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Problem Sheet 4

C3.1: Algebraic Topology

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Convention: All spaces are topological spaces. Maps of spaces are always continuous.

Question 1

a) For M an oriented closed connected n-manifold, prove that

- $H^n(M) \cong \mathbb{Z}$;
- $H_{n-1}(M)$ has no torsion;
- There exists a generator $\boldsymbol{\omega_M} \in H^n(M)$ with $\boldsymbol{\omega_M}([M]) = 1$.

(You may use Poincaré duality and universal coefficients theorems.)

b) For M, N oriented closed connected n-manifolds, and $f : M \to N$, prove that

$$\begin{aligned} f^* : \quad & H^n(N) \longrightarrow H^n(M) \\ & \omega_N \longmapsto \deg f \cdot \omega_M \end{aligned}$$

c) Let $f : S^n \to T^n$, $n \geqslant 2$. Prove that $\deg f = 0$. Construct a map $T^n \to S^n$ of non-zero degree.

Proof. a) Since M is an oriented compact connected n-manifold, by Poincaré duality, $H^k(M) \cong H_{n-k}(M)$ for $0 \leqslant k \leqslant n$.

- $H^n(M) \cong H_0(M) \cong \mathbb{Z}$;
- $H_{n-1}(M) \cong H^1(M)$. By universal coefficient theorem for cohomology, we have a split short exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_{\mathbb{Z}}(H_0(M), \mathbb{Z}) \longrightarrow H^1(M) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(H_1(M), \mathbb{Z}) \longrightarrow 0$$

  Note that $\operatorname{Ext}^1_{\mathbb{Z}}(H_0(M), \mathbb{Z}) \cong \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) = 0$. We have $H^1(M) \cong \operatorname{Hom}_{\mathbb{Z}}(H_1(M), \mathbb{Z})$. We know that dualisation kills torsion. More specifically, suppose that $n \in \mathbb{Z} \setminus \{0\}$ and $\varphi \in \operatorname{Hom}_{\mathbb{Z}}(H_1(M), \mathbb{Z}) \setminus \{0\}$ are such that $n\varphi = 0$. We take $x \in H_1(M) \setminus \{0\}$ such that $\varphi(x) \neq 0$. Then $n\varphi(x) \cancel{\neq} = 0$ since $\mathbb{Z}$ is an integral domain. This is a contradiction. We conclude that $H_{n-1}(M) \cong H^1(M)$ is torsion-free.

- By universal coefficient theorem for cohomology, we have a split short exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_{\mathbb{Z}}(H_{n-1}(M), \mathbb{Z}) \longrightarrow H^n(M) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(H_n(M), \mathbb{Z}) \longrightarrow 0$$

  Since M is compact, the homology group $H_{n-1}(M)$ is finitely generated. We have proven that it is torsion-free. Then by the structure theorem for finitely generated Abelian groups, $H_{n-1}(M) \cong \mathbb{Z}^k$ for some k. In particular $H_{n-1}(M)$ is free. Hence $\operatorname{Ext}^1_{\mathbb{Z}}(H_{n-1}(M), \mathbb{Z}) = 0$. We have $H^n(M) \cong \operatorname{Hom}_{\mathbb{Z}}(H_n(M), \mathbb{Z})$. Since $[M]$ generates $H_n(M)$, there exists $\omega_M \in H^n(M)$ such that $\omega_M([M]) = 1$.

b) We have $H_n(M)^\vee \cong H^n(M)$, and ω_M is a dual basis of $[M]$. Then $f^* : H^n(N) \to H^n(M)$ is the dual map of $f_* : H_n(M) \to H_n(N)$, $[M] \mapsto \deg f \cdot [N]$. Then by linear algebra we have $f^* : \omega_N \mapsto \deg f \cdot \omega_M$.
not vector spaces (result is still true though)

c) Since $T^n = S^1 \times \cdots \times S^1$, By Künneth Theorem, we have the following group isomorphism, which is also a ring homorphism into $H^\bullet(T^n)$.

$$\begin{aligned} \bigotimes_{i=1}^{n} H^1(S^1) & \xrightarrow{\ \cong\ } H^n(T^n) \\ e_1 \otimes \cdots \otimes e_n & \longmapsto p_1^*(e_1) \smile \cdots \smile p_n^*(e_n) \end{aligned}$$

where $p_i^* : H^1(S^1) \to H^1(T^n)$ is the pull-back ~~of~~ via the projection $p_i : T^n \to S^1$.

Consider $f : S^n \to T^n$. The pull-back $f^* : H^\bullet(T^n) \to H^\bullet(S^n)$ is a ring homomorphism. Then

$$f^*\left(p_1^*(e_1) \smile \cdots \smile p_n^*(e_n)\right) = f^* p_1^*(e_1) \smile \cdots \smile f^* p_n^*(e_n)$$

Note that each $f^* p_i^*(e_i) \in H^1(S^n) = 0$. So we must have $f^*\left(p_1^*(e_1) \smile \cdots \smile p_n^*(e_n)\right) = 0$. Since $p_1^*(e_1) \smile \cdots \smile p_n^*(e_n)$ generates $H^n(T^n)$, we conclude that $\deg f = 0$. ✓

Finally we construct a map $f : T^n \to S^n$ with non-zero degree. Choose $p \in T^n$. Since T^n is a manifold, there exists a neighbourhood $U \subseteq T^n$ of p such that $U \cong \mathbb{D}^n$. Consider the quotient map

$$f\colon T^n \to T^n/(T^n \setminus U) \cong S^n$$

We claim that $f_* : H_n(T^n) \to H_n(S^n)$ is a isomorphism, and hence have degree ± 1. Consider the long exact sequence of relative homology

$$0 \longrightarrow H_n(T^n \setminus U) \longrightarrow H_n(T^n) \xrightarrow{p_n} H_n(T^n, T^n \setminus U) \xrightarrow{\delta_n} H_{n-1}(T^n \setminus U) \longrightarrow \cdots$$

Note that $T^n \setminus U \simeq S^1 \vee \cdots \vee S^1$ (view T^n as I^n with edge identifications, and retract $T^n \setminus U$ onto the edges). So $H_n(T^n \setminus U) = 0$ and $H_{n-1}(T^n \setminus U) = 0$ ($n \geqslant 3$). Then $p_n\colon H_n(T^n) \to H_n(T^n, T^n \setminus U)$ is an isomorphism. Since $(T^n, T^n \setminus U)$ is a good pair, we have another isomorphism $\varphi : H_n(T^n, T^n \setminus U) \to H_n(T^n/(T^n \setminus U)) \cong H_n(S^n)$. The composite $p_n \circ \varphi = f_*$. This proves that f_* is an isomorphism. perfect □

Question 2

Show that any matrix $A \in M_{n\times n}(\mathbb{Z})$ defines a map $f : T^n \to T^n$ on the ***n*-torus T^n**$= \mathbb{R}^n/\mathbb{Z}^n \cong S^1 \times \cdots \times S^1$.

