Peize Liu
St. Peter's College
University of Oxford

Problem Sheet 2

B2: Symmetry & Relativity

Well done! Everything is essentially correct. However, may I suggest a slight change of attitude in your comments?
"I believe a better way to phrase this may be..." sounds much nicer than "the statement is INCORRECT"
I don't really mind, but this might be helpful for you, in the future :)

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Indeed, it is an abuse of notation, but it is a commonly accepted and understood one in the Physics literature

Conventions: Greek indices take values 0 through 3, while Latin indices take values 1 through 3. 3-vectors can also be indicated by boldface, e.g. **a**. Minkowski metric $g^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$.

Remark. From my perspective, saying that "$X^\mu = (ct, x, y, z)$ is a 4-vector" is an abuse of notation. X^μ should always mean the μ-th component of the 4-vector X (unless we are using the system of Abstract Index Notation, which should not be confused with the component index notation).

Question 1

Show, using algebra, a spacetime diagram, or otherwise,

(i) the temporal order of two events is the same in all reference frames if and only if they are separated by a time-like interval;

(ii) there exists a reference frame in which two events are simultaneous if and only if they are separated by a space-like interval.

(iii) for any time-like vector there exists a frame in which its spatial part is zero;

(iv) any vector orthogonal to a time-like vector must be space-like;

(v) with one exception, any vector orthogonal to a null vector is space-like, and describe the exception.

(vi) the instantaneous 4-velocity of a particle is parallel to the worldline (i.e., demonstrate that you understand the meaning of this claim - if you do then it is obvious);

(vii) if the 4-displacement between any two events is orthogonal to an observer's worldline, then the events are simultaneous in the rest frame of that observer.

Proof. I shall use an algebraic approach in this question.

For 4-vectors X and Y, we use $g(X, Y)$ to denote the bilinear form $g_{\mu\nu} X^\mu Y^\nu$. Then a 4-vector X is

- space-like, if $g(X, X) > 0$,
- light-like or null, if $g(X, X) = 0$,
- time-like, if $g(X, X) < 0$.

Any (proper orthochronous) Lorentz transformation $L \in \mathrm{SO}^+(1,3)$ is a composition of spatial rotations and standard Lorentz boosts (in x-direction). More specifically,

$$L = \begin{pmatrix} 1 & 0 \\ 0 & H \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & H' \end{pmatrix}$$

where $H, H' \in \mathrm{SO}(3)$.

(i) The statement is **incorrect**. The correct statement is: the temporal order of two events is the same in all reference frames if and only if they are separated by a time-like *or light-like* interval.

"$\Longleftarrow$": Let X be a future-pointing time-like or light-like vector between the two events. Write $X = (ct, x, y, z)$ where $t > 0$ and $g(X, X) \leqslant 0$. It is clear that if $L \in \mathrm{SO}^+(1,3)$ is a spatial rotation, then $(X')^0 = L^0_\mu X^\mu > 0$, so X' is still future-pointing. For a Lorentz boost $L \in \mathrm{SO}^+(1,3)$:

$$(X')^0 = L^0_\mu X^\mu = \gamma(ct - \beta x)$$

$g(X, X) \leqslant 0$ implies that $c^2 t^2 \geqslant X^a X_a \geqslant x^2$, which implies that $ct \geqslant |x| > \beta x$ as $|\beta| < 1$. Hence $(X')^0 > 0$. X' is future-pointing. We deduce that Lorentz transformations do not change the temporal sign of a 4-vector. In physics, this implies that the temporal order of two events is the same in all reference frames.

"$\Longrightarrow$": Suppose that $X = (ct, x, y, z)$ is a future pointing 4-vector which is future pointing under any Lorentz transformations. There exists a spatial rotation L_1 such that $X' = L_1 X = (ct, \sqrt{x^2 + y^2 + z^2}, 0, 0)$. Let L_2 be a standard Lorentz boost. Then

$$(X'')^0 = (L_2)^0_\mu (X')^\mu = \gamma\left(ct - \beta\sqrt{x^2 + y^2 + z^2}\right) > 0$$

Then $c^2t^2 > \beta^2(x^2+y^2+z^2)$ for any $|\beta|<1$. We deduce that $g(X,X) = -c^2t^2+x^2+y^2+z^2 \leqslant 0$. Hence X is either time-like or light-like.

