A Nice solutions.

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Problem Sheet 4

B2.1: Introduction to Representation Theory

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Some general remarks about induced modules:

We consider the tensor product of modules over non-commutative rings:

Definition. Tensor Product.

Suppose that R, S are rings, M is a (R,S)-bimodule, and N is a left S-module. Then $M \otimes_S N$ is a left R-module satisfying the universal property:

For any Abelian group P and balanced map $\varphi : M \times N \to P$, there exists a unique group homomorphism $\widetilde{\varphi} : M \otimes_S N \to P$ such that the following diagram commutes:

$$\begin{tikzcd} M \times N \arrow[r, "\varphi"] \arrow[d, "\sigma"] & P \\ M \otimes_S N \arrow[ur, dashed, "\exists!\, \widetilde{\varphi}"'] & \end{tikzcd}$$

Suppose that k is a field, G is a group, $H \leqslant G$, and W is a $k[H]$-module. It is not hard to verify that the induced module satisfies the universal property, so that we have:

$$\operatorname{Ind}_H^G W = k[G] \otimes_{k[H]} W$$

I believe that this is the standard way to define induced modules in most textbooks. The tensor product is useful because we know the identities:

$$\left(\bigoplus_{i=1}^{n} M_i\right) \otimes_S N \cong \bigoplus_{i=1}^{n} (M_i \otimes_S N) \qquad M \otimes_S S \cong S \otimes_S M \cong M \qquad (M \otimes_R N) \otimes_S P \cong M \otimes_R (N \otimes_S P)$$

which makes the transitivity of induced modules a trivial fact:

$$\operatorname{Ind}_H^G \operatorname{Ind}_J^H W = k[G] \otimes_{k[H]} \big(k[H] \otimes_{k[J]} W\big) \cong \big(k[G] \otimes_{k[H]} k[H]\big) \otimes_{k[J]} W \cong k[G] \otimes_{k[J]} W = \operatorname{Ind}_J^G W$$

Question 1 A+

Find the character table of the alternating group A_5. It may be helpful to remember that A_5 acts as a group of rotations of the regular icosahedron.

Proof. Recall from Part A Group Theory (Lemma 44) that the conjugacy classes of A_5 are:

- the identity;
- all 20 3-cycles;
- all 15 double transpositions;
- 12 of the 5-cycles;
- the remaining 12 of the 5-cycles, these being the squares of those in the previous class.

We choose the representatives $e, (123), (12)(34), (12345), (13245)$. So A_5 have 5 non-isomorphic irreducible representations. The first one would be the trivial representation $\mathbf{1}$ on $\mathbb{C}$.

Since A_5 is the symmetry group of the regular icosahedron, we have a representation $\rho : A_5 \to \mathrm{GL}_3(\mathbb{R}) \leqslant \mathrm{GL}_3(\mathbb{C})$. Using the method established in Question 4 of Sheet 3, for $g \in A_5$ with $\mathrm{o}(g) = n$, we have $\chi_\rho(g) = 1 + 2\operatorname{Re}\zeta_n$, where ζ_n is a primitive n-th root of unity. Then we have:

$$\begin{aligned} \chi_\rho(e) &= 1 + 2 = 3 \\ \chi_\rho((123)) &= 1 + 2\operatorname{Re}\zeta_3 = 1 + (-1) = 0 \\ \chi_\rho((12)(34)) &= 1 + 2\operatorname{Re}\zeta_2 = 1 + (-2) = -1 \\ \chi_\rho((12345)) &= 1 + 2\operatorname{Re}\zeta_5 = \frac{1+\sqrt{5}}{2} \text{ or } \frac{1-\sqrt{5}}{2} \\ \chi_\rho((13245)) &= 1 + 2\operatorname{Re}\zeta_5 = \frac{1+\sqrt{5}}{2} \text{ or } \frac{1-\sqrt{5}}{2} \end{aligned}$$

In fact there are two distinct 3-dimensional irreducible representations, as we compute the inner product and invoke the row orthogonality relation.

Now we have 1 1-dimensional and 2 3-dimensional irreducible representations. By Artin-Weddernburn theorem, the remaining two irreducible representations satisfy

$$(\deg\rho_4)^2 + (\deg\rho_5)^2 = |A_5| - 1^2 - 2\times 3^2 = 41 \implies \deg\rho_4 = 4,\ \deg\rho_5 = 5$$

We have the following incomplete character table:

A_5	e	(123)	$(12)(34)$	(12345)	(13245)
$\lvert g^G\rvert$	1	20	15	12	12
χ_1	1	1	1	1	1
χ_2	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_3	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
χ_4	4	a_{42}	a_{43}	a_{44}	a_{45}
χ_5	5	a_{52}	a_{53}	a_{54}	a_{55}

