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Problem Sheet 3

Quantum Field Theory

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Question 1. Noether's Theorem

The Lagrangian density for a complex scalar field is

$$\mathcal{L} = \partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi + \mathcal{L}_{\text{Int}}\left(\phi^\dagger, \phi\right)$$

(a) Show that

$$\mathcal{L}_{\text{Int}} = -\frac{\lambda}{4}\left(\phi^\dagger \phi\right)^2$$

does not change the conserved current arising from the global phase symmetry in the free field case.

(b) Find the conserved current when

$$\mathcal{L}_{\text{Int}} = \lambda\left(\phi^\dagger \partial^\mu \phi\right)\left(\phi^\dagger \partial_\mu \phi\right)^\dagger$$

(c) What is the symmetry of the Lagrangian if

$$\mathcal{L}_{\text{Int}} = \lambda\left(\phi^2 + \phi^{\dagger 2}\right)^2$$

Is there a conserved current?

(d) Use the phase symmetry to find the conserved current for the Dirac Lagrangian

$$\mathcal{L} = \overline{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi$$

Proof. Consider the U(1) symmetry:

$$\phi(x,t) \mapsto \mathrm{e}^{\mathrm{i}\alpha(x)}\,\phi(x,t), \qquad \phi(x,t)^\dagger \mapsto \mathrm{e}^{-\mathrm{i}\alpha(x)}\,\phi(x,t)^\dagger$$

where $\alpha(x)$ is a real-valued function.

(a) The variation in the interacting Lagrangian density is given by

$$\begin{aligned}\delta\mathcal{L}_{\text{int}} &= -\frac{\lambda}{4}\left(\phi^\dagger\,\mathrm{e}^{-\mathrm{i}\alpha(x)}\,\mathrm{e}^{\mathrm{i}\alpha(x)}\,\phi\right)^2 - \left(-\frac{\lambda}{4}(\phi^\dagger\phi)^2\right)\\ &= \left(-\frac{\lambda}{4}(\phi^\dagger\phi)^2\right) - \left(-\frac{\lambda}{4}(\phi^\dagger\phi)^2\right)\\ &= 0\end{aligned}$$

Hence the U(1) symmetry does not change the interacting Lagrangian, which implies that $\delta S = \delta S_{\text{free}}$. Therefore we shall obtain the same conserved current as that in the free field.

(b) The variation in the interacting Lagrangian density is given by

$$\begin{aligned}\delta\mathcal{L}_{\text{int}} &= \lambda\left(\phi^\dagger\,\mathrm{e}^{-\mathrm{i}\alpha}\,\partial^\mu\left(\mathrm{e}^{\mathrm{i}\alpha}\,\phi\right)\right)\left(\phi^\dagger\,\mathrm{e}^{-\mathrm{i}\alpha}\,\partial_\mu\left(\mathrm{e}^{\mathrm{i}\alpha}\,\phi\right)\right)^\dagger - \lambda\left(\phi^\dagger\partial^\mu\phi\right)\left(\phi^\dagger\partial_\mu\phi\right)^\dagger\\ &= \lambda\left(\phi^\dagger\,\mathrm{e}^{-\mathrm{i}\alpha}\,\mathrm{i}\partial^\mu\alpha\,\mathrm{e}^{\mathrm{i}\alpha}\,\phi\right)\left(\phi^\dagger\partial_\mu\phi\right)^\dagger + \lambda\left(\phi^\dagger\partial^\mu\phi\right)\left(\phi^\dagger\,\mathrm{e}^{-\mathrm{i}\alpha}\,\mathrm{i}\partial_\mu\alpha\,\mathrm{e}^{\mathrm{i}\alpha}\,\phi\right)^\dagger\\ &\quad + \lambda\left(\phi^\dagger\,\mathrm{e}^{-\mathrm{i}\alpha}\,\mathrm{i}\partial^\mu\alpha\,\mathrm{e}^{\mathrm{i}\alpha}\,\phi\right)\left(\phi^\dagger\,\mathrm{e}^{-\mathrm{i}\alpha}\,\mathrm{i}\partial_\mu\alpha\,\mathrm{e}^{\mathrm{i}\alpha}\,\phi\right)^\dagger\\ &= \mathrm{i}\lambda\partial^\mu\alpha\phi^\dagger\left(\phi\partial_\mu\phi^\dagger - \partial_\mu\phi\phi^\dagger\right)\phi + \mathcal{O}(\alpha^2)\end{aligned}$$

From the lectures we know that the variation of the free field Lagrangian is given by

$$\delta\mathcal{L}_0 = \mathrm{i}\partial^\mu\alpha\left(\partial_\mu\phi^\dagger\phi - \phi^\dagger\partial_\mu\phi\right)$$

Therefore

$$\begin{aligned}\delta S &= \int_{M^4} \mathrm{d}^4x\ (\delta\mathcal{L}_0 + \delta\mathcal{L}_{\text{int}}) \\ &= \int_{M^4} \mathrm{d}^4x\, \mathrm{i}\partial^\mu\alpha \left(\lambda\phi^\dagger\left(\phi\partial_\mu\phi^\dagger - \partial_\mu\phi\phi^\dagger\right)\phi + \left(\partial_\mu\phi^\dagger\phi - \phi^\dagger\partial_\mu\phi\right)\right) \\ &= \int_{M^4} \mathrm{d}^4x\, \mathrm{i}\partial^\mu\alpha \left(\left(1+\lambda\phi^\dagger\phi\right)\partial_\mu\phi^\dagger\phi - \phi^\dagger\partial_\mu\phi\left(1+\lambda\phi^\dagger\phi\right)\right) \\ &= -\mathrm{i}\int_{M^4} \mathrm{d}^4x\, \alpha\partial^\mu \left(\left(1+\lambda\phi^\dagger\phi\right)\partial_\mu\phi^\dagger\phi - \phi^\dagger\partial_\mu\phi\left(1+\lambda\phi^\dagger\phi\right)\right)\end{aligned}$$