(ii) "$\Longleftarrow$": Let X be a future-pointing space-like vector between the two events. Write $X=(ct,x,y,z)$ where $t>0$ and $g(X,X)>0$. As above there exists a spatial rotation L_1 such that $X' = L_1X = (ct, \sqrt{x^2+y^2+z^2}, 0, 0)$. $g(X,X)>0$ implies that

$$\beta := \frac{ct}{\sqrt{x^2+y^2+z^2}} \in (0,1)$$

Hence the standard Lorentz boost with β transforms X' to a 4-vector X'' with $(X'')^0 = ct - \beta\sqrt{x^2+y^2+z^2} = 0$. In physics, it suggests that there exists an inertial frame such that the two events are simultaneous.

"$\Longrightarrow$": Suppose that there exists an inertial frame such that the two events are simultaneous. Then in that frame, the 4-vector between the two events is $X=(0,x,y,z)$. Since $g(X,X) = x^2+y^2+z^2>0$, X is space-like.

(iii) Let $X=(ct,x,y,z) = (ct,\mathbf{r})$ be a time-like vector. Write $\mathbf{r} = r\mathbf{e}_r$. There exists $\mathbf{a},\mathbf{b}\in\mathbb{R}^3$ such that $\{\mathbf{e}_r,\mathbf{a},\mathbf{b}\}$ forms an orthonormal basis of $\mathbb{R}^3$. Then the following 4-vectors

$$\frac{1}{\sqrt{-g(X,X)}}(ct, r\mathbf{e}_r), \quad \frac{1}{\sqrt{-g(X,X)}}(r, ct\mathbf{e}_r), \quad (0,\mathbf{a}), \quad (0,\mathbf{b})$$

form a pseudo-orthonormal basis of the Minkowski space. Hence it defines an inertial frame, in which

$$X' = \left(\sqrt{-g(X,X)}, 0, 0, 0\right)$$

has zero spatial part.

(iv) **This statement is true for non-zero 4-vectors.**

Let X be a time-like 4-vector and $Y\neq 0$ be a 4-vector pseudo-orthogonal to X. By (iii) there exists an inertial frame in which $X = \left(\sqrt{-g(X,X)},0,0,0\right)$. Let $Y=(ct,x,y,z)$ in that frame. Then

$$g(X,Y) = 0 \implies ct\sqrt{-g(X,X)} = 0 \implies t=0$$

By (ii) we deduce that Y is space-like.

(v) Let $X=(ct,\mathbf{x})$ be a null 4-vector and $X'=(ct,\mathbf{x}')$ be a 4-vector pseudo-orthogonal to X. We have

$$g(X,X') = 0 \implies c^2tt' = \mathbf{x}\cdot\mathbf{x}' \implies c^4t^2t'^2 = (\mathbf{x}\cdot\mathbf{x}')^2 \leqslant \|\mathbf{x}\|^2\left\|\mathbf{x}'\right\|^2 \qquad \text{(Cauchy-Schwarz inequality)}$$

Since X is null, $\|\mathbf{x}\|^2 = c^2t^2$. We deduce that $c^2t'^2 \leqslant \left\|\mathbf{x}'\right\|^2$. So X' is either space-like or null.

If X' is null, then $c^2t'^2 = \left\|x'\right\|^2$ and $\mathbf{x}$ is is colinear with $\mathbf{x}'$. Then X and X' are colinear. This is the only exceptional case.

(vi) Let $\eta : [0,1]\to(\mathbb{R}^4,g)$ be a (not necessarily straight) worldline in the Minkowski space. Recall that the length element (i.e. the metric tensor) in the Minkowski space is given by

$$\mathrm{d}\ell^2 = g_{\mu\nu}\mathrm{d}X^\mu\mathrm{d}X^\nu$$

The **proper time** on the worldline is defined by

$$\tau(s) = \frac{1}{c}\int_{\eta|_{(0,s)}}\sqrt{-\mathrm{d}\ell^2} = \frac{1}{c}\int_0^s\sqrt{-g_{\mu\nu}\frac{\mathrm{d}X^\mu}{\mathrm{d}s}\frac{\mathrm{d}X^\nu}{\mathrm{d}s}}\,\mathrm{d}s$$

Now we can reparametrize the worldline by proper time: $\tilde{\eta} : [0,\tau(1)]\to M$, $\tilde{\eta} = \eta\circ\tau^{-1}$. On the worldline, a tangent vector V at $p\in\operatorname{Im}\eta$ is given by

$$V(p) = \left.\frac{\mathrm{d}\tilde{\eta}}{\mathrm{d}\tau}\right|_p \implies V^\mu(p) = \left.\frac{\mathrm{d}X^\mu}{\mathrm{d}\tau}\right|_p \qquad \text{where } \tilde{\eta}(\tau) = \left(X^0(\tau), X^1(\tau), X^2(\tau), X^3(\tau)\right)$$

which is exactly the definition of a 4-velocity at p. So 4-velocities are tangent vectors on the worldline. By definition they are "parallel" to the worldline locally.