The table can be completed by using column orthogonality relation successively:

$$\begin{aligned}
1+4a_{42}+5a_{52}=0,\quad 1+|a_{42}|^2+|a_{52}|^2=3 &\implies a_{42}=1,\ a_{52}=-1\\
1+a_{43}-a_{53}=0,\quad 1-3-3+4a_{43}+5a_{53}=0 &\implies a_{43}=0,\ a_{53}=1\\
1+a_{44}-a_{54}=0,\quad 1+3\times\frac{1+\sqrt{5}}{5}+3\times\frac{1-\sqrt{5}}{2}+4a_{44}+5a_{54}=0 &\implies a_{44}=-1,\ a_{54}=0\\
1+a_{45}-a_{55}=0,\quad 1+3\times\frac{1-\sqrt{5}}{5}+3\times\frac{1+\sqrt{5}}{2}+4a_{45}+5a_{55}=0 &\implies a_{45}=-1,\ a_{55}=0
\end{aligned}$$

The complete character table is shown as follows:

A_5	e	(123)	$(12)(34)$	(12345)	(13245)
$\lvert g^G\rvert$	1	20	15	12	12
χ_1	1	1	1	1	1
χ_2	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_3	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
χ_4	4	1	0	-1	-1
χ_5	5	-1	1	0	0

Good. To save some calculations, look @ Sheet 3, Q6, to construct a 4dim$^{\underline{n}}$ for free ($\chi(g) = |\mathrm{Fix}(g)| - 1$)

□

A

Question 2

Let G be a finite group with an irreducible representation $\rho : G \to \mathrm{GL}_2(\mathbb{C})$.

(a) Prove that G has an element a of order 2.

(b) For a as above show that either $\det\rho(a) \neq 1$ or else $\rho(a)$ is central in $\mathrm{GL}_2(\mathbb{C})$.

(c) Deduce that a finite simple group cannot have an irreducible representation of degree 2.

Maybe $\rho(a) = I_2$? Not true

Proof. (a) By Frobenius divisibility, we know that $\dim \mathrm{GL}_2(\mathbb{C}) = 2$ divides $|G|$. By Cauchy's Theorem, G has elements of order 2.

(b) Suppose that $\det\rho(a) = 1$. Since $a \in G$ is of order 2, $\rho(a)$ is of order 2 in $\mathrm{GL}_2(\mathbb{C})$. Hence the eigenvalues of $\rho(a)$ are in $\{1, -1\}$. But this implies that the only eigenvalue of $\rho(a)$ is -1, and since $\rho(a)$ is diagonalisable, $\rho(a) = -\mathrm{id}$. Hence $\rho(a)$ is central in $\mathrm{GL}_2(\mathbb{C})$.

(or, $\rho(a) = \mathrm{id}$)

(c) Suppose that G is simple and has an irreducible representation of degree 2. Then $\ker\rho \lhd G$ is either $\{e\}$ or G. In the latter case ρ is not irreducible. Therefore ρ is faithful. For $g \in G$, we have

$$\rho([a,g]) = [\rho(a), \rho(g)] = \mathrm{id} \implies [a,g] = e$$

good.

Hence $a \in Z(G)$ and, by simplicity of G, $Z(G) = G$. Hence G is Abelian. But this contradicts the result in Question 3.(a) of Sheet 3, which says that every complex irreducible representation of an Abelian group is 1-dimensional. □

Great.

Question 3 BT

Let G be a finite group and suppose that V is a simple $\mathbb{C}G$-module.

(a) Prove that $e_V = \dfrac{\dim V}{|G|} \sum_{g\in G} \overline{\chi_V(g)} g$ is an element of the centre of $\mathbb{C}G$.

(b) Let V' be another simple $\mathbb{C}G$-module. Prove that e_V kills V' if V' is not isomorphic to V, and that e_V acts as the identity on V.

(c) Let $V_1, \cdots, V_r$ be the simple $\mathbb{C}G$-modules (up to isomorphism) and let $e_i := e_{V_i}$ for $i = 1, \cdots, r$. Prove that $e_i e_j = \delta_{i,j} e_i$ for all $i, j = 1, \cdots, r$, and that $e_1 + \cdots + e_r = 1$.

Proof. (a) For $h \in G$, using the fact that χ_V is a class function, we have

$$he_V = \frac{\dim V}{|G|}\sum_{g\in G}\overline{\chi_V(g)}hg = \frac{\dim V}{|G|}\sum_{hgh^{-1}\in G}\overline{\chi_V(g)}hgh^{-1}\cdot h = \frac{\dim V}{|G|}\sum_{g\in G}\overline{\chi_V(h^{-1}gh)}gh = \frac{\dim V}{|G|}\sum_{g\in G}\overline{\chi_V(g)}gh = e_V h$$

(actually, I think it never is)

By extending linearly in $\mathbb{C}[G]$, we deduce that $e_V \in Z(\mathbb{C}[G])$.

