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Problem Sheet 3

C3.11: Riemannian Geometry

A+

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Section A: Introductory

Question 1

Let E_1, E_2, E_3 be vector fields on $\mathcal{S}^3$ such that $[E_i, E_j] = -2\epsilon_{ijk}E_k$. For $\lambda > 0$, let

$$X_1 = \lambda E_1, \quad X_2 = E_2, \quad X_3 = E_3$$

and define a Riemannian metric g on $\mathcal{S}^3$ by the condition that

$$g\left(X_i, X_j\right) = \delta_{ij}$$

(a) Show that $\left(\mathcal{S}^3, g\right)$ is Einstein if and only if $\lambda = 1$.

(b) Find a necessary and sufficient condition on λ so that the scalar curvature of $\left(\mathcal{S}^3, g\right)$ is zero.

Question 2

Let $(\mathcal{S}^n, g)$ be the round n-sphere and let h be the product metric on $\mathcal{S}^n \times \mathcal{S}^n$.

Show that $(\mathcal{S}^n \times \mathcal{S}^n, h)$ is Einstein with non-negative sectional curvature.

Question 3

(a) Show that the induced metric on an oriented minimal hypersurface in $\left(\mathbb{R}^{n+1}, g_0\right)$ is flat if and only if the minimal hypersurface is totally geodesic.

(b) Let

$$M = \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2| = \frac{1}{\sqrt{2}}\right\} \subseteq \mathcal{S}^3$$

and let g be the induced metric on M from the round metric on $\mathcal{S}^3$. Show that (M, g) is flat and that M is a minimal hypersurface in $\mathcal{S}^3$ which is not totally geodesic.

Section B: Core

Question 4

Let M be $\mathrm{SO}(n), \mathrm{O}(n), \mathrm{SU}(m)$ or $\mathrm{U}(m)$ and let g be the bi-invariant metric on M given by

$$g_A(B, C) = -\operatorname{tr}\left(A^{-1}BA^{-1}C\right)$$

for all $A \in M$ and $B, C \in T_AM$. Let $L_A : M \to M$ denote left-multiplication by A and let

$$\mathcal{X} = \{\text{vector fields } X \text{ on } M : (L_A)_* X = X \;\forall\, A \in M\}$$

(a) Show that, for all $X, Y \in \mathcal{X}$,

$$\nabla_X Y = \frac{1}{2}[X, Y]$$

[*You may assume that* $[X,Y](I)$ *is the matrix commutator of* $X(I)$ *and* $Y(I)$*, where* I *is the identity matrix.*]

(b) Show that the sectional curvatures of (M, g) are non-negative and that (M, g) is flat if and only if $n = 2$ or $m = 1$.

(c) Let $m > 1$ and define a submanifold D of $\mathrm{U}(m)$ by

$$D = \left\{\mathrm{diag}\left(\mathrm{e}^{\mathrm{i}\theta_1}, \dots, \mathrm{e}^{\mathrm{i}\theta_m}\right) : \theta_1, \dots, \theta_m \in \mathbb{R}\right\} \subseteq \mathrm{U}(m)$$

Show that D is a flat totally geodesic submanifold in $(\mathrm{U}(m), g)$.

Proof. (a) For any $X, Z \in \mathcal{X}$, the Koszul formula is given by

$$\begin{aligned} g(\nabla_X X, Z) &= \frac{1}{2}\left(X(g(X,Z)) + X(g(X,Z)) - Z(g(X,X)) - g(X,[X,Z]) + g(X,[Z,X]) + g(Z,[X,X])\right) \\ &= X(g(X,Z)) - \frac{1}{2}Z(g(X,X)) + g(X,[Z,X]) \end{aligned}$$

Since X and Z are invariant, and g is bi-invariant, we have

$$g_A(X,Z) = g_A((L_A)_* X, (L_A)_* Z) = ((L_A)^* g)_I(X,Z) = g_I(X,Z)$$

for all $A \in M$. Hence $g(X,Z)$ is constant on M, and $X(g(X,Z)) = 0$. Similarly $Z(g(X,X)) = 0$. Therefore we have $g(\nabla_X X, Z) = g(X,[Z,X])$.