Hence the Noether current is given by

$$j^\mu = -\mathrm{i}\left(\left(1+\lambda\phi^\dagger\phi\right)\partial^\mu\phi^\dagger\phi - \phi^\dagger\partial^\mu\phi\left(1+\lambda\phi^\dagger\phi\right)\right)$$

(c) Now we consider the global U(1) symmetry

$$\phi(x,t) \mapsto \mathrm{e}^{\mathrm{i}\alpha}\,\phi(x,t), \qquad \phi(x,t)^\dagger \mapsto \mathrm{e}^{-\mathrm{i}\alpha}\,\phi(x,t)^\dagger$$

where $\alpha \in \mathbb{R}$. Then under this transformation,

$$\mathcal{L}_{\text{int}}[\alpha] = \lambda\left(\mathrm{e}^{2\mathrm{i}\alpha}\,\phi^2 + \mathrm{e}^{-2\mathrm{i}\alpha}\,\phi^{\dagger 2}\right)^2 \neq \mathcal{L}_{\text{int}}$$

The Lagrangian does not exhibit global U(1) symmetry, so there is no corresponding conserved current.

(d) Under a global U(1) symmetry,

$$\mathcal{L}[\alpha] = \mathrm{e}^{-\mathrm{i}\alpha}\,\psi^\dagger\gamma^0\left(\mathrm{i}\gamma^\mu\partial_\mu - m\right)\mathrm{e}^{\mathrm{i}\alpha}\,\psi = \overline{\psi}\left(\mathrm{i}\gamma^\mu\partial_\mu - m\right)\psi = \mathcal{L}$$

Hence the spinor field has global U(1) symmetry. Under a local transformation,

$$\begin{aligned}\delta\mathcal{L} &= \mathrm{e}^{-\mathrm{i}\alpha}\,\overline{\psi}\left(\mathrm{i}\gamma^\mu\partial_\mu - m\right)\left(\mathrm{e}^{\mathrm{i}\alpha}\,\psi\right) - \overline{\psi}\left(\mathrm{i}\gamma^\mu\partial_\mu - m\right)\psi \\ &= \mathrm{e}^{-\mathrm{i}\alpha}\,\overline{\psi}\cdot\mathrm{i}\gamma^\mu\partial_\mu(\mathrm{e}^{\mathrm{i}\alpha})\psi \\ &= -\partial_\mu\alpha\alpha\overline{\psi}\gamma^\mu\psi\end{aligned}$$

Therefore

$$\delta S = \int_{M^4} \mathrm{d}^4x\,\partial_\mu\alpha\left(-\overline{\psi}\gamma^\mu\psi\right) = \int_{M^4} \mathrm{d}^4x\,\alpha\partial_\mu\left(\overline{\psi}\gamma^\mu\psi\right)$$

The Noether current is given by

$$j^\mu = \overline{\psi}\gamma^\mu\psi$$

□

Question 2. The Quantized Dirac Field

Here is some practice at manipulating anti-commutators. The quantized Dirac field and its Hamiltonian are given by

$$\begin{aligned}\psi(t,\boldsymbol{x}) &= \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=\pm}\left(e^{-ip\cdot x}a^s_{\boldsymbol{p}}u^s(\boldsymbol{p}) + e^{ip\cdot x}b^{s\dagger}_{\boldsymbol{p}}v^s(\boldsymbol{p})\right) \\ H &= \int \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}E_{\boldsymbol{p}}\sum_{s=\pm}\left(a^{s\dagger}_{\boldsymbol{p}}a^s_{\boldsymbol{p}} + b^{s\dagger}_{\boldsymbol{p}}b^s_{\boldsymbol{p}}\right)\end{aligned}$$

where annihilation and creation operators now satisfy anti-commutation rules. The vacuum state satisfies

$$a^s_{\boldsymbol{p}}|0\rangle = b^s_{\boldsymbol{p}}|0\rangle = 0$$

(a) Check explicitly that

$$i\frac{\partial\psi}{\partial t} = [\psi, H]$$

(b) Show that for a general eigenstate of the Hamiltonian, $|\psi\rangle$,

$$H a_{\boldsymbol{p}}^{s\dagger}|\psi\rangle = (E_\psi + E_{\boldsymbol{p}})\, a_{\boldsymbol{p}}^{s\dagger}|\psi\rangle$$

and similarly for $b_{\mathrm{p}}^{s\dagger}|\psi\rangle$. Hence show that the spectrum of the Hamiltonian contains the states $\left|\{\{\boldsymbol{p}_i, s_i\}, i = 1\ldots n\}, \left\{\overline{\{\boldsymbol{p}_i, s_i\}}, i = n+1\ldots n+m\right\}\right\rangle$ with the constraint that no two particles / anti-particles occupy the same state (here the overline denotes antiparticle states). Show that the eigenvalues are $\sum_{i=1}^{n+m} E_{\boldsymbol{p}_i}$.