(b) For simplicity we use the notation in part (c). By row orthogonality relation, we have

This question is only valid for $g \in G$.

e_i might not be a group element!

$$\chi_j(e_i) = \frac{\dim V_i}{|G|}\sum_{g\in G}\overline{\chi_i(g)}\chi_j(g) = \dim V_i \delta_{ij}$$

For $i = j$, $\chi_i(e_i) = \dim V_i = \chi_i(e_G)$. In Question 5.(a) of Sheet 3 we have proven that $\chi_i(e_i) = \chi_i(e_G)$ implies that $e_i \in \ker\rho_i$, where $\rho_i : G \to \mathrm{GL}(V_i)$ is the representation afforded by V_i. Hence $\rho_i(e_i) = \mathrm{id}_{V_i}$. ✗

For $i \neq j$, $\chi_j(e_i) = 0$. Note that by Schur's Lemma e_i acting on V_j is a scalar. So we have in fact $e_i \cdot v = 0$ for any $v \in V_j$.

Instead, for (b), use that e_i acts as a scalar in all $\tilde{\rho}_i : \mathbb{C}G \to GL(V_i)$

(c) Consider the Artin-Weddernburn decomposition of $\mathbb{C}[G]$ into simple submodules:

$$\mathbb{C}[G] = \bigoplus_{i=1}^{r} V_i^{\dim V_i}, \qquad \forall\, v \in \mathbb{C}[G]: \ v = \sum_{i=1}^{r} v_i, \ \ v_i \in V_i^{\dim V_i}$$

Then by (b) $e_i \cdot v = v_i$ for any $v \in \mathbb{C}[G]$. In particular, $e_i \cdot 1 = e_i \in V_i$. Hence

$$1 = \sum_{i=1}^{r} e_i, \ e_i \in V_i$$

and $e_i e_j = \delta_{ij} e_i$. $\{e_1, ..., e_r\}$ is a set of (primitive) central idempotent. □

Question 4 A

A conjugacy class g^G of a finite group G is called real if g is conjugate to g^{-1}. A character χ of G is called real if $\chi(g) \in \mathbb{R}$ for all $g \in G$. By considering the vector space

$$V := \{f : G \to \mathbb{C} : f(g) = f\left(h^{-1}gh\right) = f\left(g^{-1}\right) \quad \text{for all} \quad g, h \in G\}$$

or otherwise, prove that the number of real conjugacy classes in G is equal to the number of irreducible real characters.

Proof. First we claim that χ is an irreducible character of G if and only if $\overline{\chi} : g \mapsto \overline{\chi(g)}$ is an irreducible character of G.

Let V be a $\mathbb{C}[G]$-module. Then the dual space V' is naturally a $\mathbb{C}[G]$-module which affords the dual representation of V. By Proposition 5.21 we know that $\chi_{V'} = \overline{\chi_V}$. Suppose that V is not simple. Let U be a non-trivial sub-$\mathbb{C}[G]$-module of V. By Maschke's Theorem we have $V = U \oplus W$ for some sub-$\mathbb{C}[G]$-module $W \leqslant V$. Then $V' \cong U' \oplus W'$ is reducible. For the other direction, we use the canonical isomorphism $V \cong V''$ and repeat the same proof. This proves the claim.

(or quicker $\langle \chi, \chi \rangle = \overline{\langle \chi, \chi \rangle} = \langle \overline{\chi}, \overline{\chi} \rangle$) :)

$(\in \mathbb{N})$

Since $\chi = \overline{\chi}$ if and only if χ is real, by row orthogonality relation, we have

$$\langle \chi, \overline{\chi} \rangle = \begin{cases} 1, & \chi \text{ is real} \\ 0, & \chi \text{ is not real} \end{cases}$$

Let $\mathrm{Irr}(G)$ be the set of irreducible characters of G. Then the number of irreducible real characters is

$$n = \sum_{\chi \in \mathrm{Irr}(G)} \langle \chi, \overline{\chi} \rangle = \frac{1}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \sum_{g \in G} \overline{\chi(g)\chi(g)} = \frac{1}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \sum_{g \in G} \overline{\chi(g)}\chi(g^{-1})$$

By column orthogonality relation, we have

$$\sum_{\chi \in \mathrm{Irr}(G)} \overline{\chi(g)}\chi(g^{-1}) = \begin{cases} |C_G(g)|, & g^{-1} \in g^G \\ 0, & g^{-1} \notin g^G \end{cases}$$

nice

Let $g_1, \ldots, g_r$ be the representatives of the conjugacy classes of G. Then the number of real conjugacy classes is

$$m = \sum_{i=1}^{r} \frac{1}{|C_G(g_i)|} \sum_{\chi \in \mathrm{Irr}(G)} \overline{\chi(g_i)}\chi(g_i^{-1}) = \sum_{i=1}^{r} \frac{|g_i^G|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \overline{\chi(g_i)}\chi(g_i^{-1}) = \frac{1}{|G|} \sum_{g \in G} \sum_{\chi \in \mathrm{Irr}(G)} \overline{\chi(g)}\chi(g^{-1})$$

It is clear that $m = n$. The number of irreducible real characters equals to the number of real conjugacy classes. □

Question 5 A+

Prove that every finite group has a faithful representation. Which finite abelian groups have a faithful irreducible representation?