Next, we claim that $g(X,[Z,X]) = 0$. Since this is constant on M, it suffices to look at the identity $I \in M$. By the hint we have

$$g(X,[Z,X]) = g_I(X,[Z,X]) = g_I(X, ZX - XZ) = \mathrm{tr}(XZX - XXZ) = \mathrm{tr}(XZX) - \mathrm{tr}(XZX) = 0$$

We deduce that $g(\nabla_X X, Z) = 0$. We know that $\mathcal{X} \cong \mathrm{T}_I M$ is the Lie algebra of M. In particular $\mathcal{X}$ spans $\Gamma(\mathrm{T}M)$ as a $\mathrm{C}^\infty(M)$-module. This is enough to deduce that $\nabla_X X = 0$. Then for $X, Y \in \mathcal{X}$

$$0 = \nabla_{X+Y}(X+Y) = \nabla_X X + \nabla_Y Y + \nabla_X Y + \nabla_Y X = \nabla_X Y + \nabla_Y X$$

And finally

$$\nabla_X Y = \frac{1}{2}(\nabla_X Y - \nabla_Y X) = \frac{1}{2}[X,Y]$$

✓

(b) For $X, Y, Z \in \mathcal{X}$, the Riemann curvature is given by

$$\begin{aligned} R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &= \frac{1}{4}[X,[Y,Z]] - \frac{1}{4}[Y,[X,Z]] - \frac{1}{2}[[X,Y],Z] \\ &= -\frac{1}{4}[[X,Y],Z] \end{aligned}$$

For a plane $\sigma \subseteq \mathrm{T}_A M$, we can find $X, Y \in \mathcal{X}$ such that $X|_A$ and $Y|_A$ form a orthonormal basis of σ. Then the sectional curvature is given by

$$K(\sigma) = \frac{R(X,Y,Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2} = g(R(X,Y)Y,X) = -\frac{1}{4}g([[X,Y],Y],X)$$

Using the same method as above (looking at the identity), we can prove that $g([[X,Y],Y],X) = g([X,Y],[Y,X])$. Hence

$$K(\sigma) = \frac{1}{4}g([X,Y],[X,Y]) \geqslant 0$$

by the positivity of g. Hence all sectional curvatures are non-negative. ✓

Suppose that (M, g) are flat. Then $K(\sigma) = 0$ for all $\sigma \subseteq \mathrm{T}_A M$ and $A \in M$. For orthonormal $X, Y \in \mathcal{X}$, $[X, Y] = 0$. Hence $\mathcal{X}$ is an Abelian Lie algebra. The exponential map $\exp_I : \mathcal{X} \to M$ is surjective onto the connected component of $I \in M$, which is hence an Abelian subgroup of M. From linear algebra this implies that $n = 2$ or $m = 1$, because $\mathrm{SO}(n)$ and $\mathrm{SU}(n)$ are non-Abelian for $n > 2$ and $m > 1$.

Conversely, suppose that $n = 2$ or $m = 1$. We know that M is Abelian as a Lie group. Hence the Lie algebra $\mathcal{X}$ is Abelian. $[X, Y] = 0$ for all $X, Y \in \mathcal{X}$. This implies that $K(\sigma) = 0$ for all σ. By Proposition 4.6, $R = 0$ on M and hence (M, g) is flat. ✓

(c) Note that D is an Abelian subgroup of M with the induced bi-invariant metric. By the same reasoning in (b), we can show that D is flat. ✓

it works just as before, given $A \in D$ $L_A : M \to M$ is an isometry and sends D into D
(L_A sends geod. in geodesics / the exp map in the exp. map)

To show that D is totally geodesic, we need to show that for $A \in D$ and $X \in \mathrm{T}_A D \subseteq \mathrm{T}_A M$, the geodesic $\gamma(t) = \exp_A(tX) \in D$ for all $t \in \mathbb{R}$. In fact it suffices to prove this for $A = I$. (*This is some intuition from bi-invariance. I am not sure...*) We work in the local coordinates $(\theta_1, ..., \theta_m)$ on D with the frame vector fields $\partial_1, ..., \partial_m$. For $X = X^i \partial_i \in \mathrm{T}_I \cancel{M}$, $\mathrm{T}_I D$

$$\exp_I(tX) = \mathrm{diag}(\mathrm{e}^{itX^1}, ..., \mathrm{e}^{itX^m}) \in D$$ [1]

I don't get what you mean with the remark.
The coordinates $\theta_1 \dots \theta_m$ induce the frame of $\mathrm{T}_I D$
$\{ i E_j \mid j = 1 \dots m \}$ where $E_j = \mathrm{diag}(0 \dots \underset{j}{1} \dots 0)$
Then by definition $\exp_I(tX) = \exp_I(\mathrm{diag}(itX^1 | \dots | itX^m))$
$= \mathrm{diag}(e^{itX^1} \dots e^{itX^m})$
and this $\in D$ by definition of D.

which proves that D is geodesic at I, and hence is totally geodesic. □

Perfect!