Describe $f_* : H_1(T) \to H_1(T)$ in terms of explicit generators. Show that $\deg f = \det A \in \mathbb{Z}$.

Cutural Remark. Any Lie group homomorphism $\varphi : T^n \to T^n$ gives rise to such a Lie algebra homomorphism $D_1\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that $D_1\varphi|_{\mathbb{Z}^n} = A$.

Proof. Let $q : \mathbb{R}^n \to (\mathbb{R}/\mathbb{Z})^n \cong \mathbb{R}^n/\mathbb{Z}^n$ be the quotient map. Then $q \circ A : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$. For any $v \in \mathbb{Z}^n$, as $Av \in \mathbb{Z}^n$, ✓ then $q \circ Av = 0$. Hence $q \circ A$ induces a map $\widetilde{A} : \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$. ✓ $\widetilde{A}$ can be viewed as an endomorphism on the n-torus T^n.

$H^1(S^1)$ for ✓ they're maps $I \to \mathbb{R}/\mathbb{Z} \cong S^1$, you could just work with them like that

Let $e_1, \ldots, e_n$ be the generators of each S^1. Their representatives in $\mathbb{R}^n$ form a basis of $\mathbb{R}^n$. We know that $H_1(T^n) \cong \mathbb{Z}^n$ by Künneth's Theorem. ✓ $e_1, \ldots, e_n \in H_1(T^n)$ in fact form a basis of $H_1(T^n)$. The map $f_* : H_1(T^n) \to H_1(T^n)$ is given by $e_i \mapsto Ae_i \in H_1(T^n)$.

$I \to (\mathbb{R}/\mathbb{Z})^n$
$e_i : t \mapsto (0, \ldots, t, \ldots, 0)$
then this follows easily.

From the lectures, we know that $H^n(T^n)$ is generated by $p_1^*(e_1) \wedge \cdots \wedge p_n^*(e_n)$. ✓ Then $f^* : H^n(T^n) \to H^n(T^n)$ is given by $p_1^*(e_1) \wedge \cdots \wedge p_n^*(e_n) \mapsto p_1^*(Ae_1) \wedge \cdots \wedge p_n^*(Ae_n)$. By the definition of determinant,

$$\det A = \frac{Ae_1 \wedge \cdots \wedge Ae_n}{e_1 \wedge \cdots \wedge e_n} = \frac{p_1^*(Ae_1) \wedge \cdots \wedge p_n^*(Ae_n)}{p_1^*(e_1) \wedge \cdots \wedge p_n^*(e_n)} = \deg f$$

good! □

these fractions aren't the best notation

Question 3

a) For M, N compact connected orientable n-manifolds, prove that $M \# N$ is also a compact connected orientable n-manifold, and that

$$H_\bullet(M \# N) \cong H_\bullet(M) \oplus H_\bullet(N) \quad \text{for } 1 \leqslant \bullet \leqslant n-1$$

b) Formulate and prove such an isomorphism on cohomology, as a ring isomorphism.

c) What can you say about the case $\bullet = n$, and cup products of $H^\bullet(M), H^\bullet(N)$ classes that land in $H^n(M \# N)$?

d) Deduce what $H_\bullet(\Sigma_g)$, $\chi(\Sigma_g)$ and the ring $H^\bullet(\Sigma_g)$ are, for the genus g surface Σ_g.

$\cong \mathbb{D}^n$

Proof. a) The connected sum can be defined in the following way. Choose $x \in M$ and $y \in N$. Let $U \in M$ and $V \in N$ be charts containing x and y respectively. We can identify ∂U and ∂V via $\partial U \cong S^{n-1} \cong \partial V$. Then we define the connected sum to be $M \# N := ((M \setminus U) \cup (N \setminus V))/(\partial U \sim \partial V)$.

It is clear from definition that $M \# N$ is connected and compact (being the quotient of a compact space). $M \# N$ is orientable, as we can pick an isomorphism $\partial U \cong \partial V$ such that the local orientations on them agree.
Also show that $M \# N$ is a manifold using collar neighbourhood theorem.
Let U' be a neighbourhood of $\overline{U}$ in M and V' be a neighbourhood of $\overline{V}$ in N. Consider $A := (M \setminus U) \cup (V' \setminus V)$ and $B = (N \setminus V) \cup (U' \setminus U)$ as subspaces of $M \# N$. Then $M \# N = A^\circ \cup B^\circ$. The Mayor-Vietoris sequence is given by

$$\cdots \longrightarrow \widetilde{H}_k(A \cap B) \longrightarrow \widetilde{H}_k(A) \oplus \widetilde{H}_k(B) \longrightarrow \widetilde{H}_k(M \# N) \longrightarrow \widetilde{H}_{k-1}(A \cap B) \longrightarrow \cdots$$

needs some more specific choices for U', V'

Note that $A \cap B \simeq S^{n-1}$. Then $\widetilde{H}_k(A \cap B) = 0$ for $k \leqslant n-2$. In such case, we have $\widetilde{H}_k(M \# N) \cong \widetilde{H}_k(A) \oplus \widetilde{H}_k(B)$ from the long exact sequence.

We have $M \simeq A \cup_\varphi \mathbb{D}^n$, where φ identifies $\partial \mathbb{D}^n \cong S^{n-1}$ with $\partial V'$. From Question 6 of Sheet 3, we have $H_k(A) \cong H_k(M)$ for $k \leqslant n-2$. Similarly $H_k(B) \cong H_k(N)$. Hence $H_k(M \# N) \cong H_k(M) \oplus H_k(N)$ for $1 \leqslant k \leqslant n-2$.

For $k = n-1$, since M and N are orientable, we use Poincaré duality: $H_{n-1}(M \# N) \cong H^1(M \# N)$, $H_{n-1}(M) \cong H^1(M)$, and $H_{n-1}(N) \cong H^1(N)$. Using the cohomology version of Mayer-Vietoris sequence we can prove that $H^1(M \# N) \cong H^1(M) \oplus H^1(N)$. Hence $H_{n-1}(M \# N) \cong H_{n-1}(M) \oplus H_{n-1}(N)$. This concludes the proof.