Proof. Suppose that G is a finite group. The regular representation afford by the regular $k[G]$-module $k[G]$ is faithful.

Suppose that G is Abelian. We claim that G has a faithful irreducible complex representation if and only if G is cyclic.

" $\Longrightarrow$ ": Suppose that G has a faithful irreducible representation $\rho : G \to \mathrm{GL}(V)$. Since G is Abelian, by Schur's Lemma any $\rho(g)$ acting on V is a scalar $g_V \in \mathbb{C}$. Moreover, since $o(g) < \infty$, g_V is a root of unity. Hence we have a group monomorphism $\rho : G \to \mathbb{C}^\times$. Hence G is isomorphic to a subgroup of $\mathbb{C}^\times$ of finite order, which is cyclic.

" $\Longleftarrow$ ": Suppose that $G = \langle g \rangle$ is cyclic. Consider the set of $|G|$-th roots of unity in $\mathbb{C}$, $H := \{\omega \in \mathbb{C}^\times : \omega^{|G|} = 1\}$. It is clear that $G \cong H$ and $H \leqslant \mathrm{GL}(\mathbb{C})$. This gives a faithful irreducible representation $\rho : G \to \mathrm{GL}(V)$. □

Question 6

Let H be a cyclic subgroup of $G := S_4$ and let $\varphi : H \to \mathbb{C}^\times$ be a faithful linear character. Write $\mathrm{Ind}_H^G \varphi$ as a sum of irreducible characters of G when (a) $H = \langle (1234) \rangle$, and (b) $H = \langle (123) \rangle$.

Proof. A+ First we write down the character table of $G = S_4$ from Example 5.24:

	e	(12)	(12)(34)	(123)	(1234)
$\lvert g^G \rvert$	1	3	8	6	6
$\lvert C_G(g) \rvert$	24	8	3	4	4
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	2	-1	0
χ_4	3	1	-1	0	-1
χ_5	3	-1	-1	0	1

By Frobenius reciprocity, we have

$$\mathrm{Ind}_H^G \varphi = \sum_{i=1}^{5} \chi_i \langle \chi_i, \mathrm{Ind}_H^G \varphi \rangle_G = \sum_{i=1}^{5} \chi_i \langle \mathrm{Res}_H^G \chi_i, \varphi \rangle_H$$

(a) For $H = \langle(1234)\rangle = \{e, (1234), (13)(24), (1432)\}$, we can write down the following table (two elements in S_4 are conjugate if and only if they have the same cycle type):

H	e	(1234)	(13)(24)	(1432)
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	2	0	2	0
χ_4	3	-1	-1	-1
χ_5	3	1	-1	1
φ	1	i	-1	$-$i

Hence

$$\langle \mathrm{Res}_H^G \varphi, \chi_1\rangle_H = 0, \quad \langle \mathrm{Res}_H^G \varphi, \chi_2\rangle_H = 0, \quad \langle \mathrm{Res}_H^G \varphi, \chi_3\rangle_H = 0, \quad \langle \mathrm{Res}_H^G \varphi, \chi_4\rangle_H = 1, \quad \langle \mathrm{Res}_H^G \varphi, \chi_5\rangle_H = 1,$$

We conclude that $\mathrm{Ind}_H^G \varphi = \chi_4 + \chi_5$.

(b) For $H = \langle(123)\rangle = \{e, (123), (132)\}$, we can write down the following table:

H	e	(123)	(132)
χ_1	1	1	1
χ_2	1	1	1
χ_3	2	-1	-1
χ_4	3	0	0
χ_5	3	0	0
φ	1	ω	ω^2

where $\omega = \dfrac{1+\sqrt{3}\mathrm{i}}{2}$ is a primitive third root of unity. Hence

$$\langle \mathrm{Res}_H^G \varphi, \chi_1\rangle_H = 0, \quad \langle \mathrm{Res}_H^G \varphi, \chi_2\rangle_H = 0, \quad \langle \mathrm{Res}_H^G \varphi, \chi_3\rangle_H = 1, \quad \langle \mathrm{Res}_H^G \varphi, \chi_4\rangle_H = 1, \quad \langle \mathrm{Res}_H^G \varphi, \chi_5\rangle_H = 1,$$

We conclude that $\mathrm{Ind}_H^G \varphi = \chi_3 + \chi_4 + \chi_5$. □

Question 7

(a) Let V be a simple $\mathbb{C}G$-module and let W be a simple $\mathbb{C}H$-module. Construct a linear $G \times H$-action on $V \otimes W$ and prove that the resulting $\mathbb{C}(G \times H)$-module is simple.