(vii) Assume that the observer is stationary in some inertial frame so the worldline of it is a straight line. Let $\gamma(\tau) = A+B\tau$ be the parametrization of the worldline, where $A,B\in(\mathbb{R}^4,g)$. Let X be the 4-vector between the two events. We have

$g(X,B)=0$. Let L be the Lorentz transformation to the frame of the observer. Since the observer is stationary in his rest frame, we have $B' = LB = (\sqrt{-g(B,B)}, 0, 0, 0)$. Since Lorentz transformations preserves bilinear form g, we have

$$g(LX, LB) = g(X,B) = 0 \implies (LX)^0 \sqrt{-g(B,B)} = 0 \implies (X')^0 = (LX)^0 = 0$$

Hence in the rest frame of the observer, the two events are simultaneous. □

Question 2

Define proper time. A worldline (not necessarily straight) may be described as a locus of time-like separated events specified by $X^\mu = (ct, x, y, z)$ in some inertial reference frame. Show that the increase of proper time τ along a given worldline is related to reference frame time t by $dt/d\tau = \gamma$.

Two particles have 3-velocities $\mathbf{u}$ and $\mathbf{v}$ in some reference frame. The Lorentz factor for their relative 3-velocity $\mathbf{w}$ is given by

$$\gamma_w = \gamma_u \gamma_v \left(1 - \mathbf{u}\cdot\mathbf{v}/c^2\right)$$

Prove this twice, by using each of the following two methods:

(i) In the given frame, the worldline of the first particle is $X^\mu = (ct, \mathbf{u}t)$. Transform to the rest frame of the other particle to obtain

$$t' = \gamma_v t \left(1 - \mathbf{u}\cdot\mathbf{v}/c^2\right)$$

Obtain dt'/dt and apply the result of the first part of this question.

(ii) Use the invariant $U^\mu V_\mu$, first showing that it is equal to $-c^2\gamma_w$.

Proof. In Question 1.(vi) I have defined the proper time before defining the 4-velocity. I copy it here:

Let $\eta : [0,1] \to (\mathbb{R}^4, g)$ be a worldline in the Minkowski space. The **proper time** on the worldline is defined by

$$\tau(s) = \frac{1}{c}\int_{\eta|_{(0,s)}} \sqrt{-\mathrm{d}\ell^2} = \frac{1}{c}\int_0^s \sqrt{-g_{\mu\nu}\frac{\mathrm{d}X^\mu}{\mathrm{d}s}\frac{\mathrm{d}X^\nu}{\mathrm{d}s}}\,\mathrm{d}s$$

The 3-speed is given by

$$v = \|\mathbf{v}\| = \sqrt{\frac{\mathrm{d}X^a}{\mathrm{d}t}\frac{\mathrm{d}X_a}{\mathrm{d}t}} \implies \mathrm{d}X^a \mathrm{d}X_a = v^2 \mathrm{d}t^2$$

Then

$$\mathrm{d}\tau = \frac{1}{c}\sqrt{c^2\mathrm{d}t^2 - \mathrm{d}X^a\mathrm{d}X_a} = \frac{1}{c}\sqrt{c^2\mathrm{d}t^2 - v^2\mathrm{d}t^2} = \sqrt{1-v^2/c^2}\,\mathrm{d}t = \gamma^{-1}\mathrm{d}t$$

Hence $\mathrm{d}t/\mathrm{d}\tau = \gamma$.

(i) Let $Y = (ct, \mathbf{v}t)$ be the 4-vector of the second particle. The transformation L to the rest frame of the second particle is given by

$H \in SO(3,\mathbb{R})$

$$L = \begin{pmatrix} \gamma_v & -\gamma_v \mathbf{v}^{\mathrm{T}}/c \\ \gamma_v \mathbf{v}/c & H \end{pmatrix}$$

for some $H \in M_{3\times 3}(\mathbb{R})$. Then

$$(X')^0 = L^0_\mu X^\mu = \gamma_v ct - \gamma_v \mathbf{u}\cdot\mathbf{v}t/c = \gamma_v ct(1 - \mathbf{u}\cdot\mathbf{v}/c^2)$$

Hence $t' = \gamma_v t(1 - \mathbf{u}\cdot\mathbf{v}/c^2)$. Now

$$\gamma_w = \frac{\mathrm{d}t'}{\mathrm{d}\tau} = \frac{\mathrm{d}t'}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma_u\gamma_v(1 - \mathbf{u}\cdot\mathbf{v}/c^2)$$

(ii) The 4-velocities of the two particles are given by $U = \gamma_u(c, \mathbf{u})$ and $V = \gamma_v(c, \mathbf{v})$.