(c) Find the three-particle wavefunction

$$\langle 0|\psi(x)\psi(y)\psi(z)|\{\boldsymbol{p}_1, s_1\}, \{\boldsymbol{p}_2, s_2\}, \{\boldsymbol{p}_3, s_3\}\rangle$$

Show that it can be written as a determinant (this is an example of the Slater Determinant) and hence is totally antisymmetric under exchange.

Proof. (a) The anti-commutation relations for the ladder operators are given by

$$\left\{a_{\boldsymbol{p}}^r, a_{\boldsymbol{q}}^{s\dagger}\right\} = \left\{b_{\boldsymbol{p}}^r, b_{\boldsymbol{q}}^{s\dagger}\right\} = (2\pi)^3\delta^{rs}\delta(\boldsymbol{p}-\boldsymbol{q})$$

To compute $[\psi, H]$, we compute $[a_{\boldsymbol{p}}^s, H]$. We need the following observation:

If A, B, C belong to a non-commutative algebra, then

$$[A, BC] = \{A, B\}\,C - B\,\{A, C\}$$

In particular, if $\{A, B\} = \{A, C\} = 0$, then $[A, BC] = 0$. Following the observation we have

$$\begin{aligned}
[a_{\boldsymbol{p}}^s, H] &= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} E_{\boldsymbol{q}} \sum_{r=\pm}\left([a_{\boldsymbol{p}}^s, a_{\boldsymbol{q}}^{r\dagger}a_{\boldsymbol{q}}^r] + [a_{\boldsymbol{p}}^s, b_{\boldsymbol{q}}^{r\dagger}b_{\boldsymbol{q}}^r]\right)\\
&= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} E_{\boldsymbol{q}}\left(\left\{a_{\boldsymbol{p}}^s, a_{\boldsymbol{q}}^{s\dagger}\right\}a_{\boldsymbol{q}}^s - a_{\boldsymbol{q}}^{s\dagger}\left\{a_{\boldsymbol{p}}^s, a_{\boldsymbol{q}}^s\right\}\right)\\
&= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} E_{\boldsymbol{q}}(2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{q})a_{\boldsymbol{q}}^s\\
&= E_{\boldsymbol{p}}a_{\boldsymbol{p}}^s
\end{aligned}$$

Similarly we have

$$[a_{\boldsymbol{p}}^{s\dagger}, H] = -E_{\boldsymbol{p}}a_{\boldsymbol{p}}^{s\dagger}, \qquad [b_{\boldsymbol{p}}^s, H] = E_{\boldsymbol{p}}b_{\boldsymbol{p}}^s, \qquad [b_{\boldsymbol{p}}^{s\dagger}, H] = -E_{\boldsymbol{p}}b_{\boldsymbol{p}}^{s\dagger}$$

Therefore

$$\begin{aligned}
[\psi, H] &= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=\pm}\left(\mathrm{e}^{-\mathrm{i}p_\mu x^\mu}\,u^s(\boldsymbol{p})[a_{\boldsymbol{p}}^s, H] + \mathrm{e}^{\mathrm{i}p_\mu x^\mu}\,v^s(\boldsymbol{p})[b_{\boldsymbol{p}}^{s\dagger}, H]\right)\\
&= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=\pm}\left(E_{\boldsymbol{p}}\,\mathrm{e}^{-\mathrm{i}(E_{\boldsymbol{p}}t-\boldsymbol{p}\cdot\boldsymbol{x})}\,u^s(\boldsymbol{p})a_{\boldsymbol{p}}^s - E_{\boldsymbol{p}}\,\mathrm{e}^{\mathrm{i}(E_{\boldsymbol{p}}t-\boldsymbol{p}\cdot\boldsymbol{x})}\,v^s(\boldsymbol{p})b_{\boldsymbol{p}}^{s\dagger}\right)\\
&= \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=\pm}\left(\mathrm{i}\frac{\partial}{\partial t}\,\mathrm{e}^{-\mathrm{i}(E_{\boldsymbol{p}}t-\boldsymbol{p}\cdot\boldsymbol{x})}\,u^s(\boldsymbol{p})a_{\boldsymbol{p}}^s + \mathrm{i}\frac{\partial}{\partial t}\,\mathrm{e}^{\mathrm{i}(E_{\boldsymbol{p}}t-\boldsymbol{p}\cdot\boldsymbol{x})}\,v^s(\boldsymbol{p})b_{\boldsymbol{p}}^{s\dagger}\right)\\
&= \mathrm{i}\frac{\partial\psi}{\partial t}
\end{aligned}$$

(b) Using the result in (a), we have

$$Ha_{\boldsymbol{p}}^{s\dagger}|\psi\rangle = \left(a_{\boldsymbol{p}}^{s\dagger}H + [H, a_{\boldsymbol{p}}^{s\dagger}]\right)|\psi\rangle = a_{\boldsymbol{p}}^{s\dagger}(H + E_{\boldsymbol{p}})|\psi\rangle = (E_{\psi} + E_{\boldsymbol{p}})\, a_{\boldsymbol{p}}^{s\dagger}|\psi\rangle$$
$$Hb_{\boldsymbol{p}}^{s\dagger}|\psi\rangle = \left(b_{\boldsymbol{p}}^{s\dagger}H + [H, b_{\boldsymbol{p}}^{s\dagger}]\right)|\psi\rangle = b_{\boldsymbol{p}}^{s\dagger}(H + E_{\boldsymbol{p}})|\psi\rangle = (E_{\psi} + E_{\boldsymbol{p}})\, b_{\boldsymbol{p}}^{s\dagger}|\psi\rangle$$