(b) Let V be a simple $\mathbb{C}G$ -module and let Z be the centre of G. Show that for each $m \geqslant 1$, the subgroup $D_m := \{(z_1, \cdots, z_m) \in Z^m : z_1 \cdots z_m = 1\}$ of Z^m acts trivially on $V^{\otimes m}$.

(c) By considering large values of m, deduce that $\dim V$ divides $|G/Z|$.

A

Proof. (a) For $(g, h) \in G \times H$, we define

$$(g, h) \cdot (v \otimes_{\mathbb{C}} w) := (g \cdot v) \otimes_{\mathbb{C}} (h \cdot w)$$

for $v \in V$ and $w \in W$, and then extend linearly on $\mathbb{C}[G \times H]$ and on $V \otimes_{\mathbb{C}} W$. It is easy to verify that this defines a left $\mathbb{C}[G \times H]$-module structure on $V \otimes_{\mathbb{C}} W$:

- For $(g_1, h_1), (g_2, h_2) \in G \times H$ and $v \otimes_{\mathbb{C}} w \in V \otimes_{\mathbb{C}} W$:

$$(g_1, h_1) \cdot \big((g_2, h_2) \cdot (v \otimes_{\mathbb{C}} w)\big) = (g_1 g_2 \cdot v) \otimes_{\mathbb{C}} (h_1 h_2 \cdot w) = (g_1 g_2, h_1 h_2) \cdot (v \otimes_{\mathbb{C}} w) = \big((g_1, h_1) \cdot (g_2, h_2)\big) \cdot (v \otimes_{\mathbb{C}} w)$$

  By extending linearly on $\mathbb{C}[G \times H]$ and on $V \otimes_{\mathbb{C}} W$ we can prove associativity.

- The distributivity is satisfied automatically when we make the definition and extend them linearly.

- For $v \otimes_{\mathbb{C}} w \in V \otimes_{\mathbb{C}} W$,

$$(e_G, e_H) \cdot (v \otimes_{\mathbb{C}} w) = v \otimes_{\mathbb{C}} w$$

  By extending linearly on $V \otimes_{\mathbb{C}} W$ we can prove the identity action.

Follows from homomorphism property!

Let χ_V be the character afforded by V and χ_W be the character afforded by W. By the same proof of Proposition 5.21.(c), we have $\chi_{V\otimes_{\mathbb{C}}W} = \chi_V\chi_W$. Since χ_V and χ_W are irreducible characters, we have

$$\begin{aligned}\langle \chi_{V\otimes_{\mathbb{C}}W}, \chi_{V\otimes_{\mathbb{C}}W}\rangle &= \frac{1}{|G||H|}\sum_{g\in G}\sum_{h\in H}\overline{\chi_V(g)\chi_W(h)}\chi_V(g)\chi_W(h) = \left(\frac{1}{|G|}\sum_{g\in G}\overline{\chi_V(g)}\chi_V(g)\right)\left(\frac{1}{|H|}\sum_{h\in H}\overline{\chi_W(h)}\chi_W(h)\right)\\ &= \langle\chi_V,\chi_V\rangle\langle\chi_W,\chi_W\rangle = 1\end{aligned}$$

Hence $\chi_{V\otimes_{\mathbb{C}}W}$ is an irreducible character of $G\times H$, and $V\otimes_{\mathbb{C}} W$ is a simple $\mathbb{C}[G\times H]$-module.

(b) By Schur's Lemma, the action of $Z(G)$ on V is scalar multiplication. We denote this by the central character $\varphi : Z(G)\to\mathbb{C}$. For $(z_1,\dots,z_m)\in Z^m$ and $v_1\otimes_{\mathbb{C}}\cdots\otimes_{\mathbb{C}} v_m \in V^{\otimes m}$,

$$\begin{aligned}(z_1,\dots,z_m)\cdot(v_1\otimes_{\mathbb{C}}\cdots\otimes_{\mathbb{C}} v_m) &= z_1\cdot v_1\otimes_{\mathbb{C}}\cdots\otimes_{\mathbb{C}} z_m\cdot v_m = \varphi(z_1)v_1\otimes_{\mathbb{C}}\cdots\otimes_{\mathbb{C}}\varphi(z_m)v_m\\ &= \varphi(z_1\cdots z_m)v_1\otimes_{\mathbb{C}}\cdots\otimes_{\mathbb{C}} v_m = v_1\otimes_{\mathbb{C}}\cdots\otimes_{\mathbb{C}} v_m\end{aligned}$$

By extending linearly on $V^{\otimes m}$ we deduce that Z^m acts trivially on $V^{\otimes m}$.