There's still some difficulty involved.

b) We have a group isomorphism at each grading of the cohomology ring:

$$H^k(M \# N) \cong H^k(M) \oplus H^k(N), \qquad 1 \leqslant k \leqslant n-1$$

which is proven by the same Mayer-Vietoris technique. The cup product is computed component-wise for $1 \leqslant k + \ell \leqslant n-1$. That is, for $(\alpha_k, \beta_k) \in H^k(M) \oplus H^k(N) \cong H^k(M \# N)$ and $(\alpha_\ell, \beta_\ell) \in H^\ell(M) \oplus H^\ell(N) \cong H^k(M \# N)$,

$$(\alpha_k, \beta_k) \smile (\alpha_\ell, \beta_\ell) = (\alpha_k \smile \alpha_\ell, \beta_k \smile \beta_\ell) \in H^{k+\ell}(M) \oplus H^{k+\ell}(N) \cong H^{k+\ell}(M \# N)$$

c) At degree n, the good pair $(M \# N, S^{n-1})$ gives the long exact sequence

$$\cdots \longrightarrow H^{n-1}(S^{n-1}) \longrightarrow H^n(M \# N, S^{n-1}) \longrightarrow H^n(M \# N) \longrightarrow 0$$

Note that $H^n(M \# N, S^{n-1}) \cong H^n(M \# N / S^{n-1}) \cong H^n(M \vee N) \cong H^n(M) \oplus H^n(N)$.

Since $H^n(M) \oplus H^n(N) \cong \mathbb{Z}^2$, $H^n(M \# N) \cong \mathbb{Z}$, the map $H^n(M) \oplus H^n(N) \to H^n(M \# N)$ is given by $(\omega_M, 0) \mapsto \omega_{M\#N}$ and $(0, \omega_N) \mapsto \omega_{M\#N}$. Therefore, for $k + \ell = n$, $(\alpha_k, \beta_k) \in H^k(M \# N)$ and $(\alpha_\ell, \beta_\ell) \in H^\ell(M \# N)$, we have

$$(\alpha_k, \beta_k) \smile (\alpha_\ell, \beta_\ell) = \alpha_k \smile \alpha_\ell + \beta_k \smile \beta_\ell \in H^n(M \# N)$$

can say more

The rings $H^\bullet(M \# N)$ and $H^\bullet(M) \times H^\bullet(N)$ are certainly not isomorphic.

d) From the lectures and previous problem sheets, we know that for $\Sigma_1 = T^2$,

$$H_n(T^2) \cong \begin{cases} \mathbb{Z}, & n = 0 \\ \mathbb{Z}^2, & n = 1 \\ \mathbb{Z}, & n = 2 \\ 0, & \text{otherwise} \end{cases}, \qquad H^n(T^2) \cong \begin{cases} \mathbb{Z}, & n = 0 \\ \mathbb{Z}^2, & n = 1 \\ \mathbb{Z}, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

Since $\Sigma_g = T^2 \# \cdots \# T^2$, inductively we have

$$H_n(\Sigma_g) \cong \begin{cases} \mathbb{Z}, & n=0 \\ \mathbb{Z}^{2g}, & n=1 \\ \mathbb{Z}, & n=2 \\ 0, & \text{otherwise} \end{cases}, \qquad H^n(\Sigma_g) \cong \begin{cases} \mathbb{Z}, & n=0 \\ \mathbb{Z}^{2g}, & n=1 \\ \mathbb{Z}, & n=2 \\ 0, & \text{otherwise} \end{cases}$$

The Euler characteristic

$$\chi(\Sigma_g) = \sum_{n=0}^{\infty} (-1)^n \operatorname{rank} H_n(\Sigma_g) = 2-2g$$

Let a_i, b_i be the generators of $H^1(T^2)$ for $i = 1,\dots,g$. Then $H^1(\Sigma_g)$ is generated by $a_1, b_1, \dots, a_g, b_g$, with the cup product structure

$$a_i \smile a_j = 0, \qquad b_i \smile b_j = 0, \qquad a_i \smile b_j = 0, \qquad a_1 \smile b_1 = \cdots = a_g \smile b_g \text{ generates } H^2(\Sigma_g) \cong \mathbb{Z}$$

This completely describes the ring structure of $H^\bullet(\Sigma_g)$. □

Question 4

a) Verify that $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}; G) = 0$ and $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/d; G) \cong G/dG$ for any Abelian group G.

b) Use the universal coefficients theorem to compute $H^\bullet(\mathbb{R}P^3; \mathbb{Q}/\mathbb{Z})$.

c) Compute $H^{\mathrm{CW}}_\bullet(\mathbb{R}P^3; \mathbb{Q}/\mathbb{Z})$ and $H^\bullet_{\mathrm{CW}}(\mathbb{R}P^3; \mathbb{Q}/\mathbb{Z})$ directly.

d) We typically expect the torsion of $H_\bullet$ to move up by 1 in $H^\bullet$. How come that failed in (c)?

Proof. a) Since $\mathbb{Z}$ is free, it is projective, and the functor $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, -)$ is exact. Then the right derived functors $\operatorname{Ext}^k_{\mathbb{Z}}(\mathbb{Z}, -) := \mathrm{R}^k \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, -) = 0$ for $k \geqslant 1$. In particular $\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}, G) = 0$.

The following exact sequence is a free resolution of $\mathbb{Z}/d$:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d} \mathbb{Z} \longrightarrow \mathbb{Z}/d \longrightarrow 0$$

Applying the functor $\operatorname{Hom}_{\mathbb{Z}}(-, G)$ to the unaugmented chain:

$$0 \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, G) \xrightarrow{d} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, G) \longrightarrow 0$$

As $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, G) \cong G$, after taking cohomology we obtain that

$$\operatorname{Ext}^k_{\mathbb{Z}}(\mathbb{Z}/d, G) = \begin{cases} \{g \in G : dg = 0\}, & k=0 \\ G/dG, & k=1 \\ 0, & \text{otherwise} \end{cases}$$

b) The universal coefficient theorem for cohomology:

$$0 \longrightarrow \operatorname{Ext}^1_{\mathbb{Z}}(H_{n-1}(\mathbb{R}P^3); \mathbb{Q}/\mathbb{Z}) \longrightarrow H^n(\mathbb{R}P^3; \mathbb{Q}/\mathbb{Z}) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(H_n(\mathbb{R}P^3), \mathbb{Q}/\mathbb{Z}) \longrightarrow 0$$