Let $|\Omega\rangle$ be the vacuum state satisfying

$$a_{\boldsymbol{p}}^{s}|\Omega\rangle = b_{\boldsymbol{p}}^{s}|\Omega\rangle = 0$$

A general eigenstate of the Hamiltonian is given by

$$\left|(\boldsymbol{p}_1, s_1), ..., (\boldsymbol{p}_n, s_n), \overline{(\boldsymbol{p}_{n+1}, s_{n+1})}, ..., \overline{(\boldsymbol{p}_{n+m}, s_{n+m})}\right\rangle = \prod_{i=1}^{k}\sqrt{2E_{\boldsymbol{p_i}}}\, a_{\boldsymbol{p}_1}^{s\dagger}\cdots a_{\boldsymbol{p}_n}^{s\dagger} b_{\boldsymbol{p}_{n+1}}^{s\dagger}\cdots b_{\boldsymbol{p}_{n+m}}^{s\dagger}|\Omega\rangle$$

Since the Hamiltonian commutes with all creation operators in the sense of the results in (a), we have

$$\begin{aligned} &H\left|(\boldsymbol{p}_1, s_1), ..., (\boldsymbol{p}_n, s_n), \overline{(\boldsymbol{p}_{n+1}, s_{n+1})}, ..., \overline{(\boldsymbol{p}_{n+m}, s_{n+m})}\right\rangle \\ &= \sum_{i=1}^{n+m} E_{\boldsymbol{p_i}}\left|(\boldsymbol{p}_1, s_1), ..., (\boldsymbol{p}_n, s_n), \overline{(\boldsymbol{p}_{n+1}, s_{n+1})}, ..., \overline{(\boldsymbol{p}_{n+m}, s_{n+m})}\right\rangle \end{aligned}$$

(c) We shall prove the following lemma:

Lemma 1. Slant Determinant

Let $a_1, ..., a_n, b_1, ..., b_n$ belong to a non-commutative unital algebra, with the anti-commutation relations

$$\{a_i, a_j\} = \{b_i, b_j\} = 0, \qquad \{a_i, b_j\} = c_{ij}\,\mathrm{id}$$

Moreover, suppose that there is a state $|0\rangle$ with $a_i|0\rangle = 0$ for all i. Then we have

$$a_1\cdots a_n b_1\cdots b_n|0\rangle = (-1)^{\frac{n(n-1)}{2}}\det(c_{ij})_{i,j=1}^{n}|0\rangle$$

We prove by induction on n. For $n = 1$ this is trivial. Suppose that the result holds for $n - 1$. We have

$$\begin{aligned} a_1\cdots a_n b_1\cdots b_n|0\rangle &= (-a_1\cdots a_{n-1}b_1 a_n b_2\cdots b_n + c_{1n}a_1\cdots a_{n-1}b_2\cdots b_n)|0\rangle \\ &= -a_1\cdots a_{n-1}b_1 a_n b_2\cdots b_n|0\rangle + (-1)^{\frac{(n-1)(n-2)}{2}}c_{n1}\det C_{n1}|0\rangle \end{aligned}$$

Here we used the induction hypothesis to $a_1\cdots a_{n-1}b_2\cdots b_n|0\rangle$. C_{1n} is the matrix $(c_{ij})_{i,j=1}^{n}$ deleting the first row and the n-th column. Repeating the process until we move a_n to the rightmost of the product.

$$\begin{aligned} a_1\cdots a_n b_1\cdots b_n|0\rangle &= (-1)^n a_1\cdots a_{n-1}b_1 b_2\cdots b_n a_n|0\rangle + \sum_{i=1}^{n}(-1)^{\frac{(n-1)(n-2)}{2}}(-1)^{i+1}c_{ni}\det C_{ni}|0\rangle \\ &= (-1)^{\frac{(n-1)(n-2)}{2}+(n-1)}\sum_{i=1}^{n}(-1)^{i+n}c_{ni}\det C_{ni}|0\rangle \\ &= (-1)^{\frac{n(n-1)}{2}}\det(c_{ij})_{i,j=1}^{n}|0\rangle \qquad \text{(Laplace expansion of the determinant)} \end{aligned}$$

which completes the induction.

Return to the problem in (c). In the following derivation, we can just throw away all the $b_{\boldsymbol{q}}^{s\dagger}$ because

