Topology & Groups

Michaelmas 2016

Question Sheet 6

Questions with an asterisk $*$ beside them are optional.

1. Recall that $G * H$ denotes the free product of groups G and H. Let $\alpha\colon G \to G * H$ be one of the canonical homomorphisms. Find a homomorphism $\pi\colon G * H \to G$ such that $\pi\alpha = \mathrm{id}_G$. Deduce that α is injective.

$*$ 2. Any element of $G * H$ is represented by a word in the alphabet $G \cup H$. We may perform the following operations to such a word, without changing the element of $G * H$ that it represents:

   (I) if successive letters g_1 and g_2 belong to G (or they both belong to H), then amalgamate them to form the letter g_3, where $g_3 = g_1 g_2$ in G (or H);

   (II) if some letter is the identity in G or H, remove it.

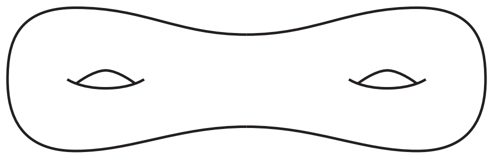
   Each of these operations shortens the word, and so eventually we will reach a stage a where they cannot be performed any further. The resulting word is $g_1 h_1 g_2 h_2 \dots g_n h_n$, where $g_i \in G$ and $h_i \in H$, and each g_i and each h_i is non-trivial, except possibly g_1 and/or h_n. We then say that this word is *reduced*. Prove that each element of $G * H$ has a *unique* reduced representative. [Hint: emulate the proof of IV.8 by formulating and proving a suitable version of IV.9.]

3. (i) Let T be the torus, which is obtained from the square by the usual side identifications. Let D be a small open disc at the centre of the square. Let X be the space obtained from T by removing D. Let ∂D be the boundary curve of D, and let b be a basepoint on ∂D. Prove that $\pi_1(X, b)$ is isomorphic to a free group on two generators.

   (ii) What word in these generators does the loop ∂D spell?

   (iii) Now let S be the 'two-holed torus' which is the surface shown on the following page. Show that S can be obtained by taking two copies of X and gluing them along the two copies of ∂D.

   (iv) Deduce that $\pi_1(S)$ is an amalgamated free product.



4. Construct simply-connected covering spaces of the following spaces:

   (i) the Möbius band,

   (ii) $S^2 \vee S^1$,

   (iii) $\mathbb{R}^2 - \{\text{point}\}$.

Topology & Groups 6

Peize Liu

1. ~~Let f~~ Suppose there are presentations :

$$G = \langle X_1 | R_1 \rangle, \; H = \langle X_2 | R_2 \rangle, \; X_1 \cap X_2 = \emptyset .$$

Let $f : X_1 \cup X_2 \to G$ defined by :

$$f(a) = a, \; f(b) = e_G \quad \forall a \in X_1, \forall b \in X_2 .$$

This induces group homomorphism : [illegible]!

$$\varphi : F(X_1 \cup X_2) \to G .$$

$\forall r_1 \in R_1, \; \varphi(r_1) = r_1 = e_G ;$

$\forall r_2 \in R_2, \; \varphi(r_2) = e_G .$

By Lemma 5.11, φ induces a group homomorphism :

$$\pi : \langle X_1 \cup X_2 | R_1 \cup R_2 \rangle \to G .$$

(By definition, $G * H \cong \langle X_1 \cup X_2 | R_1 \cup R_2 \rangle$)

We shall show that $\pi \circ \alpha = id_G$:

~~For $g \in G$, g is a word in the alphabet X_1. The canonical inclusion sends g to $\alpha(g) \in G * H$. Since $f|_{X_1} = id_{X_1}$, $\pi \circ \alpha(g)$ sends every word in X_1 to the~~

For $g \in G$, $g = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$ with $x_1, \cdots, x_n \in X_1$ and $\varepsilon_1, \cdots, \varepsilon_n \in \{1, -1\}$. $\alpha(g) = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} \in G * H$.