Note that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module (since it is divisible), and hence is an acyclic object with respect to the left exact functor $\operatorname{Hom}_{\mathbb{Z}}(H_{n-1}(\mathbb{R}P^3), -)$. The extension module $\operatorname{Ext}^1_{\mathbb{Z}}(H_{n-1}(\mathbb{R}P^3); \mathbb{Q}/\mathbb{Z}) = 0$. Hence we have

$$H^n(\mathbb{R}P^3; \mathbb{Q}/\mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(H_n(\mathbb{R}P^3), \mathbb{Q}/\mathbb{Z})$$

It remains to compute the homology groups $H_n(\mathbb{R}P^3)$. From the computation in Question 5 of Sheet 3, we know that the cellcular chain complex of $\mathbb{R}P^3$ is given by

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

Taking homology we have

$$H_n(\mathbb{R}P^3) = \begin{cases} \mathbb{Z}, & n = 0,3 \\ \mathbb{Z}/2, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

It is clear that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}$. To compute $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Q}/\mathbb{Z})$, we note that for $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Q}/\mathbb{Z})$, we must have $2\varphi(1) = 0 \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Q}/\mathbb{Z})$. Hence $\varphi(1) = 0$ or $1/2$. We deduce that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Z}/2$. In summary, the cohomology groups are given by

$$H^n(\mathbb{R}P^3; \mathbb{Q}/\mathbb{Z}) = \begin{cases} \mathbb{Q}/\mathbb{Z}, & n = 0,3 \\ \mathbb{Z}/2, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

c) The cellcular chain complex of $\mathbb{R}P^3$ with coefficients in $\mathbb{Q}/\mathbb{Z}$ is given by

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{0} \mathbb{Q}/\mathbb{Z} \xrightarrow{2} \mathbb{Q}/\mathbb{Z} \xrightarrow{0} \mathbb{Q}/\mathbb{Z}$$

Taking the homology we obtain

$$H_n^{\mathrm{CW}}(\mathbb{R}P^3; \mathbb{Q}/\mathbb{Z}) = \begin{cases} \mathbb{Q}/\mathbb{Z}, & n = 0,3 \\ \mathbb{Z}/2, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

Dualising the chain complex and taking the cohomology, we have

This shows why you can't use the theorem in the notes, but its proof is actually slightly stronger/needs adaptation anyway. Here $\mathbb{Q}/\mathbb{Z}$ is $\mathbb{Z}$-mod, not a ring.

$$H^n_{\mathrm{CW}}(\mathbb{R}P^3; \mathbb{Q}/\mathbb{Z}) = \begin{cases} \mathbb{Q}/\mathbb{Z}, & n = 0,3 \\ \mathbb{Z}/2, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

Proof doesn't work because $\mathbb{Q}/\mathbb{Z}$ has torsion

d) The torsion shift holds only if $H_n(\mathbb{R}P^3; \mathbb{Q}/\mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module for all $n \in \mathbb{N}$. But for $n = 0$, $H_0(\mathbb{R}P^3; \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$ is not finitely generated as a $\mathbb{Z}$-module.

Technically the torsion shift only cares about H_i & H_{i+1}, so a priori there still could be a shift from H_1 to H_2. There isn't because of

Let me prove this. Suppose that $\mathbb{Q}/\mathbb{Z}$ is finitely generated. Since it is divisible, we have $\langle 2 \rangle_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = \mathbb{Q}/\mathbb{Z}$. Then by Nakayama Lemma, there exists $n \in \mathbb{Z}$ odd, such that $n\mathbb{Q}/\mathbb{Z} = 0$, which is impossible. □

Question 5

Let X be the **Moore space $M(\mathbb{Z}/m, n)$** $= S^n \cup_\varphi \mathbb{D}^{n+1}$, where the attaching map $\varphi : \partial\mathbb{D}^{n+1} = S^n \to S^n$ has degree m.

a) Show that the quotient map $X \to X/S^n \cong S^{n+1}$ is zero on $\widetilde{H}_\bullet$ but non-zero on $\widetilde{H}^\bullet$.

b) Deduce that in the universal coefficient theorem the splitting cannot be natural.

Proof. a) The good pair (X, S^n) induces the long exact sequence of the reduced homology

$$\cdots \longrightarrow \widetilde{H}_k(S^n) \xrightarrow{i_k} \widetilde{H}_k(X) \xrightarrow{q_k} \widetilde{H}_k(S^{n+1}) \xrightarrow{\delta_k} \widetilde{H}_{k-1}(S^n) \longrightarrow \cdots$$

Since $\widetilde{H}_k(S^{n+1}) = 0$ for $k \neq n+1$, then $q_k = 0$ for $k \neq n+1$. But, for $k = n+1$, $\widetilde{H}_{n+1}(X) = 0$ (Question 2 and 8 of Sheet 3) and hence $q_{n+1} = 0$. This implies that the ~~push-outs~~ pushforwards of the quotient map $q : X \to S^{n+1}$ are zero on the homology groups.

Similarly, we have the long exact sequence of the ~~relative~~ reduced cohomology

$$\cdots \longrightarrow \widetilde{H}^{k-1}(S^n) \xrightarrow{\delta^{k-1}} \widetilde{H}^k(S^{n+1}) \xrightarrow{q^k} \widetilde{H}^k(X) \xrightarrow{i^k} \widetilde{H}^k(S^n) \longrightarrow \cdots$$

For $k \neq n+1$, $\widetilde{H}^k(S^{n+1}) = 0$ and hence $q^k = 0$. For $k = n+1$, from Question 8 of Sheet 3 we know that $\widetilde{H}^{n+1}(X) \cong \mathbb{Z}/m$.

Furthermore, since $\widetilde{H}^{n+1}(S^n) = 0$, then $i^k = 0$. By exactness at $\widetilde{H}^{n+1}(X)$, $\operatorname{im} q^k = \ker i^k = \widetilde{H}^{n+1}(X)$. In particular $q^k \neq 0$. This implies that the pull-back of the quotient map q is non-zero on $\widetilde{H}^{n+1}$.