$\langle 0|\, b_{\boldsymbol{q}}^{s\dagger} = 0.$

The n-particle wave function is given by

$$\begin{aligned}
&\langle 0|\, \psi(x_1)\cdots\psi(x_n)\, |(\boldsymbol{p}_1, s_1), ..., (\boldsymbol{p}_n, s_n)\rangle \\
&= \prod_{i=1}^{k} \sqrt{2E_{\boldsymbol{p_i}}}\, \langle 0|\, \psi(x_1)\cdots\psi(x_n) a_{\boldsymbol{p}_1}^{s_1\dagger}\cdots a_{\boldsymbol{p}_n}^{s_n\dagger}\, |0\rangle \\
&= \langle 0| \prod_{i=1}^{n} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{q}_i}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{q}_i}}} \mathrm{e}^{-\mathrm{i}(q_i)_\mu (x_i)^\mu} \sum_{r_i=\pm} u^{r_i}(\boldsymbol{q}_i) a_{\boldsymbol{q}_i}^{r_i} \prod_{j=1}^{n} \sqrt{2E_{\boldsymbol{p}_j}} a_{\boldsymbol{p_j}}^{s_j^\dagger}\, |0\rangle \\
&= \langle 0| \prod_{i=1}^{n} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\boldsymbol{q}_i}{(2\pi)^3} \frac{1}{\sqrt{2E_{\boldsymbol{q}_i}}} \mathrm{e}^{-\mathrm{i}(q_i)_\mu (x_i)^\mu} \prod_{j=1}^{n} \sqrt{2E_{\boldsymbol{p}_j}} \sum_{r_i=\pm} u^{r_i}(\boldsymbol{q}_i)(-1)^{\frac{n(n-1)}{2}} (2\pi)^{3n} \det(\delta^{r_k s_\ell}\delta(\boldsymbol{p}_k - \boldsymbol{q}_\ell))_{k,\ell=1}^{n}\, |0\rangle \\
&= (-1)^{\frac{n(n-1)}{2}} \det\Big(\mathrm{e}^{-\mathrm{i}(p_i)_\mu (x_j)^\mu}\Big)_{i,j=1}^{n} \prod_{i=1}^{n} u^{s_i}(\boldsymbol{p}_i)
\end{aligned}$$

Finally, we take $n = 3$ and obtain the 3-particle wave functon:

$$\begin{aligned}
&\langle 0|\, \psi(x_1)\psi(y)\psi(z)\, |(\boldsymbol{p}_1, s_1), (\boldsymbol{p}_2, s_2), (\boldsymbol{p}_3, s_3)\rangle \\
&= -u^{s_1}(\boldsymbol{p}_1) u^{s_2}(\boldsymbol{p}_2) u^{s_3}(\boldsymbol{p}_3) \det \begin{pmatrix} \mathrm{e}^{-\mathrm{i}(p_1)_\mu x^\mu} & \mathrm{e}^{-\mathrm{i}(p_2)_\mu x^\mu} & \mathrm{e}^{-\mathrm{i}(p_3)_\mu x^\mu} \\ \mathrm{e}^{-\mathrm{i}(p_1)_\mu y^\mu} & \mathrm{e}^{-\mathrm{i}(p_2)_\mu y^\mu} & \mathrm{e}^{-\mathrm{i}(p_3)_\mu y^\mu} \\ \mathrm{e}^{-\mathrm{i}(p_1)_\mu z^\mu} & \mathrm{e}^{-\mathrm{i}(p_2)_\mu z^\mu} & \mathrm{e}^{-\mathrm{i}(p_3)_\mu z^\mu} \end{pmatrix}
\end{aligned}$$

(*Note that $\psi(x)$ is a vector operator. So $\psi(x)\psi(y)\psi(z)$ should be unterstood as component-wise multiplication. Similarly for $u^{s_1}(\boldsymbol{p}_1)u^{s_2}(\boldsymbol{p}_2)u^{s_3}(\boldsymbol{p}_3)$.*) □

Question 3. Cancellation of vacuum bubble diagrams

Consider the real scalar field with interaction $\frac{1}{4!}\lambda\phi^4$ and the vacuum expectation value

$$G_K = \left\langle \Omega \middle| T \prod_{i=1}^{2K} \phi(x_i) \middle| \Omega \right\rangle = \frac{\left\langle 0 \middle| T \left(\prod_{i=1}^{2K} \phi_I(x_i) \right) \exp\left(-\mathrm{i} \int_{-\infty(1-\mathrm{i}\epsilon)}^{\infty(1-\mathrm{i}\epsilon)} \mathrm{d}t\, H_{\mathrm{Int}}(\phi_I) \right) \middle| 0 \right\rangle}{\left\langle 0 \middle| T \exp\left(-\mathrm{i} \int_{-\infty(1-\mathrm{i}\epsilon)}^{\infty(1-\mathrm{i}\epsilon)} \mathrm{d}t\, H_{\mathrm{Int}}(\phi_I) \right) \middle| 0 \right\rangle}$$

where K is an integer. Show that the contribution to G_K at any order λ^L, L integer, in which all external points are connected to a single cluster, contains no bubble diagrams.

Proof. The idea is that, the denominator contains the exponentials of all the contributions of bubble diagrams, which cancels those in the numerator, leaving the contributions of connected diagrams[1] only in the expression of G_K.

The expension of $\langle 0|\, \mathcal{T}\left\{ \prod_{i=1}^{2K} \phi_I(x_i) \exp\left(-\mathrm{i} \int_{-T(1-\mathrm{i}\epsilon)}^{T(1-\mathrm{i}\epsilon)} \mathrm{d}t\, H_{\mathrm{int}}(\phi_I) \right) \right\} |0\rangle$ at order λ^L is given by

$$\langle 0|\, \frac{1}{L!} \left(-\frac{\mathrm{i}\lambda}{4!} \right)^L \mathcal{T}\left\{ \prod_{i=1}^{2K} \prod_{j=1}^{L} \phi_I(x_i) \int_{[-T(1-\mathrm{i}\varepsilon), T(1-\mathrm{i}\varepsilon)]\times\mathbb{R}^3} \mathrm{d}^4 z_j\, \phi_I(z_j)^4 \right\} |0\rangle \qquad (*)$$

In terms of the Feynman diagrams, the expression $(*)$ is a summation of all possible diagrams with external points $x_1, ..., x_{2K}$ and internal points $z_1, ..., z_L$. Let Γ_L be the set of all such diagrams. Let Λ_L be set of

[1] Here connectedness means that every internal point is path-connected to the external points.

bubble diagrams arising as path-components of the digrams in Γ_L. Let $\Lambda := \bigcup_{L\in\mathbb{N}} \Lambda_L$ and $\Gamma := \bigcup_{L\in\mathbb{N}} \Gamma_L$. It is clear that Λ is countable. Let $\{V_i\}_{i\in\mathbb{N}}$ be an enumeration of Λ.