(c) Since Z^m acts trivially on $V^{\otimes m}$, the $\mathbb{C}[G^m]$-module structure on $V^{\otimes m}$ descends to a $\mathbb{C}[G^m/D_m]$-module via

$$\forall\, g\in G^m\ \forall\, v\in V^{\otimes m}\quad gD_m\cdot v := g\cdot v$$

By Frobenius divisibility, $\dim V^{\otimes m}$ divides $|G^m/D_m|$.

It is clear from the definition of D_m that $|D_m| = |Z|^{m-1}$, because the value of z_m is fixed by $z_1,\dots,z_{m-1}\in Z$, which are arbitrary. On the other hand, $\dim V^{\otimes m} = (\dim V)^m$. Hence $(\dim V)^m$ divides $[G:Z]^m|Z|$. Let $\alpha := \dfrac{[G:Z]}{\dim V}$. Then $|Z|^{-1}$ divides α^m for any $m\in\mathbb{Z}_+$. We deduce that

"divides"
but maybe here $|Z|^{-1} \notin \mathbb{Z}$?

I suppose you mean: $\alpha^n \in \frac{1}{|Z|}\mathbb{Z}$?

$$\mathbb{Z}[\alpha]\subseteq\frac{1}{|Z|}\mathbb{Z}\subseteq\mathbb{C}$$

It is clear that $\dfrac{1}{|Z|}\mathbb{Z}$ is a finitely generated $\mathbb{Z}$-module. Since $\mathbb{Z}$ is a principle ideal domain, $\mathbb{Z}[\alpha]$ is also a finitely generated $\mathbb{Z}$-module. Since $\alpha\cdot\mathbb{Z}[\alpha]\subseteq\mathbb{Z}[\alpha]$, by Proposition 7.4, α is an algebraic integer. But α is also rational, and $\mathbb{Z}$ is integrally closed. Therefore $\alpha\in\mathbb{Z}$ and $\dim V$ divides $[G:Z]$. □

Question 8

Prove that induction is transitive: if k is a field and $J\subseteq H$ are subgroups of G, then

$$\mathrm{Ind}_H^G\left(\mathrm{Ind}_J^H V\right)\cong\mathrm{Ind}_J^G V$$

as kG-modules, for every kJ-module V.

Proof. See the general remark at the beginning of this sheet. □

Question 9

Suppose that V is a faithful representation of G. Prove that every simple $\mathbb{C}G$-module W appears as a direct summand of some tensor power $V^{\otimes n}$ of V, by considering the infinite series

$$\sum_{n\geqslant 0}\langle\chi_W,\chi_{V^{\otimes n}}\rangle t^n$$

where t is an indeterminate.

Proof. Consider the power series in $\mathbb{C}[[t]]$:

$$f(t) = \sum_{n=0}^{\infty}\langle\chi_W,\chi_{V^{\otimes n}}\rangle t^n = \sum_{n=0}^{\infty}\frac{1}{|G|}\overline{\chi_W(g)}\chi_V(g)^n t^n = \frac{1}{|G|}\overline{\chi_W(g)}\sum_{n=0}^{\infty}(\chi_V(g)t)^n$$

For sufficiently small $t \in \mathbb{C}$, the sum converges to

$$\frac{1}{|G|}\sum_{g\in G}\frac{\overline{\chi_W(g)}}{1-\chi_V(g)t} = \frac{1}{|G|}\left(\frac{\dim W}{1-\dim V\cdot t}+\sum_{g\neq e}\frac{\overline{\chi_W(g)}}{1-\chi_V(g)t}\right)$$

Since the representation afforded by V is faithful, by the proof of Question 5.(a) in Sheet 3, $\chi_V(g) \neq \dim V$ for all $g \in G\setminus\{e\}$. Then $f(t)$ contains a non-zero term whose denominator is $1-\dim V\cdot t$. In particular $f(t)$ is not the zero function. Hence there exists $n \in \mathbb{N}$ such that $\langle \chi_W, \chi_{V^{\otimes n}}\rangle \neq 0$.

Since W is a simple $\mathbb{C}[G]$-module, and $\langle \chi_W, \chi_{V^{\otimes n}}\rangle \neq 0$, we have

!!

$V^{\otimes n}$ contains W

$$\chi_{V^{\otimes n}} = \langle \chi_W, \chi_{V^{\otimes n}}\rangle\chi_W + \sum_{i=1}^{r} a_i\chi_i$$

good but dont need to justify (!!).

where $\chi_1,\dots,\chi_r$ are other irreducible characters of G besides χ_W. Passing to $\mathbb{C}[G]$-modules,

$$V^{\otimes n} = W^{\langle \chi_W, \chi_{V^{\otimes n}}\rangle} \oplus \left(\bigoplus_{i=1}^{r} V_i^{a_i}\right)$$

Hence W is a direct summand of $V^{\otimes n}$. □

Question 10

Construct the character table of A_6 as follows.