Question 5

(a) Let $\gamma : [0, L] \to (M, g)$ be a geodesic and let $f : (-\epsilon, \epsilon) \times [0, L] \to M$ be a variation of γ so that the curve $\gamma_s : [0, L] \to (M, g)$ given by $\gamma_s(t) = f(s, t)$ is a geodesic for all $s \in (-\epsilon, \epsilon)$. Show that the variation field V_f of f is a Jacobi field along γ.

(b) Let

$$\mathcal{H}^n = \left\{ (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} x_i^2 - x_{n+1}^2 = -1, x_{n+1} > 0 \right\}$$

and let g be the restriction of $h = \sum_{i=1}^n \mathrm{d}x_i^2 - \mathrm{d}x_{n+1}^2$ on $\mathbb{R}^{n+1}$ to $\mathcal{H}^n$. Given that the normalized geodesics γ in $(\mathcal{H}^n, g)$ with $\gamma(0) = x$ and $\gamma'(0) = X$ are given by

$$\gamma(t) = x \cosh t + X \sinh t$$

show that $(\mathcal{H}^n, g)$ has constant sectional curvature -1.

Proof. (a) This is the proof of Lemma 6.3 verbatim. We have

$$\begin{aligned}
\frac{\mathrm{D}^2}{\mathrm{D}t^2} \frac{\partial f}{\partial s} &= \nabla_{\partial_t f} \nabla_{\partial_t f} \frac{\partial f}{\partial s} = \nabla_{\partial_t f} \nabla_{\partial_s f} \frac{\partial f}{\partial t} && \text{(symmetry lemma)} \\
&= \nabla_{\partial_s f} \nabla_{\partial_t f} \frac{\partial f}{\partial t} - R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right) \frac{\partial f}{\partial t} && \text{(Lemma 6.1)} \\
&= -R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right) \frac{\partial f}{\partial t} && \left(\nabla_{\partial_t f} \frac{\partial f}{\partial t} = \nabla_{\gamma_s'} \gamma_s' = 0\right)
\end{aligned}$$

Take $s = 0$ in the above equation. We have $\dfrac{\partial f}{\partial s} = V_f$ and $\dfrac{\partial f}{\partial t} = \gamma'$. Hence $V_f'' + R(V_f, \gamma')\gamma' = 0$. This implies that V_f is a Jacobi field. ✓

[1] This is again intuition from matrix exponentiation. I don't know if there is a clean way to deduce it rigourously.

(b) Fix $x \in \mathcal{H}^n$. Let σ be a plane in $\mathrm{T}_x\mathcal{H}^n$. Let X, Y be an orthonormal basis of σ. Let $\gamma : [0, L] \to \mathcal{H}^n$ be a geodesic such that $\gamma(0) = x$ and $\gamma'(0) = X$. Let $\alpha : [0, L] \to \mathcal{H}^n$ be a geodesic such that $\alpha(0) = x$ and $\alpha'(0) = Y$. Then we know that $\gamma(t) = x \cosh t + X \sinh t$ and $\alpha(s) = x \cosh s + Y \sinh s$. We define $f : [0, L] \times [0, L] \to \mathcal{H}^n$ by

$$f(s,t) := \alpha(s) \cosh t + X \sinh t = (x \cosh s + Y \sinh s) \cosh t + X \sinh t$$

Then $f(0,t) = \gamma(t)$, and $\gamma_s : t \mapsto f(s,t)$ is a geodesic for each fixed s. By (a), V_f is a Jacobi field along γ:

$$V_f(t) = \frac{\partial f}{\partial s}(0,t) = (x \sinh s + Y \cosh s) \cosh t + X \sinh t$$