Consider a diagram $\gamma \in \Gamma_L$. We decompose it into a disjoint union of path-connected sub-diagrams. For each i, let n_i be the number of copies of V_i as path-components of γ. Only finitely many of the n_i is non-zero. If $v(V_i)$ denotes the value of the Wick's contraction represented by the diagram V_i, then the n_i copies contribute $\frac{1}{n_i!}v(V_i)^{n_i}$ to $v(\gamma)$, where $1/n_i!$ is a symmetry factor. We have

$$v(\gamma) = \prod_k v(A_{\gamma,k}) \cdot \prod_{i\in\mathbb{N}} \frac{1}{n_i!} v(V_i)^{n_i}$$

where $\{A_{\gamma,k}\}$ is the set of non-bubble path-components of γ.

Now the value of expression $(*)$ is just the sum

$$\sum_{\gamma\in\Gamma_L} v(\gamma) = \sum_{\gamma\in\Gamma_L} \prod_k v(A_{\gamma,k}) \cdot \prod_{i\in\mathbb{N}} \frac{1}{n_i!} v(V_i)^{n_i}$$

Following *Peskin & Schroder* we sum over all order L:

$$\sum_{L\in\mathbb{N}} \sum_{\gamma\in\Gamma_L} \prod_k v(A_{\gamma,k}) \cdot \prod_{i\in\mathbb{N}} \frac{1}{n_i!} v(V_i)^{n_i}$$

Peskin & Schroder p.97 (4,52) claims that the expression above equals to

$$\mathscr{C}(x_1, ..., x_{2K}) \cdot \sum_{\{n_i\}} \prod_{i\in\mathbb{N}} \frac{1}{n_i!} v(V_i)^{n_i} = \mathscr{C}(x_1, ..., x_{2K}) \cdot \exp\left(\sum_{i\in\mathbb{N}} v(V_i)\right)$$

where $\mathscr{C}(x_1, ..., x_{2K})$ represents the contributions of the non-bubble diagrams, and the sum is over all finite sequences $\{n_i\}_{i\in\mathbb{N}}$. *I am not convinced by this argument.*

The similar argument claims that the denominator of G_K is given by

$$\lim_{T\to\infty} \langle 0| \mathcal{T}\left\{\exp\left(-\mathrm{i}\int_{-T(1-\mathrm{i}\epsilon)}^{T(1-\mathrm{i}\epsilon)} \mathrm{d}t\, H_{\text{int}}(\phi_I)\right)\right\} |0\rangle = \exp\left(\sum_{i\in\mathbb{N}} v(V_i)\right)$$

Therefore the contributions $\exp\left(\sum_{i\in\mathbb{N}} v(V_i)\right)$ cancels in the fraction. Only the connected diagrams have contributions to G_K. □

Proof. (***A better proof from the model solution.***)

Let us focus first on the numerator of the vacuum expectation value G_K

$$\widetilde{G}_K \equiv \lim_{T\to\infty(1-i\epsilon)} \left\langle 0 \middle| \mathcal{T}\left\{\prod_{i=1}^{2K} \phi_i \exp\left(-i\int_{-T}^{T} \mathrm{d}t \int \mathrm{d}^3\boldsymbol{y} \frac{\lambda}{4!}\phi^4(y)\right)\right\} \middle| 0 \right\rangle$$

If we introduce the shorthand notation

$$\left[-i\int H(\phi_j)\right] \equiv -i\int \mathrm{d}y_j \frac{\lambda}{4!}\phi^4(y_j)$$

the expansion in powers of λ, i.e. in powers of the Hamiltonian, reads

$$\widetilde{G}_K \equiv \left\langle 0 \left| \mathcal{T} \left\{ \prod_{i=1}^{2K} \phi_i \sum_{n=0}^{\infty} \frac{1}{n!} \prod_{j=1}^{n} \left[-i \int H\left(\phi_j\right) \right] \right\} \right| 0 \right\rangle$$

According to Wick's theorem, for a fixed order in n, only the fully contracted diagrams with $2K$ external points and n internal ones contribute. Therefore, for given n we will have a sum of diagrams which contain either connected graphs only (i.e. no vacuum graphs) or connected and disconnected subgraphs. Thus

$$\begin{aligned}
&\left\langle 0 \left| \mathcal{T} \left\{ \prod_{i=1}^{2K} \phi_i \prod_{j=1}^{n} \left[-i \int H\left(\phi_j\right) \right] \right\} \right| 0 \right\rangle \\
&= \sum_{m=0}^{n} \binom{n}{m} \left\langle 0 \left| \mathcal{T} \left\{ \prod_{i=1}^{2K} \phi_i \prod_{j=1}^{m} \left[-i \int H\left(\phi_j\right) \right] \right\} \right| 0 \right\rangle_c \left\langle 0 \left| \mathcal{T} \left\{ \prod_{j=m+1}^{n} \left[-i \int H\left(\phi_j\right) \right] \right\} \right| 0 \right\rangle
\end{aligned}$$

where with the subscript c we denote the connected subgraphs. The combinatorial factor comes from the fact that there are $\binom{n}{m}$ ways to choose m internal points out of n to fully contract with the external fields. If we now sum over all n, the numerator takes the form

