard notation and results for reference: t_0 is the current age of the universe, $R = (t/t_0)^{2/3}$ for E-dS, $H_0 \equiv \dot{R}_0$.

a.) The quantity $d\varpi/dt$ measures the bullet's velocity relative to expanding, comoving observers who are all moving away. Show that if the bullet has a measured velocity V_1 at some instant when it passes one such observer, then when the bullet overtakes another observer, a tiny distance d ϖ farther away, the velocity V_2 this observer measures is

$$V_2 = V_1 - \frac{\dot{R} d\varpi}{R}\left(1 - \frac{V_1^2}{c^2}\right)$$

to first order in $d\varpi$. (You will need the special relativity velocity addition formula and Hubble's law. Full special relativity works locally because $d\varpi$ is tiny, and in this tiny comoving frame special relativity holds. The relativity only matters when V_1 is comparable to c.) From this equation, show that the rate at which the measured $V = d\varpi/dt$ is changing with cosmic time is given by the differential equation

$$\frac{\dot{V}}{V\left(1 - V^2/c^2\right)} = -\frac{\dot{R}}{R}$$

where $\dot{V} = (V_2 - V_1)/dt = dV/dt$. Solve this equation and show that with $V = V_0$ at $t = t_0$, the solution is

$$\frac{V}{\sqrt{1 - V^2/c^2}} = \frac{U_0}{R}$$

where U_0 is the spatial component of the bullet 4 -velocity corresponding to V_0 at time t_0. (N.B.: In this problem, subscript 0 will always denote "current time," not the 4-vector time-like component.)

b.) The result of (3a.) shows that the product $\mathscr{P}R$ is constant, where $\mathscr{P}$ is the spatial component of the bullet 4-momentum. Show that, in this form, this is equivalent to an adiabatic expansion, either of photons (extreme relativistic particles), or classical particles (classical nonrelativistic gas). [Cosmic adiabatic expansion for photons correponds to the temperature T obeying $TR \sim$ constant, while for a classical gas, adiabatic behaviour is $T\rho^{-2/3} \sim$ constant, where ρ is the mass (or in this case number) density. In other words, a gas of bullets would "cool" like an ordinary gas!

c.) Solve the equation $d\varpi/dt = V(R)$ for the comoving coordinate r in an E-dS universe to obtain for our problem:

$$r(R) = \frac{c}{H_0}\int_0^R \frac{dx}{\left[x + c^2 x^3/U_0^2\right]^{1/2}}$$

and show therefore that as $R \to \infty$, the comoving coordinate $r \to r_{\max}$, where

$$r_{\max} = \frac{3.708\sqrt{U_0 c}}{H_0}$$

The numerical factor is

$$3.708 = \int_0^\infty \frac{dy}{\left(y + y^3\right)^{1/2}}$$

Even after an infinite amount of time, and even though this universe is decelerating, a fired bullet only reaches a finite value of comoving coordinate r for any finite U_0. But the bullet can reach arbitrarily large r, if V_0 approaches the speed of light.

Proof. a.)

b.)

□

Question 4. Schwarzschild and FRW geometries.

How long does it take a classical matter dominated closed universe to collapse, starting at its maximum extent? Express your answer two ways: in terms of the current value of the density ρ_0 and Ω_{m0}, and then in terms of the density at maximum extent ρ_m. Now, suppose we take all the mass in a small sphere of radius r_0 with density ρ_m (the sphere is small so that we don't have to worry about non-Euclidian curvature: the mass is just $4\pi r_0^3\rho_m/3$), and turn the matter into a Schwarzschild black hole. Calculate the proper time for a test particle to fall into the hole from radial coordinate r_0 in a Schwarzschild geometry. You should find exactly the same answer for the universe as a whole. (Sections 6.5 and 10.5 in the notes will be useful.) Can you account for this amazing agreement in a simple way?

Proof. 1. Recall from §10.5 in the notes that

$$R(\eta) = \frac{1-\sin\eta}{2(1-\Omega_{M0}^{-1})}, \qquad H_0 t(\eta) = \frac{\eta-\cos\eta}{2\Omega_{M0}^{1/2}(1-\Omega_{M0}^{-1})^{3/2}}$$

where $t=0$ represents the beginning of the universe (i.e. the *big bang*). The cosmological time in terms of Ω_{M0} and ρ_0 is given by

$$t = \frac{\eta-\sin\eta}{2\Omega_{M0}^{1/2}(1-\Omega_{M0}^{-1})^{3/2}H_0} = \frac{\eta-\sin\eta}{2(1-\Omega_{M0}^{-1})^{3/2}}\sqrt{\frac{3}{8\pi G\rho_0}}$$

Since the scale factor R measures the extent of the universe, we note that the universe is at the largest when $\eta=\pi$, and collapses into a singular point when $\eta=2\pi$. Therefore the time to collapse is given by

$$t_c = t(2\pi)-t(\pi) = \frac{1}{(1-\Omega_{M0}^{-1})^{3/2}}\sqrt{\frac{3\pi}{32G\rho_0}}$$

When $\rho=\rho_m$, $\dot{R}=0$ and hence the Hubble constant $H_m=0$. We find that

$$\Omega_{Mm}^{-1} = \frac{3H_m^2}{8\pi G\rho_m} = 0$$

Therefore

$$t_c = \sqrt{\frac{3\pi}{32G\rho_m}}$$

2. Recall from Question 5 in Sheet 2 that the radial orbit equation in the Schwarzschild metric is given by

$$\frac{1}{2}\dot{r}^2 + \frac{J^2}{2r^2} - \left(\frac{GM}{r}+\frac{GMJ^2}{c^2r^3}\right) = \frac{E^2-c^4}{2c^2}$$