Obviously $V_f(0) = Y$ and $V_f'' = V_f$. The Jacobi equation $V_f'' + R(V_f, \gamma')\gamma' = 0$ at $t = 0$ gives $Y + R(Y,X)X = 0$. Since X and Y are orthonormal, we have

$$K(\sigma) = K(X,Y) = g(R(Y,X)X, Y) = g(-Y, Y) = -1$$

We conclude that $\mathcal{H}^n$ has constant sectional curvature -1. ✓ Perfect! □

Section C: Optional

Question 6

Let $(\mathcal{S}^{2n+1}, g)$ be the round $(2n+1)$-sphere, view $\mathcal{S}^{2n+1} \subseteq \mathbb{C}^{n+1}$ and let $\pi : \mathcal{S}^{2n+1} \to \mathbb{CP}^n$ be the projection map. For $z \in \mathcal{S}^{2n+1}$ we have $E(z) = iz$ (identifying tangent vectors in $\mathbb{C}^n$ with $\mathbb{C}^n$), $\ker \mathrm{d}\pi_z = \mathrm{Span}\{E(z)\}$ and we let $H_z = \{X \in T_z\mathcal{S}^{2n+1} : g(X, E(z)) = 0\}$ and $\Phi_z = \mathrm{d}\pi_z : H_z \to T_{\pi(z)}\mathbb{CP}^n$ The Fubini-Study metric h on $\mathbb{CP}^n$ is then given by

$$h_{\pi(z)}(X,Y) = g_z\left(\Phi_z^{-1}(X), \Phi_z^{-1}(Y)\right)$$

(a) For any vector field X on $\mathbb{CP}^n$ we define a vector field $\widehat{X}$ on $\mathcal{S}^{2n+1}$ by

$$\widehat{X}(z) = \Phi_z^{-1}(X(\pi(z)))$$

If $\widehat{\nabla}$ is the Levi-Civita connection of g and ∇ is the Levi-Civita connection of h, show that, for all vector fields X, Y on $\mathbb{CP}^n$

$$\widehat{\nabla}_{\widehat{X}}\widehat{Y} = \widehat{\nabla_X Y} + \frac{1}{2} g([\widehat{X}, \widehat{Y}], E)E$$

[*Hint: Show that* $[\widehat{X}, \widehat{Y}] - \widehat{[X,Y]}$ *and* $[\widehat{X}, E]$ *are multiples of* E.]

(b) Show that $\gamma : (-\epsilon, \epsilon) \to (\mathbb{CP}^n, h)$ is a geodesic with $\gamma(0) = \pi(z)$ if and only if $\gamma = \pi \circ \widehat{\gamma}$ where $\widehat{\gamma} : (-\epsilon, \epsilon) \to (\mathcal{S}^{2n+1}, g)$ is a geodesic with $\widehat{\gamma}(0) = z$ and $\widehat{\gamma}'(0) \in H_z$.

(c) Since $X \in H_z$ if and only if $iX \in H_z$, we can define $J = J_{\pi(z)} : T_{\pi(z)}\mathbb{CP}^n \to T_{\pi(z)}\mathbb{CP}^n$ by

$$J(X) = \mathrm{d}\pi_z\left(i\Phi_z^{-1}(X)\right)$$

which then extends to a map J from vector fields to vector fields on $\mathbb{CP}^n$. Let $X, Y \in T_{\pi(z)}\mathbb{CP}^n$ be orthogonal unit vectors and write $Y = \cos\alpha Z + \sin\alpha JX$ where Z is orthogonal to JX and unit length. Show that the sectional curvature K of $(\mathbb{CP}^n, h)$ satisfies

$$K(X,Y) = 1 + 3\sin^2\alpha$$

[*Hint: Let* γ *be a geodesic in* $(\mathbb{CP}^n, h)$ *with* $\gamma(0) = \pi(z)$ *and* $\gamma'(0) = X$*, and consider a variation* $f(s,t)$ *of* γ *so that* $\gamma_s(t) = f(s,t)$ *is geodesic for all* s *such that* $\gamma_s(0) = \pi(z)$ *and* $\gamma_s'(0) = \cos sX + \sin sY$*. You may want to consider the cases* $\sin\alpha = 0$ *and* $\cos\alpha = 0$ *first.*]