$$g(U,V) = \gamma_u\gamma_v(-c^2 + \mathbf{u}\cdot\mathbf{v})$$

In the rest frame of the second particle, $V' = (c, \mathbf{0})$ and $U' = \gamma_w(c, \mathbf{w})$. By invariance of the bilinear form we have

$$-\gamma_w c^2 = g(U', V') = g(U,V) = \gamma_u\gamma_v(-c^2 + \mathbf{u}\cdot\mathbf{v}) \implies \gamma_w = \gamma_u\gamma_v(1 - \mathbf{u}\cdot\mathbf{v}/c^2)$$ □

Question 3

Derive a formula for the frequency ω of light waves from a moving source, in terms of the proper frequency ω_0 in the source frame and the angle in the observer's frame, θ, between the direction of observation and the velocity of the source.

A galaxy with a negligible speed of recession from Earth has an active nucleus. It has emitted two jets of hot material with the same speed v in opposite directions, at an angle θ to the direction to the Earth. A spectral line in singly-ionised Mg (proper wavelength $\lambda_0 = 448.1\text{nm}$) is emitted from both jets. Show that the wavelengths $\lambda_\pm$ observed on Earth from the two jets are given by

$$\lambda_\pm = \lambda_0 \gamma (1 \pm (v/c)\cos\theta)$$

(you may assume the angle subtended at Earth by the jets is negligible). If $\lambda_+ = 420.2\text{nm}$ and $\lambda_- = 700.1\text{nm}$, find v and θ.

In some cases, the receding source is difficult to observe. Suggest a reason for this.

Proof. The wave 4-vector $K = (\omega/c, \mathbf{k})$ is a 4-vector in the Minkowski space. In the lab frame, suppose that the light source travels in the x-direction and emits light at an angle θ. Then the wave 4-vector in the lab frame is given by

$$K = \frac{\omega}{c}(1, \cos\theta, \sin\theta, 0)$$

Consider the Lorentz transformation to the rest frame of the light source, we have

$$\frac{\omega_0}{c} = \gamma\left(\frac{\omega}{c} - \frac{v}{c}\frac{\omega}{c}\cos\theta\right) \implies \frac{\omega}{\omega_0} = \frac{1}{\gamma(1 - v\cos\theta/c)}$$

The corresponding rule for the wavelength is given by

$$\frac{\lambda}{\lambda_0} = \gamma(1 - v\cos\theta/c)$$

For the emission problem, the angles of the two jets are θ and $\theta + \pi$ respectively. So their wavelengths observed on the Earth are $\lambda_\pm = \lambda_0 \gamma(1 \mp v\cos\theta/c)$.

From the given data, we find that

$$\gamma = \frac{\lambda_+ + \lambda_-}{2\lambda_0} = 1.25 \implies v = 0.6c \qquad \theta = \arccos\left(\frac{\lambda_- - \lambda_+}{2\lambda_0 \beta\gamma}\right) = 65.4^\circ$$

□

Question 4

The 4-angular momentum of a single particle about the origin is defined as

$$L^{\mu\nu} \equiv X^\mu P^\nu - X^\nu P^\mu$$

(i) Prove that, in the absence of forces, $dL^{\mu\nu}/d\tau = 0$.

(ii) Exhibit the relationship between the space-space part L^{ij} and the 3-angular momentum vector $\mathbf{L} = \mathbf{x} \wedge \mathbf{p}$.

(iii) The total angular momentum of a collection of particles about the pivot R^λ is defined as

$$L^{\mu\nu}_{\text{tot}}\left(R^\lambda\right) = \sum_i \left(X_i^\mu - R^\mu\right) P_i^\nu - \left(X_i^\nu - R^\nu\right) P_i^\mu$$

where the sum runs over the particles (that is, X^μ and P^μ are 4-vectors, not 2nd-rank tensors; i here labels the particles). Show that the 3-angular momentum in the CM frame is independent of the pivot.

Proof. (i) In the absence of force, we have conservation of momentum: $\dfrac{\mathrm{d}P^\mu}{\mathrm{d}\tau} = 0$. Since $\dfrac{\mathrm{d}X^\mu}{\mathrm{d}\tau} = V^\mu = \dfrac{P^\mu}{m}$,

$$\frac{\mathrm{d}L^{\mu\nu}}{\mathrm{d}\tau} = \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau}P^\nu - \frac{\mathrm{d}X^\nu}{\mathrm{d}\tau}P^\mu = \frac{1}{m}\left(P^\mu P^\nu - P^\nu P^\mu\right) = 0$$

(ii) $\mathbf{L} = \mathbf{x} \wedge \mathbf{p} \implies L_k = \varepsilon_{ijk} X^i P^j$. L^{ij} forms an anti-symmetric type (2,0) tensor:

$$L = \begin{pmatrix} 0 & L_3 & -L_2 \\ -L_3 & 0 & L_1 \\ L_2 & -L_1 & 0 \end{pmatrix}$$

The corresponding between L_k and L^{ij} is due to the isomorphism of vector spaces: $\mathbb{R}^3 = \bigwedge^1 \mathbb{R}^3 \cong \bigwedge^2 \mathbb{R}^3$.