As the test particle starts at rest at $r=r_0$, the angular momentum $J=0$. So

$$\frac{1}{2}\dot{r}^2 - \frac{GM}{r} = -\frac{GM}{r_0}$$

Substituting $M=\frac{4}{3}\pi r^3\rho_m$ into the equation:

$$\dot{r}^2 = \frac{8\pi G\rho_m r_0^3}{3r} - \frac{8\pi G\rho_m r_0^2}{3}$$

Integration:

$$\int_0^{r_0}\left(\frac{8\pi G\rho_m r_0^3}{3}\left(\frac{1}{r}-\frac{1}{r_0}\right)\right)^{-1/2}\mathrm{d}r = t_c$$

The integral is exactly the same as in §10.5. We shall use the formula

$$\int\frac{\mathrm{d}x}{\sqrt{B/x-A}} = \frac{B}{2A^{3/2}}(2\theta-\sin 2\theta)+\text{const} \qquad x=\frac{B}{A}\sin^2\theta$$

We have

$$t_c = \sqrt{\frac{3}{32\pi G\rho_m}}(2\theta_0 - \sin 2\theta_0), \quad r_0 = r_0 \sin^2\theta_0$$

Taking $\theta_0 = \pi/2$ we obtain the same result as in (1):

$$t_c = \sqrt{\frac{3\pi}{32G\rho_m}}$$

The universe collapses due to the gravity of ordinary matter. A particle feels the same gravity when the matter is uniformly distributed in a maximally symmetric 3-space and when the matter is concentrated in a singular point. □

Question 5. There and back again: a photon's tale.

a.) For a closed, matter-dominated universe with current mass density ρ_0, show that

$$H_0^2(\Omega_{M0} - 1) = c^2/a^2$$

where

$$\Omega_{M0} = \frac{8\pi G\rho_0}{3H_0^2}$$

b.) Consider the path of a photon (null geodesic) through this universe. With η defined in §10.5 in the notes:

$$R = \frac{1-\cos\eta}{2\left(1-\Omega_{M0}^{-1}\right)}$$

show that

$$\eta = \sin^{-1}(r/a)$$

where r follows the proper coordinate of the photon. In other words, r goes from zero to a and back again to zero (and R goes from zero to a maximum), as η advances by π. How many times could a photon travel around such a universe?

Proof. a.) We start from the Friedmann equation

$$H^2 = \frac{\dot{R}^2}{R^2} = \frac{8\pi G\rho_M}{3} - \frac{c^2}{a^2}$$

Then

$$H^2(1-\Omega_M) = H^2\left(1 - \frac{8\pi G\rho_M}{3H^2}\right) = -\frac{c^2}{a^2}$$

Evaluating at current time $t = t_0$ we have

$$H_0^2(\Omega_{M0} - 1) = \frac{c^2}{a^2}$$

b.) The FRW metric:

$$g_{ab} = -\mathrm{d}t_{ab}^2 + R(t)^2\left(\frac{\mathrm{d}r_{ab}^2}{1-r^2/a^2} + r^2\mathrm{d}\theta_{ab}^2 + r^2\sin^2\theta\mathrm{d}\varphi_{ab}^2\right)$$

Following a null geodesic parametrised by η in the FRW spacetime, we have

$$g_{ab}\left(\frac{\mathrm{d}x}{\mathrm{d}\eta}\right)^a\left(\frac{\mathrm{d}x}{\mathrm{d}\eta}\right)^b = 0$$

Assuming that the photon goes along the radial direction, we have

$$-c^2\left(\frac{\mathrm{d}t}{\mathrm{d}\eta}\right)^2 + \frac{R^2}{1-r^2/a^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\eta}\right)^2 = 0 \qquad (*)$$

First we shall derive that $\mathrm{d}t/\mathrm{d}\eta = |a|R/c$.

Recall from §10.5 in the notes that

$$R(\eta) = \frac{1-\sin\eta}{2(1-\Omega_{M0}^{-1})}, \qquad H_0\,t(\eta) = \frac{\eta-\cos\eta}{2\Omega_{M0}^{1/2}(1-\Omega_{M0}^{-1})^{3/2}}$$

Differentiate the second equation by η:

$$H_0\frac{\mathrm{d}t}{\mathrm{d}\eta} = \frac{1-\sin\eta}{2\Omega_{M0}^{1/2}(1-\Omega_{M0}^{-1})^{3/2}} = \frac{R}{2\Omega_{M0}^{1/2}(1-\Omega_{M0}^{-1})^{1/2}}$$

Using $1-\Omega_{M0}^{-1} = \dfrac{c^2}{a^2H_0^2\Omega_{M0}}$, which is from part (a), we have

$$H_0\frac{\mathrm{d}t}{\mathrm{d}\eta} = \frac{H_0R|a|}{c}$$

Hence $\mathrm{d}t/\mathrm{d}\eta = |a|R/c$. Substituting into the equation $(*)$ we obtain

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\eta}\right)^2 = a^2 - r^2$$

By inverse function theorem and integration we obtain

$$\eta(r) = \arcsin(r/a) + \text{const}$$

□