(a) Use the conjugation action of A_5 on its set of Sylow 5-subgroups to construct an injective homomorphism $\sigma : A_5 \to A_6$, and prove that its image contains no 3-cycles.

(b) Use the left-multiplication action of A_6 on $A_6/\sigma(A_5)$ to construct an automorphism $\tau : A_6 \to A_6$ and prove that τ swaps the two conjugacy classes in A_6 consisting of elements of order 3.

(c) Use the natural 2-transitive action on A_6 on $\{1,2,3,4,5,6\}$ together with part (b) to write down two irreducible characters χ_2 and χ_3 of A_6, each of degree 5.

(d) Use $\Lambda^2\chi_2$ and $\chi_2\chi_3$ and the Orthogonality Theorems to complete the character table of A_6.

Proof. (a) Note that $|A_5| = 60 = 5\times 12$. By Sylow 1st theorem, A_5 has Sylow 5-subgroups. By Sylow 3rd theorem, the number a of Sylow 5-subgroups satisfies

$$a \equiv 1 \bmod 5, \qquad a \mid 12$$

which implies that $a = 1$ or 6.

Note that A_5 has 24 elements of order 5, each of which generates a cycle subgroup of A_5 of order 5. Hence A_5 has exactly 6 Sylow 5-subgroups. Consider the action of A_5 on $\mathrm{Syl}_5(A_5)$ by conjugation:

$$g\cdot H := gHg^{-1}$$

This defines a group homomorphism $\sigma : A_5 \to S_6$. It is clear that σ is non-trivial. Since A_5 is simple, σ is injective. On the other hand, we know that A_5 is generated by all 3-cycles, whose image under σ is of order 3. The order 3 elements in S_6 are a product of disjoint 3-cycles, and hence are elements of A_6. We deduce that $\sigma(A_5) \subseteq A_6$. Hence we have an injective homomorphism $\sigma : A_5 \to A_6$.

By Sylow 2nd theorem, the action of A_5 on $\mathrm{Syl}_5(A_5)$ is transitive. By orbit-stabliser theorem, the stabliser of any $H \in \mathrm{Syl}_5(A_5)$ is the identity e. Hence for any $g \in \sigma(A_5)\setminus\{e\}$, g fixes no points in $\{1,2,3,4,5,6\}$. In particular, $\sigma(A_5)$ contains no 3-cycles.

(b) The left multiplication action of A_6 on $A_6/\sigma(A_5)$ gives a group homomorphism $\tau : A_6 \to \mathrm{Sym}(A_6/\sigma(A_5)) \cong S_6$. It is clear that τ is non-trivial. Since A_6 is simple, τ is injective. Hence $\tau : A_6 \to A_6 \leqslant S_6$ is an automorphism.

A_6 has two conjugacy classes whose elements are of order 3: one is the set of all 3-cycles; the other is the set of all products of two disjoint 3-cycles. Each of them has 40 elements. Since τ is an automorphism of A_6, it either preserves the two classes, or swaps the two classes.

From part (a) we know that all elements of order 3 in $\sigma(A_5)$ are of the form $(abc)(def)$, where $\{a,b,c,d,e,f\} = \{1,2,3,4,5,6\}$. Take $(abc)(def) \in \sigma(A_5)$. We note that $(abc)(def)\sigma(A_5) = \sigma(A_5)$, so $(abc)(def)$ fixes $\sigma(A_5) \in A_6/\sigma(A_5)$. $\tau((abc)(def))$ has a fixed point, and hence can only be a 3-cycle in A_6. We deduce that τ swaps the two conjugacy classes consisting of elements of order 3.

(c) First we consider the permutation representation of S^6 on $V = \mathbb{C}^6$. By Question 2 in Sheet 1, it is the direct sum of the simple sub-$\mathbb{C}[S_6]$-modules U and W, where

$$U := \left\langle \sum_{i=1}^{6} x_i \right\rangle, \qquad W := \left\{ \sum_{i=1}^{6} a_i x_i : \sum_{i=1}^{6} a_i = 0 \right\}$$

It is clear that $\mathrm{Res}_{A_6}^{S_6} U$ can only be the trivial representation. Hence $\chi_2 := \mathrm{Res}_{A_6}^{S_6} \chi_W = \mathrm{Res}_{A_6}^{S_6} \chi_V - 1$. (The restriction $\mathrm{Res}_{A_6}^{S_6} W$ is not necessarily irreducible, but once we calculate the character it will be clear). By Question 6.(a) in Sheet 3, $\chi_2(g) = \mathrm{Fix}(g) - 1$.