(iii) In the CM frame $\sum_i P_i^\mu = 0$. Hence the spatial parts Only $\vec{P}_{tot} = 0$

$$L^{ab}_{\text{tot}}(R) = \sum_i \left(\left(X_i^a - R^a\right) P_i^b - \left(X_i^b - R^b\right) P_i^a \right) = \sum_i \left(X_i^a P_i^b - X_i^b P_i^a \right) - R^a \sum_i P_i^b + R^b \sum_i P_i^a = \sum_i \left(X_i^a P_i^b - X_i^b P_i^a \right) = L^{ab}_{\text{tot}}(0)$$

are independent of R. □

Question 5

The 4-vector field F^μ is given by $F^\mu = 2x^\mu + k^\mu (x^\nu x_\nu)$ where k^μ is a constant 4-vector and $x^\mu = (ct, x, y, z)$ is the 4-vector displacement in spacetime. Evaluate the following:

(i) $\partial_\lambda x^\lambda$

(ii) $\partial^\mu \left(x_\lambda x^\lambda\right)$

(iii) $\partial^\mu \partial_\mu x^\nu x_\nu$

(iv) $\partial_\lambda F^\lambda$

(v) $\partial^\mu \left(\partial_\lambda F^\lambda\right)$

(vi) $\partial^\mu \partial_\mu \sin\left(k_\lambda x^\lambda\right)$

(vii) $\partial^\mu x^\nu$

Proof. (i) $\partial_\lambda x^\lambda = \partial x^\lambda / \partial x^\lambda = 4$.

(ii) $\partial^\mu x_\lambda x^\lambda = x_\lambda \partial^\mu x^\lambda + x^\lambda \partial^\mu x_\lambda = x_\lambda g^{\mu\nu} \partial_\nu x^\lambda + x^\lambda \partial^\mu x_\lambda = x_\lambda g^{\mu\nu} \dfrac{\partial x^\lambda}{\partial x^\nu} + x^\lambda \dfrac{\partial x_\lambda}{\partial x_\mu} = x_\lambda g^{\mu\nu} \delta^\lambda_\nu + x^\lambda \delta^\mu_\lambda = x^\mu + x^\mu = 2x^\mu$.

(iii) $\partial^\mu \partial_\mu x^\nu x_\nu = \partial^\mu g_{\mu\lambda} \partial^\lambda x^\nu x_\nu = \partial^\mu g_{\mu\lambda} (2x^\lambda) = 2\partial_\lambda x^\lambda = 8$.

(iv) $\partial_\lambda F^\lambda = 2\partial_\lambda x^\lambda + k^\lambda \partial_\lambda x^\nu x_\nu = 8 + 2k^\lambda x_\lambda$.

(v) $\partial^\mu \partial_\lambda F^\lambda = \partial^\mu (8 + 2k^\lambda x_\lambda) = 2k^\lambda \partial^\mu x_\lambda = 2k^\lambda \delta^\mu_\lambda = 2k^\mu$.

(vi) $\partial^\mu \partial_\mu \sin(k_\lambda x^\lambda) = \partial^\mu \left(k_\lambda \cos(k_\lambda x^\lambda) \partial_\mu x^\lambda\right) = \partial^\mu k_\lambda \cos(k_\lambda x^\lambda) \delta^\lambda_\mu = k_\mu \partial^\mu \cos(k_\lambda x^\lambda) = k^\mu \partial_\mu \cos(k_\lambda x^\lambda) = -k^\mu k_\lambda \sin(k_\lambda x^\lambda) \delta^\lambda_\mu$
$= -k^\mu k_\mu \sin(k_\lambda x^\lambda)$.

(vii) $\partial^\mu x^\nu = g^{\mu\lambda} \partial_\lambda x^\nu = g^{\mu\lambda} \dfrac{\partial x^\nu}{\partial x^\lambda} = g^{\mu\lambda} \delta^\nu_\lambda = g^{\mu\nu}$. □

Question 6

A particle of rest mass m and kinetic energy $3mc^2$ strikes a stationary particle of rest mass $2m$ and combines with it while still conserving energy and momentum. Find the rest mass and speed of the composite particle.