$X \xrightarrow{q} S^{n+1}$

$H^*(S^{n+1})$

$\downarrow q^*$

$H^*(X)$

H^* is contravariant

b) The universal coefficient theorems for cohomology for X and S^{n+1} give split short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}^1_{\mathbb{Z}}(\widetilde{H}_n(X), \mathbb{Z}) \arrow[r] & \widetilde{H}^{n+1}(X) \arrow[r] \arrow[d, "q^{n+1}"] & \operatorname{Hom}_{\mathbb{Z}}(\widetilde{H}_{n+1}(X), \mathbb{Z}) \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Ext}^1_{\mathbb{Z}}(\widetilde{H}_n(S^{n+1}), \mathbb{Z}) \arrow[r] & \widetilde{H}^{n+1}(S^{n+1}) \arrow[r] & \operatorname{Hom}_{\mathbb{Z}}(\widetilde{H}_{n+1}(S^{n+1}), \mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

Suppose that the splitting is functorial. Since $\widetilde{H}_{n+1}(X) = 0$ and $\widetilde{H}_n(S^{n+1}) = 0$, we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}^1_{\mathbb{Z}}(\widetilde{H}_n(X), \mathbb{Z}) \arrow[r, "\cong"] \arrow[d, "\alpha"] & \widetilde{H}^{n+1}(X) \arrow[r] \arrow[d, "q^{n+1}"] & 0 \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \widetilde{H}^{n+1}(S^{n+1}) \arrow[r, "\cong"] & \operatorname{Hom}_{\mathbb{Z}}(\widetilde{H}_{n+1}(S^{n+1}), \mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

Then $\alpha = 0$ and $\beta = 0$. Since $q^{n+1} \neq 0$, it is impossible that the diagram is commutative. Hence the splitting in the universal coefficient theorems is not functorial. $\square$

The SESs are functorial. The splitting isn't!

Question 6

State and prove a locality theorem for cohomology when viewed as a ring.

(Hint. Naturality of the universal coefficient SES.)

Proof. Let $\mathscr{U} \subseteq \mathscr{P}(X)$ such that $X = \bigcup_{U \in \mathscr{U}} U^\circ$. Let $C_\bullet^{\mathscr{U}}(X)$ be the chain of $\mathscr{U}$-small simplicies. That is, $C_n^{\mathscr{U}}(X)$ is the free Abelian group generated by the n-simplicies σ for which $\sigma \subseteq U$ for some $U \in \mathscr{U}$. The **locality theorem for homology** states that

$$H_n(C_\bullet^{\mathscr{U}}(X)) \cong H_n(C_\bullet(X)) =: H_n(X)$$

Let $C^\bullet_{\mathscr{U}}(X)$ be the cochain complex of $C_\bullet^{\mathscr{U}}(X)$. We claim that the **locality theorem for cohomology** gives the following ring isomorphism

$$H^\bullet(C^\bullet_{\mathscr{U}}(X)) \cong H^\bullet(C^\bullet(X)) =: H^\bullet(X)$$

The universal coefficient theorem for cohomology for the cochain $C^\bullet_{\mathscr{U}}(X)$ gives the split short exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_{\mathbb{Z}}(H_{n-1}(C_\bullet^{\mathscr{U}}(X)), \mathbb{Z}) \longrightarrow H^n(C^\bullet_{\mathscr{U}}(X)) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(H_n(C_\bullet^{\mathscr{U}}(X)), \mathbb{Z}) \longrightarrow 0$$

Using the locality theorem $H_k(C_\bullet^{\mathscr{U}}(X)) \cong H_k(X)$ and 5-lemma, we have the group isomorphism $H^n(C^\bullet_{\mathscr{U}}(X)) \cong H^n(X)$ for each $n \in \mathbb{N}$.

needs commutativity, which follows from naturality of SES in the UCT

Next we consider the cup product structure on $H^\bullet(C^\bullet_{\mathscr{U}}(X))$. From the universal coefficient theorem, the inclusion of $C_\bullet^{\mathscr{U}}(X)$ into $C_\bullet(X)$ induces the isomorphism $\varphi : H^n(C^\bullet_{\mathscr{U}}(X)) \cong H^n(X)$. So by the functoriality of the cup product,

functoriality with respect to what?

$\varphi(\alpha \smile \beta) = \varphi(\alpha) \smile \varphi(\beta)$ for any $\alpha \in H^i(C^\bullet_{\mathcal{U}}(X))$ and $\beta \in H^j(C^\bullet_{\mathcal{U}}(X))$. Hence the group isomorphism $H^\bullet(C^\bullet_{\mathcal{U}}(X)) \cong H^\bullet(X)$ is indeed a ring isomorphism. □

Last part imprecise.

Question 7

Show that $S^2 \times S^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ have the same homology but have a different cup product on cohomology, where $\overline{\mathbb{C}P^2}$ is $\mathbb{C}P^2$ with opposite orientation.

(Hint. Compare quadratic forms associated to the symmetric bilinear form $H^2 \times H^2 \to H^4$.)

Explain why this argument does not work if we use $\mathbb{R}$-coefficients.

Proof. The homology groups for S^2 are given by

$$H_n(S^2) \cong \begin{cases} \mathbb{Z}, & n = 0, 2 \\ 0, & \text{otherwise} \end{cases}$$

By Künneth's Theorem, $H_n(S^2 \times S^2) \cong \bigoplus_{i+j=n} \big(H_i(S^2) \otimes H_j(S^2)\big)$. Hence the homology groups for $S^2 \times S^2$ are given by

$$H_n(S^2 \times S^2) \cong \begin{cases} \mathbb{Z}, & n = 0, 4 \\ \mathbb{Z}^2, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

Since $\mathbb{C}P^2$ is compact connected orientable 4-manifolds, By Question 3, we have

$$H_n(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}) \cong \begin{cases} \mathbb{Z}, & n = 0, 4 \\ H_n(\mathbb{C}P^2) \oplus H_n(\overline{\mathbb{C}P^2}), & 1 \leqslant n \leqslant 3 \\ 0, & \text{otherwise} \end{cases}$$

By Question 5 of Sheet 3, we know that

$$H_n(\mathbb{C}P^2) = \begin{cases} \mathbb{Z}, & n = 0, 2, 4 \\ 0, & \text{otherwise} \end{cases}$$

Hence

$$H_n(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}) \cong \begin{cases} \mathbb{Z}, & n = 0, 4 \\ \mathbb{Z}^2, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

The two spaces have the same homology groups.

Next we compute the **intersection forms** on these 4-manifolds. Since the homology groups above are all free, Poincaré duality gives non-degenerate symmetric bilinear forms

$$\begin{aligned} I \colon H^2(X) \times H^2(X) &\longrightarrow \mathbb{Z} \\ (\alpha, \beta) &\longmapsto \langle [X], \alpha \smile \beta \rangle \end{aligned}$$

For simplicity we write $X := S^2 \times S^2$ and $Y := \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.