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{m!} \left\langle 0 \left| \mathcal{T} \left\{ \prod_{i=1}^{2K} \phi_i \prod_{j=1}^{m} \left[-i \int H\left(\phi_j\right) \right] \right\} \right| 0 \right\rangle_c \left\langle 0 \left| \mathcal{T} \left\{ \frac{1}{(n-m)!} \prod_{j=m+1}^{n} \left[-i \int H\left(\phi_j\right) \right] \right\} \right| 0 \right\rangle \\
= &\sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{m!} \left\langle 0 \left| \mathcal{T} \left\{ \prod_{i=1}^{2K} \phi_i \prod_{j=1}^{m} \left[-i \int H\left(\phi_j\right) \right] \right\} \right| 0 \right\rangle_c \left\langle 0 \left| \mathcal{T} \left\{ \frac{1}{(n-m)!} \prod_{j=1}^{n-m} \left[-i \int H\left(\phi_j\right) \right] \right\} \right| 0 \right\rangle
\end{aligned}$$

which can be factorised as

$$\left(\sum_{m=0}^{\infty} \frac{1}{m!} \left\langle 0 \left| \mathcal{T} \left\{ \prod_{i=1}^{2K} \phi_i \prod_{j=1}^{m} \left[-i \int H\left(\phi_j\right) \right] \right\} \right| 0 \right\rangle_c \right) \sum_{n=0}^{\infty} \left\langle 0 \left| \mathcal{T} \left\{ \frac{1}{n!} \prod_{j=1}^{n} \left[-i \int H\left(\phi_j\right) \right] \right\} \right| 0 \right\rangle$$

The last term on the right hand side exponentiates

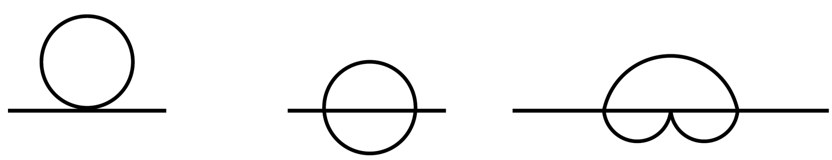
$$\left(\sum_{m=0}^{\infty} \frac{1}{m!} \left\langle 0 \left| \mathcal{T} \left\{ \prod_{i=1}^{2K} \phi_i \prod_{j=1}^{m} \left[-i \int H\left(\phi_j\right) \right] \right\} \right| 0 \right\rangle_c \right) \left\langle 0 \left| \exp\left(-i \int \mathrm{d}t H(\phi(y)) \right) \right| 0 \right\rangle$$

and this exactly cancels the denominator in the vacuum expectation value. Hence, if we expand to any $\mathcal{O}\left(\lambda^L\right)$ we can discard all diagrams containing vacuum subgraphs. □

Question 4. Feynman Diagrams

This is practice using the Feynman rules for scalar fields.

(a) Find expressions, including the combinatorial factors, corresponding to the following quantum corrections to the two-point function for a single (real) scalar field. Do not try to do the momentum integrals.



(b) Now consider the case of a complex scalar field with Lagrangian density

$$\mathcal{L} = \partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi - \frac{\lambda}{4}\left(\phi^\dagger \phi\right)^2$$

Do any of your answers in part a) change?

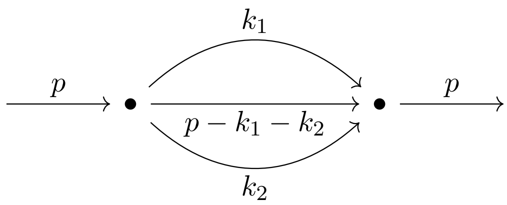
(c) Find all the distinct amputated diagrams at $O\left(\lambda^3\right)$ for the quantum corrections to the connected four-point function of the real scalar field. Try to organise your answer in an efficient manner, making full use of the structure of the graphs.

Proof. (a) We use the Feynman rules in the momentum space.

- For the first diagram, we label the external line with momentum p, and the internal line with momentum k. The combinatorial factor is $1/2$. By Feynman's rule, we have

$$\frac{1}{2}(-\mathrm{i}\lambda)\left(\frac{\mathrm{i}}{p^2 - m^2 - \mathrm{i}\epsilon}\right)^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\mathrm{i}}{k^2 - m^2 - \mathrm{i}\epsilon}$$

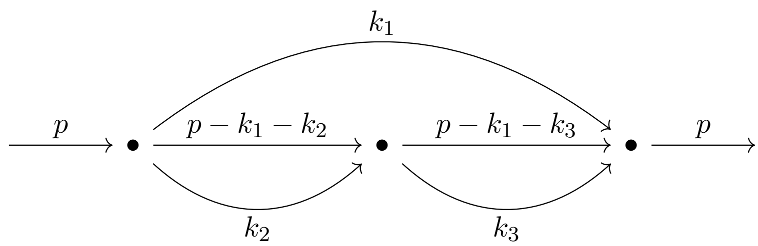
- For the second diagram, we label the edges with momenta as follows