Proof. The first particle has energy $E = mc^2 + 3mc^2 = 4mc^2$ and hence momentum $p = \sqrt{E^2/c^2 - m^2c^2} = \sqrt{15}mc$. So its 4-momentum is $P_1 = (4mc, \sqrt{15}mc, 0, 0)$. The second particle has 4-momentum $P_2 = (2mc, 0, 0, 0)$. By conservation of 4-momentum, the composite particle has 4-momentum:

$$P_1 + P_2 = (6mc, \sqrt{15}mc, 0, 0)$$

The rest mass of the composite particle is $m_3 = \sqrt{(6m)^2 - (\sqrt{15}m)^2} = \sqrt{21}m$. The speed of the particle is given by

$$\sqrt{3}mc = m_3\gamma v = \sqrt{21}mc\frac{\beta}{\sqrt{1-\beta^2}} \implies v = \frac{\sqrt{2}}{4}c \qquad \square$$

What is this?
Use $\beta = \frac{p}{E} = \frac{\sqrt{15}m}{6m} = \frac{\sqrt{15}}{6}$

Question 7

Two photons may collide to produce an electron-positron pair. If one photon has energy E_0 and the other has energy E, find the threshold value of E for this reaction, in terms of E_0 and the electron rest mass m.

High energy photons of galactic origin pass through the cosmic microwave background radiation which can be regarded as a gas of photons of energy 2.3×10^{-4}eV. Calculate the threshold energy of the galactic photons for the production of electron-positron pairs.

Proof. Let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the direction of the two photons respectively. Then they have 4-momenta:

$$P_1 = \frac{E_0}{c}(1,\mathbf{e}_1), \quad P_2 = \frac{E}{c}(1,\mathbf{e}_2)$$

In the CM frame of the electron-positron pair, the total 4-momenta of the system:

$$P = \left(\frac{E_1}{c}, \mathbf{0}\right)$$

By conservation of 4-momentum and invariance of the pseudo-norm, we have

$$(E_0+E)^2 - \|E_0\mathbf{e}_1 + E\mathbf{e}_2\|^2 = E_1^2 \geqslant (2mc^2)^2$$

Hence

$$E \geqslant \frac{(2mc^2)^2}{2E_0(1-\mathbf{e}_1\cdot\mathbf{e}_2)} \geqslant \frac{m^2c^4}{E_0} \qquad (\text{since } \mathbf{e}_1\cdot\mathbf{e}_2 \in [-1,1])$$

The threshold energy is $E = m^2c^4/E_0$.

For $E_0 = 2.3\times10^{-4}$eV and $m_e = 0.511$ MeV$/c^2$, we have $E = 1.14\times10^{15}$ eV. $\square$

Question 8

A particle Y decays into three other particles, with labels indicated by $Y \to 1+2+3$. Working throughout in the CM frame:

(i) Show that the 3-momenta of the decay products are coplanar.

(ii) Show that the energy of particle 3 is given by

$$E_3 = \frac{\left(m_Y^2 + m_3^2 - m_1^2 - m_2^2\right)c^4 - 2E_1E_2 + 2\mathbf{p}_1\cdot\mathbf{p}_2c^2}{2m_Yc^2}$$

(iii) Show that the maximum value of E_3 is

$$E_{3,\max} = \frac{m_Y^2 + m_3^2 - (m_1+m_2)^2}{2m_Y}c^2$$

and explain under what circumstances this maximum is attained.

(iv) Show that, when particle 3 has its maximum possible energy, particle 1 has the energy

$$E_1 = \frac{m_1\left(m_Yc^2 - E_{3,\max}\right)}{m_1+m_2}$$

[Hint: first argue that 1 and 2 have the same speed in this situation.]

(v) Now let's return to the more general circumstance, with E_3 not necessarily maximal. Let X be the system composed of particles 1 and 2. Show that its rest mass is given by

$$m_X^2 = m_Y^2 + m_3^2 - 2m_Y E_3/c^2$$

(vi) Write down an expression for the energy E^* of particle 2 in the rest frame of X, in terms of m_1, m_2, and m_X.

(vii) Show that, when particle 3 has an energy of intermediate size, $m_3c^2 < E_3 < E_{3,\max}$, the energy of particle 2 in the original frame (the rest frame of Y) is in the range

$$\gamma\left(E^* - \beta p^* c\right) \le E_2 \le \gamma\left(E^* + \beta p^* c\right)$$

where E^* and p^* are the energy and momentum of particle 2 in the X frame, and γ and β refer to the speed of that frame relative to the rest frame of Y.