For $X = S^2 \times S^2$, $H^2(S^2 \times S^2) \cong H_2(S^2 \times S^2)^\vee$ be universal coefficient theorem. We note that $H_2(S^2 \times S^2)$ is generated by $\alpha := [S^2] \otimes 1$ and $\beta := 1 \otimes [S^2]$. Then $H_2(S^2 \times S^2)^\vee = \mathbb{Z}\alpha^\vee \oplus \mathbb{Z}\beta^\vee$. We have $[X] = [S^2] \otimes [S^2] = \alpha \times \beta \in H_4(X)$. Then

$$[X] \frown \alpha^\vee = (\alpha \times \beta) \frown \alpha^\vee = \beta, \qquad [X] \frown \beta^\vee = \alpha$$

Hence the intersection form on X

$$I(\alpha^\vee, \alpha^\vee) = \langle [X] \frown \alpha^\vee, \alpha^\vee \rangle = \langle \beta, \alpha^\vee \rangle = 0, \qquad I(\beta^\vee, \beta^\vee) = \langle \alpha, \beta^\vee \rangle = 0, \qquad I(\alpha^\vee, \beta^\vee) = \langle \beta, \beta^\vee \rangle = 1$$

It has Gram matrix M_X with respect to the basis $\{\alpha^\vee, \beta^\vee\}$ of $H^2(X)$:

$$M_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

For $Y = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, from Question 3 we know that $H^2(Y) \cong H^2(\mathbb{C}P^2) \oplus H^2(\mathbb{C}P^2) =: \mathbb{Z}\mu \oplus \mathbb{Z}\nu$ and that $\mu \smile \nu = 0 \in H^4(Y)$. From the cup product structure on $\mathbb{C}P^2$, we know that $\mu \smile \mu = [\mathbb{C}P^2] = -[\overline{\mathbb{C}P^2}] = -\nu \smile \nu$. Hence the intersection form on Y has Gram matrix

$$M_Y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We note that M_X and M_Y are not congruent in $M_{2\times 2}(\mathbb{Z})$. We can verify this by brute computation. Suppose that $P^\top M_Y P = M_X$ for some $P \in M_{2\times 2}(\mathbb{Z})$. Then

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies a^2 + c^2 = 0 \text{ and } b^2 + d^2 = 0 \implies P = 0$$

which is impossible. In particular, the intersection forms on X and Y are distinct. Therefore $H^\bullet(S^2 \times S^2)$ and $H^\bullet(\mathbb{C}P^2 \# \overline{\mathbb{C}P^2})$ are not isomorphic as cohomology rings.

The argument fails for $\mathbb{R}$-coefficients, because M_X and M_Y are congruent in $M_{2\times 2}(\mathbb{R})$. By Sylvester's law of inertia, the congruent classes in $M_{2\times 2}(\mathbb{R})$ can be classified by the signature. Both M_X and M_Y have eigenvalues ± 1, and hence the signature $\sigma(X) = \sigma(Y) = 0$. □

Question 8

a) Let W be a compact oriented $(n+1)$-manifold with boundary $M = \partial W$. Prove that $\chi(M) = 2\chi(W)$ if n is even.

b) Can $\mathbb{R}P^2$ arise as the boundary of a compact 3-manifold?

Proof. a) The pair (W, M) induces the long exact sequence of relative homology

$$\cdots \longrightarrow H_k(M) \longrightarrow H_k(W) \longrightarrow H_k(W, M) \longrightarrow H_{k-1}(M) \longrightarrow \cdots$$

Since W is compact oriented with boundary M, by Poincaré-Lefschetz duality, $H_k(W, M) \cong H^{n+1-k}(W)$. We can take the alternating sum of the rank of the groups in the long exact sequence:

$$\sum_{k=0}^{n+1} (-1)^k \operatorname{rank} H_k(M) - \sum_{k=0}^{n+1} (-1)^k \operatorname{rank} H_k(W) + \sum_{k=0}^{n+1} (-1)^k \operatorname{rank} H^{n+1-k}(W) = 0$$

By universal coefficient theorem, $\operatorname{rank} H^{n+1-k}(W) = \operatorname{rank} H_{n+1-k}(W)$. Since n is even, we obtain that

$$\sum_{k=0}^{n+1} (-1)^k \operatorname{rank} H_k(M) - 2\sum_{k=0}^{n+1} (-1)^k \operatorname{rank} H_k(W) = 0$$

By definition of Euler characteristic, we have $\chi(M) = 2\chi(W)$ as required.

b) The homology groups of $\mathbb{R}P^2$ are given by

$$H_n(\mathbb{R}P^2) = \begin{cases} \mathbb{Z}, & n = 0 \\ \mathbb{Z}/2, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

Hence $\chi(\mathbb{R}P^2) = \sum_{n=0}^{\infty}(-1)^n \operatorname{rank} H_n(\mathbb{R}P^2) = 1$. If $\mathbb{R}P^2 = \partial X$, then by (a) $\chi(X) = 1/2$, which is impossible. Hence $\mathbb{R}P^2$ is not the boundary of a compact oriented 3-manifold. □

Question 9. Borsuk-Ulam Theorem

Prove that if $f : S^n \to S^n$ is an odd map ($f(-x) = -f(x)$) then $\deg f$ is odd. Deduce that if $g : S^n \to \mathbb{R}^n$ then there exists $x \in S^n$ with $g(x) = g(-x)$.

Hints: f induces a map $\overline{f} : \mathbb{R}P^n \to \mathbb{R}P^n$. Show that $\overline{f} : H_1(\mathbb{R}P^n) \to H_1(\mathbb{R}P^n)$ is an isomorphism (recall that $H_1(\mathbb{R}P^n) \cong \mathbb{Z}/2$ is generated by any path in S^n from a point x to $-x$), deduce that $\overline{f}^*$ is an isomorphism on $H^\bullet(\mathbb{R}P^n; \mathbb{Z}/2)$.

To show that $\deg f$ is odd, it suffices to show $H_n(S^n; \mathbb{Z}/2) \to H_n(S^n; \mathbb{Z}/2)$ sends $[S^n] \mapsto [S^n]$ (hint. universal coefficient theorem). Consider "transfer map" $C_\bullet(\mathbb{R}P^n; \mathbb{Z}/2) \to C_\bullet(S^n; \mathbb{Z}/2)$: simgular simplex $(\sigma : \Delta^n \to \mathbb{R}P^n) \mapsto \widetilde{\sigma} + a \circ \widetilde{\sigma} =$ (sum of possible "lifts" of σ to S^n). Show that it is functorial with respect to f and then consider the fundamental class $[\mathbb{R}P^n]$ over $\mathbb{Z}/2$.