The combinatorial factor is $1/3!$. By Feynman's rule, we have

$$\frac{1}{6}(-\mathrm{i}\lambda)^2\left(\frac{\mathrm{i}}{p^2 - m^2 - \mathrm{i}\epsilon}\right)^2 \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 k_2}{(2\pi)^4} \frac{\mathrm{i}}{{k_1}^2 - m^2 - \mathrm{i}\epsilon} \cdot \frac{\mathrm{i}}{{k_2}^2 - m^2 - \mathrm{i}\epsilon} \cdot \frac{\mathrm{i}}{(p - k_1 - k_2)^2 - m^2 - \mathrm{i}\epsilon}$$

- For the third diagram, we label the edges with momenta as follows



The combinatorial factor is $1/2! \cdot 1/2! = 1/4$. By Feynman's rule, we have

$$\frac{1}{4}(-\mathrm{i}\lambda)^3\left(\frac{\mathrm{i}}{p^2 - m^2 - \mathrm{i}\epsilon}\right)^2 \cdot \int \frac{\mathrm{d}^4 k_1}{(2\pi)^4} \int \frac{\mathrm{d}^4 k_2}{(2\pi)^4} \int \frac{\mathrm{d}^4 k_3}{(2\pi)^4}$$
$$\frac{\mathrm{i}}{{k_1}^2 - m^2 - \mathrm{i}\epsilon} \cdot \frac{\mathrm{i}}{{k_2}^2 - m^2 - \mathrm{i}\epsilon} \cdot \frac{\mathrm{i}}{{k_3}^2 - m^2 - \mathrm{i}\epsilon} \cdot \frac{\mathrm{i}}{(p - k_1 - k_2)^2 - m^2 - \mathrm{i}\epsilon} \cdot \frac{\mathrm{i}}{(p - k_1 - k_3)^2 - m^2 - \mathrm{i}\epsilon}$$

(b) The momentum integrals remain unchanged. The combinatorial factors are changed due to the new requirement of charge conservation. For the Feynman diagrams, this means that we need to assign each edge with a direction such that each internal vertex has two in-coming and two out-going edges. The new combinatorial factors for the diagrams in (a) are given by

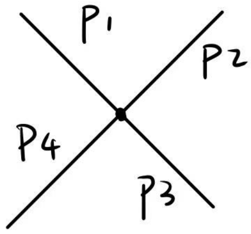
- 1;
- $1/2$;

- (1/4, 1) (The diagram for real scalar field splits into two non-homeomorphic diagrams for complex scalar field, each with a different combinatorial factor.)

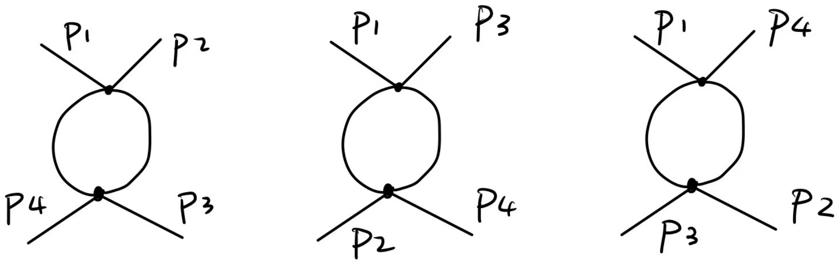
(c) I cannot give a rigorous definition for the amputated graph. So I will be hand-waving in this part.

Let n be the number of internal vertices after amputation. We note that $n \in \{1, 2, 3\}$. If $n > 1$, each internal vertex in the amputated graph cannot be directly connected to more than 2 external edges, for otherwise the rest of the internal vertices will be amputated. Let a_i be the number of external edges that are directly connected to the internal vertex i. Then each $a_i \in \{1, 2\}$, and $\sum_i a_i = 4$. We can list all non-homeomorphic graphs as follows:

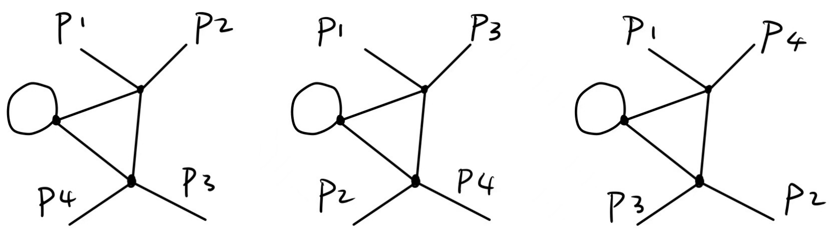
- $n = 1$:



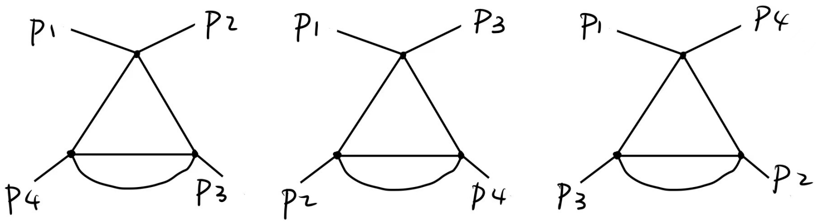
- $n = 2$, $(a_1, a_2) = (2, 2)$:



- $n = 3$, $(a_1, a_2, a_3) = (2, 0, 0)$:



- $n = 3$, $(a_1, a_2, a_3) = (2, 1, 1)$:



□