Proof. (i) Let $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ be the 3-momenta of the decay products in the CM frame. By conservation of momentum, we have $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 0$. In particular $\mathbf{p}_3 \in \operatorname{span}\{\mathbf{p}_1, \mathbf{p}_2\}$. So the 3-momenta span a subspace of $\mathbb{R}^3$ of dimension at most 2, which means they are coplanar.

(ii) Starting from $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 0$, we have

$$\|\mathbf{p}_3\|^2 = \|\mathbf{p}_1 + \mathbf{p}_2\|^2 = \|\mathbf{p}_1\|^2 + \|\mathbf{p}_2\|^2 + 2\mathbf{p}_1 \cdot \mathbf{p}_2$$

Using the energy-momentum relation $E_i^2 = \|\mathbf{p}_i\|^2 c^2 + m_i^2 c^4$, we obtain

$$E_3^2 - m_3^2 c^4 = E_1^2 + E_2^2 - m_1^2 c^4 - m_2^2 c^4 + 2\mathbf{p}_1 \cdot \mathbf{p}_2 c^2$$

By conservation of energy we have

$$m_Y c^2 = E_1 + E_2 + E_3 \implies E_1^2 + E_2^2 = (m_Y c^2 - E_3)^2 - 2E_1E_2$$

Substitution this relation into the above equation, we have

$$E_3^2 - m_3^2 c^4 = (m_Y c^2 - E_3)^2 - 2E_1E_2 - m_1^2 c^4 - m_2^2 c^4 + 2\mathbf{p}_1 \cdot \mathbf{p}_2 c^2$$

Hence

$$E_3 = \frac{\left(m_Y^2 + m_3^2 - m_1^2 - m_2^2\right)c^4 - 2E_1E_2 + 2\mathbf{p}_1 \cdot \mathbf{p}_2 c^2}{2m_Y c^2}$$

(iii) We shall prove that

$$m_1 m_2 c^4 + \mathbf{p}_1 \cdot \mathbf{p}_2 c^2 \leqslant E_1 E_2$$

with equality holds if and only if $m_1\mathbf{p}_2 = m_2\mathbf{p}_1$.

$$\begin{aligned}
\left(m_1 m_2 c^4 + \mathbf{p}_1 \cdot \mathbf{p}_2 c^2\right)^2 &\leqslant \left(m_1 m_2 c^4 + \|\mathbf{p}_1\|\,\|\mathbf{p}_2\|\,c^2\right)^2 && \text{(Cauchy-Schwarz inequality)}\\
&= m_1^2 m_2^2 c^8 + \|\mathbf{p}_1\|^2 \|\mathbf{p}_2\|^2 c^4 + 2m_1 m_2 \|\mathbf{p}_1\|\,\|\mathbf{p}_2\|\,c^6\\
&\leqslant m_1^2 m_2^2 c^8 + \|\mathbf{p}_1\|^2 \|\mathbf{p}_2\|^2 c^4 + \left(m_1\|\mathbf{p}_2\|\,c^3\right)^2 + \left(m_2\|\mathbf{p}_1\|\,c^3\right)^2 && \text{(AM-GM inequality)}\\
&= \left(m_1^2 c^4 + \|\mathbf{p}_1\|^2 c^2\right)\left(m_2^2 c^4 + \|\mathbf{p}_2\|^2 c^2\right)\\
&= E_1^2 E_2^2
\end{aligned}$$

The two inequality become equality if and only if $\mathbf{p}_1$ and $\mathbf{p}_2$ are in the same direction, and $m_1\|\mathbf{p}_2\|\,c^3 = m_2\|\mathbf{p}_1\|\,c^3$. So the overall equality holds if and only if $m_1\mathbf{p}_2 = m_2\mathbf{p}_1$.

So the energy of the third particle

$$E_3 = \frac{\left(m_Y^2 + m_3^2 - m_1^2 - m_2^2\right)c^4 - 2E_1E_2 + 2\mathbf{p}_1 \cdot \mathbf{p}_2 c^2}{2m_Y c^2} \leqslant \frac{\left(m_Y^2 + m_3^2 - m_1^2 - m_2^2\right)c^4 - 2m_1m_2}{2m_Y c^2} = \frac{m_Y^2 + m_3^2 - (m_1 + m_2)^2}{2m_Y}c^2 = E_{3,\max}$$

And $E_3 = E_{3,\max}$ iff $m_1\mathbf{p}_2 = m_2\mathbf{p}_1$.