A_6 has 7 conjugacy classes:

A_6	e	(123)	(12345)	(13524)	(12)(34)	(123)(456)	(1234)(56)
$\lvert g^G \rvert$	1	40	72	72	45	40	90
χ_2	5	2	0	0	1	-1	-1

Note that

$$\langle \chi_2, \chi_2 \rangle = \frac{1}{360}(5^2 + 2^2 \times 40 + 1 \times 45 + 1 \times 40 + 1 \times 90) = 1$$

(always true!)

Hence χ_2 is irreducible.

W has another $\mathbb{C}[A_6]$-module structure, given by $\rho(g)(x_i) := x_{\tau(g)\cdot i}$. The resulting character χ_3 swaps the value on the conjugacy classes of (123) and of (123)(456), and fixes all another values. Then χ_3 is also an irreducible character of A_6.

Now we have:

A_6	e	(123)	(12345)	(13524)	(12)(34)	(123)(456)	(1234)(56)
$\lvert g^G \rvert$	1	40	72	72	45	40	90
χ_1	1	1	1	1	1	1	1
χ_2	5	2	0	0	1	-1	-1
χ_3	5	-1	0	0	1	2	-1

(d) We compute $\bigwedge^2 \chi_2$ using Proposition 5.21.(f). For $g \in A_6$,

$$\Lambda^2 \chi_2(g) = \frac{1}{2}\left(\chi_2(g)^2 - \chi_2(g^2)\right) = \frac{1}{2}\left(\chi_2(g)^2 - \chi_V(g^2) - 1\right)$$

This gives

A_6	e	(123)	(12345)	(13524)	(12)(34)	(123)(456)	(1234)(56)
χ_2	5	2	0	0	1	-1	-1
$\bigwedge^2 \chi_2$	10	1	0	0	-2	1	0

Since $\langle \bigwedge^2 \chi_2, \bigwedge^2 \chi_2 \rangle = 1$, $\bigwedge^2 \chi_2$ is irreducible.

We compute $\mathrm{S}^2 \chi_2$ using $\mathrm{S}^2 \chi_2 = \chi_2^2 - \bigwedge^2 \chi_2$, which gives

A_6	e	(123)	(12345)	(13524)	(12)(34)	(123)(456)	(1234)(56)
χ_2	5	2	0	0	1	-1	-1
$\mathrm{S}^2 \chi_2$	15	3	0	0	3	0	1

We have

$$\langle \mathrm{S}^2 \chi_2, \mathrm{S}^2 \chi_2 \rangle = 3, \qquad \langle \mathrm{S}^2 \chi_2, \chi_1 \rangle = 1, \qquad \langle \mathrm{S}^2 \chi_2, \chi_2 \rangle = 1$$

Hence $\mathrm{S}^2 \chi_2 = \chi_1 + \chi_2 + \chi_5$, where χ_5 is an irreducible character, given by

A_6	e	(123)	(12345)	(13524)	(12)(34)	(123)(456)	(1234)(56)
χ_5	9	0	-1	-1	1	0	1

We can write down the incomplete character table as follows:

A_6	e	(123)	(12345)	(13524)	(12)(34)	(123)(456)	(1234)(56)
$\lvert g^G \rvert$	1	40	72	72	45	40	90
χ_1	1	1	1	1	1	1	1
χ_2	5	2	0	0	1	-1	-1
χ_3	5	-1	0	0	1	2	-1
$\bigwedge^2 \chi_2$	10	1	0	0	-2	1	0
χ_5	9	0	-1	-1	1	0	1
χ_6	a_{61}	a_{62}	a_{63}	a_{64}	a_{65}	a_{66}	a_{67}
χ_7	a_{71}	a_{72}	a_{73}	a_{74}	a_{75}	a_{76}	a_{77}

Since $|A_6| = 360 = 1^2 + 5^2 + 5^2 + 10^2 + 9^2 + a_{61}^2 + a_{71}^2$, we have $a_{61} = a_{71} = 8$.

We can complete the table by using column orthogonality relation successively. The complete table is given as follows:

A_6	e	(123)	(12345)	(13524)	(12)(34)	(123)(456)	(1234)(56)
$\lvert g^G \rvert$	1	40	72	72	45	40	90
χ_1	1	1	1	1	1	1	1
χ_2	5	2	0	0	1	-1	-1
χ_3	5	-1	0	0	1	2	-1
$\bigwedge^2 \chi_2$	10	1	0	0	-2	1	0
χ_5	9	0	-1	-1	1	0	1
χ_6	8	-1	$\frac{1}{2}(1+\sqrt{5})$	$\frac{1}{2}(1-\sqrt{5})$	0	-1	0
χ_7	8	-1	$\frac{1}{2}(1-\sqrt{5})$	$\frac{1}{2}(1+\sqrt{5})$	0	-1	0

□

Nice job!