Application: show that there are two antipodal points on the Earth's surface with the same temperature and barometric pressure.

Proof. • *An odd map $f : S^n \to S^n$ has odd degree.*

$f : S^n \to S^n$ induces the map $\overline{f} : \mathbb{R}P^n \to \mathbb{R}P^n$. Let $\sigma \in H_1(\mathbb{R}P^n)$ be a 1-simplex which is a path from x to $f(x)$. The push-out $\overline{f}_* : H_1(\mathbb{R}P^n) \to \mathbb{R}P^n$ sends a 1-simplex σ to $\overline{f}_*(\sigma)$, which is non-zero in $H_1(\mathbb{R}P^n)$, as it is a path from $f(x)$ to $-f(x)$. Hence $\overline{f}_*$ is an isomorphism on $H_1(\mathbb{R}P^n)$. *The same argument shows that $\overline{f}_*$ is an isomorphism on $H_1(\mathbb{R}P^n; \mathbb{Z}/2)$.*

Let $\tau : C_\bullet(\mathbb{R}P^n; \mathbb{Z}/2) \to C_\bullet(S^n; \mathbb{Z}/2)$ be the transfer map, which sends a simplex to the sum of its two distinct lifts in S^n. We have a short exact sequence of chain complexes:

$$0 \longrightarrow C_\bullet(\mathbb{R}P^n; \mathbb{Z}/2) \xrightarrow{\tau} C_\bullet(S^n; \mathbb{Z}/2) \xrightarrow{\pi} C_\bullet(\mathbb{R}P^n; \mathbb{Z}/2) \longrightarrow 0$$

This induces a long exact sequence of homology groups

$$\cdots \longrightarrow H_k(S^n; \mathbb{Z}/2) \xrightarrow{\pi_k} H_k(\mathbb{R}P^n; \mathbb{Z}/2) \xrightarrow{\delta_k} H_{k-1}(\mathbb{R}P^n; \mathbb{Z}/2) \xrightarrow{\tau_{k-1}} H_{k-1}(S^n; \mathbb{Z}/2) \longrightarrow \cdots$$

It is easy to check that f and $\overline{f}$ induce a morphism from the short exact sequence to itself:

$$\begin{tikzcd}
0 \arrow[r] & C_\bullet(\mathbb{R}P^n; \mathbb{Z}/2) \arrow[r, "\tau"] \arrow[d, "\overline{f}_\star"] & C_\bullet(S^n; \mathbb{Z}/2) \arrow[r, "\pi"] \arrow[d, "f_\star"] & C_\bullet(\mathbb{R}P^n; \mathbb{Z}/2) \arrow[r] \arrow[d, "\overline{f}_\star"] & 0 \\
0 \arrow[r] & C_\bullet(\mathbb{R}P^n; \mathbb{Z}/2) \arrow[r, "\tau"] & C_\bullet(S^n; \mathbb{Z}/2) \arrow[r, "\pi"] & C_\bullet(\mathbb{R}P^n; \mathbb{Z}/2) \arrow[r] & 0
\end{tikzcd}$$

By functoriality of the long exact sequence, we have

$$\begin{tikzcd}
\cdots \arrow[r] & H_k(S^n; \mathbb{Z}/2) \arrow[r, "\pi_k"] \arrow[d, "f_*"] & H_k(\mathbb{R}P^n; \mathbb{Z}/2) \arrow[r, "\delta_k"] \arrow[d, "\overline{f}_*"] & H_{k-1}(\mathbb{R}P^n; \mathbb{Z}/2) \arrow[r, "\tau_{k-1}"] \arrow[d, "\overline{f}_*"] & H_{k-1}(S^n; \mathbb{Z}/2) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_k(S^n; \mathbb{Z}/2) \arrow[r, "\pi_k"] & H_k(\mathbb{R}P^n; \mathbb{Z}/2) \arrow[r, "\delta_k"] & H_{k-1}(\mathbb{R}P^n; \mathbb{Z}/2) \arrow[r, "\tau_{k-1}"] & H_{k-1}(S^n; \mathbb{Z}/2) \arrow[r] & \cdots
\end{tikzcd}$$

For $1 \leqslant k \leqslant n-1$, we have $H_k(S^n;\mathbb{Z}/2) = 0$. Then δ_k is an isomorphism. In each commutative square:

$$\begin{tikzcd}
H_k(\mathbb{R}P^n;\mathbb{Z}/2) \arrow[r, "\cong"] \arrow[d, "\overline{f}_*"'] & H_{k-1}(\mathbb{R}P^n;\mathbb{Z}/2) \arrow[d, "\overline{f}_*"] \\
H_k(\mathbb{R}P^n;\mathbb{Z}/2) \arrow[r, "\cong"'] & H_{k-1}(\mathbb{R}P^n;\mathbb{Z}/2)
\end{tikzcd}$$

We can use induction to prove that $\overline{f}_* : H_k(\mathbb{R}P^n;\mathbb{Z}/2) \to H_k(\mathbb{R}P^n;\mathbb{Z}/2)$ is an isomorphism for $1 \leqslant k \leqslant n-1$.

For $k = n$, we have

$$\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{R}P^n;\mathbb{Z}/2) \arrow[r, "\tau_n"] \arrow[d, "\overline{f}_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "\pi_n"] \arrow[d, "f_*"] & H_n(\mathbb{R}P^n;\mathbb{Z}/2) \arrow[r, "\delta_n"] \arrow[d, "\overline{f}_*"] & H_{n-1}(\mathbb{R}P^n;\mathbb{Z}/2) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & H_n(\mathbb{R}P^n;\mathbb{Z}/2) \arrow[r, "\tau_n"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "\pi_n"] & H_n(\mathbb{R}P^n;\mathbb{Z}/2) \arrow[r, "\delta_n"] & H_{n-1}(\mathbb{R}P^n;\mathbb{Z}/2) \arrow[r] & 0
\end{tikzcd}$$