(iv) When $E_3 = E_{3,\max}$, $m_1\mathbf{p}_2 = m_2\mathbf{p}_1$, so $\mathbf{v}_1 = \mathbf{v}_2$. In particular, $E_1/E_2 = m_1/m_2$. Hence

$$E_1 = \frac{m_1}{m_1+m_2}(E_1+E_2) = \frac{m_1}{m_1+m_2}(m_Y c^2 - E_{3,\max})$$

(v) Using the energy-momentum relation,

$$\begin{aligned} m_X^2 &= (E_1+E_2)^2/c^4 - \|\mathbf{p}_1+\mathbf{p}_2\|^2/c^2 \\ &= (m_Y c^2 - E_3)^2/c^4 - \|\mathbf{p}_3\|^3/c^2 \\ &= (m_Y c^2 - E_3)^2/c^4 - (E_3^2/c^4 - m_3^2) \\ &= m_Y^2 + m_3^2 - 2m_Y E_3/c^2 \end{aligned}$$

(vi) Suppose that the 4-momenta of particle 1 and 2 in the rest frame of X are given by

$$P_1^* = (E_1^*, \mathbf{p}_1^*), \quad P_2^* = (E_2^*, \mathbf{p}_2^*)$$

Then $P_1^* + P_2^* = (m_X c^2, \mathbf{0})$. Using the energy-momentum relation,

$$\left(E_1^*\right)^2 - m_1^2c^4 = \|\mathbf{p}_1^*\|^2 = \|\mathbf{p}_2^*\|^2 = \left(E_2^*\right)^2 - m_2^2c^4$$

Substituting into the equation of conservation of energy:

$$\begin{aligned} & m_X c^2 = E_1^* + E_2^* \\ \implies & (m_X c^2 - E_2^*)^2 = \left(E_2^*\right)^2 - m_2^2c^4 + m_1^2c^4 \\ \implies & E_2^* = \frac{m_X^2 + m_2^2 - m_1^2}{2m_X}c^2 \end{aligned}$$

(vii) In fact this inequality has nothing to do with the collision problem. The Lorentz transformation from the rest frame of X to the rest frame of Y satisfies

$$\frac{E_2}{c} = \gamma\left(\frac{E_2^*}{c} + \beta\cdot\mathbf{p}_2^*\right)$$

So we have

$$\gamma(E_2^* - \beta p_2^* c) \leqslant E_2 \leqslant \gamma(E_2^* + \beta p_2^* c)$$

□

Question 9

Obtain the formula for the Compton effect using 4-vectors, starting from the usual energy-momentum conservation $P^\mu + P_e^\mu = (P')^\mu + (P_e')^\mu$.

[Hint: we would like to eliminate the final electron 4-momentum $(P_e')^\mu$, so make this the subject of the equation and square.]

A collimated beam of X-rays of energy 17.52keV is incident on an amorphous carbon target. Sketch the wavelength spectrum you would expect to be observed at a scattering angle of 90°, including a quantitative indication of the scale.

Proof. The incident photon has 4-momentum $P = \frac{\hbar\omega}{c}(1, \mathbf{e}_1)$. The stationary electron has 4-momentum $P_e = (m_e c, \mathbf{0})$. The scattered photon has 4-momentum $P' = \frac{\hbar\omega'}{c}(1, \mathbf{e}_2)$. The scattered electron has 4-momentum $P_e' = (E_e'/c, \mathbf{p}_e')$.

Conservation of 4-momentum:

$$P + P_e = P' + P_e'$$

Taking the pseudo-norm:

$$\begin{aligned} g(P_e', P_e') &= g(P + P_e - P', P + P_e - P') \\ &= g(P,P) + g(P',P') + g(P_e,P_e) + 2g(P,P_e) - 2g(P,P') - 2g(P_e,P') \end{aligned}$$

Since P and P' are 4-momenta of photons, $g(P,P) = g(P',P') = 0$. The scattered electron has unchanged rest mass, so $g(P'_e, P'_e) = g(P_e, P_e)$. The remaining terms are

$$g(P, P_e) - g(P, P') - g(P_e, P') = 0$$

Substituting the expressions

$$\hbar\omega m_e c^2 - \hbar\omega' m_e c^2 = \hbar^2 \omega\omega'(1 - \mathbf{e}_1 \cdot \mathbf{e}_2)$$

Rearranging the expression:

$$\frac{1}{\omega'} - \frac{1}{\omega} = \frac{\hbar}{m_e c^2}(1 - \mathbf{e}_1 \cdot \mathbf{e}_2) = \frac{\hbar}{m_e c^2}(1 - \cos\theta)$$

where θ is the scattering angle. The expression for wavelength is

$$\lambda' - \lambda = \frac{2\pi\hbar}{m_e c}(1 - \cos\theta)$$

At $\theta = 90°$, $\lambda' = \lambda + \frac{2\pi\hbar}{m_e c} = \frac{2\pi\hbar c}{E} + \frac{2\pi\hbar}{m_e c} = 7.33 \times 10^{-11}$ m. □