We note that $\pi : S^n \to \mathbb{R}P^2$ is a 2-fold covering map. Hence the induced map $\pi_n : H_n(S^n;\mathbb{Z}/2) \to H_n(\mathbb{R}P^2;\mathbb{Z}/2)$ is zero. We can split the diagram above into two commutative squares:

$$\begin{tikzcd}
H_n(\mathbb{R}P^n;\mathbb{Z}/2) \arrow[r, "\cong"] \arrow[d, "\overline{f}_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[d, "f_*"] & H_n(\mathbb{R}P^n;\mathbb{Z}/2) \arrow[r, "\cong"] \arrow[d, "\overline{f}_*"] & H_{n-1}(\mathbb{R}P^n;\mathbb{Z}/2) \arrow[d, "\cong"] \\
H_n(\mathbb{R}P^n;\mathbb{Z}/2) \arrow[r, "\cong"] & H_n(S^n;\mathbb{Z}/2) & H_n(\mathbb{R}P^n;\mathbb{Z}/2) \arrow[r, "\cong"] & H_{n-1}(\mathbb{R}P^n;\mathbb{Z}/2)
\end{tikzcd}$$

Then $\overline{f}_* : H_n(\mathbb{R}P^n;\mathbb{Z}/2) \to H_n(\mathbb{R}P^n;\mathbb{Z}/2)$ and $f_* : H_n(S^n;\mathbb{Z}/2) \to H_n(S^n;\mathbb{Z}/2)$ are isomorphisms.

Finally, by universal coefficient theorem for homology, we have the short exact sequence

$$0 \longrightarrow H_n(S^n) \otimes_{\mathbb{Z}} \mathbb{Z}/2 \longrightarrow H_n(S^n;\mathbb{Z}/2) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(S^n);\mathbb{Z}/2) \longrightarrow 0$$

It is clear that $\mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(S^n) = 0$. Then we have the natural isomorphism $H_n(S^n;\mathbb{Z}/2) \cong H_n(S^n) \otimes_{\mathbb{Z}} \mathbb{Z}/2$. By functoriality, we have a commutative diagram

$$\begin{tikzcd}
H_n(S^n) \otimes_{\mathbb{Z}} \mathbb{Z}/2 \arrow[r] \arrow[d, "\deg f \otimes \mathrm{id}"'] & H_n(S^n;\mathbb{Z}/2) \arrow[d, "f_*"] \\
H_n(S^n) \otimes_{\mathbb{Z}} \mathbb{Z}/2 \arrow[r] & H_n(S^n;\mathbb{Z}/2)
\end{tikzcd}$$

The map $H_n(S^n) \otimes_{\mathbb{Z}} \mathbb{Z}/2 \to H_n(S^n) \otimes_{\mathbb{Z}} \mathbb{Z}/2$ given by multiplication by $\deg f$ is non-zero. Hence we conclude that $\deg f$ is odd. good!

- *Proof of Borsuk-Ulam Theorem.*

  Let $f(x) = g(x) - g(-x)$. Suppose that for all $x \in S^n$, $f(x) \neq 0$. Then $h(x) := f(x)/\|f(x)\|$ is a odd map from S^n to $S^{n-1} \subseteq S^n$. The restriction $h|_{S^{n-1}} : S^{n-1} \to S^{n-1}$ has odd degree by the previous result. But $h|_{S^{n-1}}$ is null-homotopic. This is a contradiction. Hence there exists $x \in S^n$ such that $g(x) = g(-x)$.

- *There are two antipodal points on the Earth's surface with the same temperature and barometric pressure.*

  Let $(p, T) : S^2 \to \mathbb{R}^2$ represents the temperature and pressure (as scalar fields) on the Earth's surface. By Borsuk-Ulam Theorem, there exists $x \in S^2$ such that $(p(x), T(x)) = (p(-x), T(-x))$. So x and $-x$ are a pair of antipodal points on the Earth's surface that have the same T and p. $\square$

Question 10

A **good cover** of a manifold is an open cover $\{U_i\}$ such that $U_i \cong \mathbb{R}^n$ and $U_{i_1} \cap \cdots \cap U_{i_k} \cong \mathbb{R}^n$ or $\varnothing$ for all $i_1, \ldots, i_k, k$.

Fact/Example: Smooth manifolds always admit a good cover.

Prove that any manifold M which admits a finite good cover has finitely generated homology groups.

Proof. We use induction on k.

- Base case: Suppose that $M \cong \mathbb{R}^n$. Then M is contractible, with zero homology groups. ✓
- Induction case: Suppose that for any manifold M that admits a good cover of cardinality at most $k-1$, $H_m(M)$ is finitely generated for each $m \in \mathbb{N}$.

  Now suppose that M admits a good cover $\{U_1, \dots, U_k\}$. Let $N := U_1 \cup \cdots \cup U_{k-1}$. By induction hypothesis, N and $N \cap U_k = (U_1 \cap U_k) \cup \cdots \cup (U_{k-1} \cap U_k)$ have finitely generated homology groups. The Mayor-Vietoris sequence for homology is given by ✓

  $$\cdots \longrightarrow H_m(N \cap U_k) \longrightarrow H_m(N) \oplus H_m(U_k) \longrightarrow H_m(M) \longrightarrow H_{m-1}(N \cap U_k) \longrightarrow \cdots$$

  U_m is contractible and has zero homology groups for $m > 0$. Then by the following lemma we know that $H_m(M)$ is finitely generated. This completes the induction.

Lemma 1

Let R be a principal ideal domain. Suppose that the sequence of R-modules $A \xrightarrow{f} B \xrightarrow{g} C$ is exact at B. If A and C are finitely generated, then so is B.

Proof. Let $\{a_1, \dots, a_n\} \subseteq A$ generates A. Then $\{f(a_1), \dots, f(a_n)\}$ generates $\operatorname{im} f$. By exactness and the first isomorphism theorem, we have

$$\frac{B}{\langle f(a_1), \dots, f(a_n)\rangle} \cong \operatorname{im} g$$

Since C is finitely generated and $\operatorname{im} g$ is a submodule of C, $\operatorname{im} g$ is also finitely generated (Question 6 of Sheet 3 of C2.2 Homological Algebra). Suppose that $\operatorname{im} g$ ir generated by $\{c_1, \dots, c_m\} \subseteq C$. Then B is generated by $\{f(a_1), \dots, f(a_n), b_1, \dots, b_m\}$, where $b_i \in h^{-1}(c_i)$ and h is the composite map $B \longrightarrow B/\operatorname{im} f \longrightarrow \operatorname{im} g$. $\square$

We conclude that every manifold that admits a finite good cover has finitely generated homology groups. $\square$

perfect!