Since $f|_{X_1} = id_{X_1}$, we have :

$$\pi \circ \alpha(g) = f(x_1)^{\varepsilon_1} \cdots f(x_n)^{\varepsilon_n} = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} = g \in G .$$

$\Rightarrow \pi \circ \alpha = id_G$ as required.

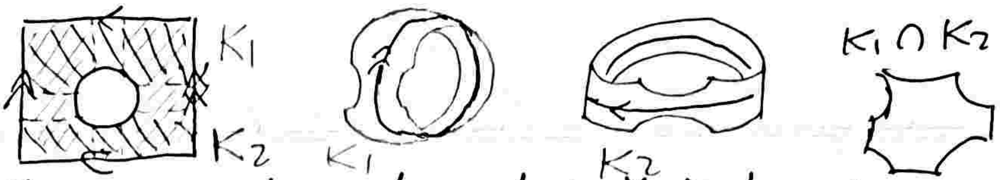
For $g_1, g_2 \in G$: $\alpha(g_1) = \alpha(g_2) \Rightarrow \pi \circ \alpha(g_1) = \pi \circ \alpha(g_2)$

$\Rightarrow g_1 = g_2$

Hence α is injective.

A-

(ii) $aba^{-1}b^{-1}$

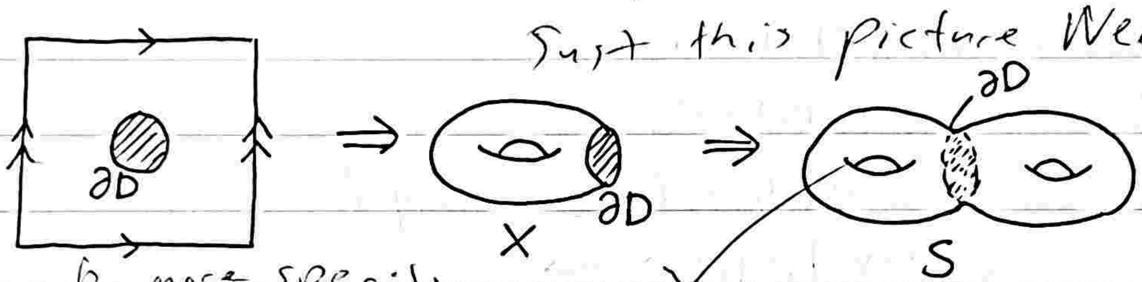


3. (i) By Q4 in Sheet 2 we have shown that X is homotopy equivalent to $S^1 \vee S^1$. Hence $\pi_1(X, b) \cong \pi_1(S^1 \vee S^1)$.
By Corollary 5.28, $\pi_1(S^1 \vee S^1) \cong F_2$.
$\Rightarrow \pi_1(X, b) \cong F_2$, the free group on 2 generators.

~~(ii) ∂D is the identity element e~~. ~~(ii) ∂D is the word a for $\pi_1(X,b)$.~~ ~~$= \langle a \rangle$~~ ~~$= \langle x, y \rangle$~~

(iii) I don't know what needs to be proven here...

Just this picture Needed



Be more specific

(iv) Consider an open set N that contains ∂D. Let the two ~~Since $N \cup$~~ copies of X be X_1 and X_2.
~~Since~~ $N \cup X_1$ and $N \cup X_2$ are path-connected and open.
X_1 is a homotopy retract of $N \cup X_1$ and X_2 is a homotopy retract of $N \cup X_2$. By Seifert-van Kampen Theorem, $\pi_1(S)$ is isomorphic to the push out of:

$$\pi_1(X_1) \xleftarrow{i_{1*}} \pi_1(N) \xrightarrow{i_{2*}} \pi_1(X_2).$$

~~$\pi_1(N) \cong \pi_1(S^1) = \mathbb{Z}$~~ ; $\pi_1(X_1) = \pi_1(X_2) = \mathbb{Z} * \mathbb{Z} = F_2$.

~~i_{1*} is induced by the inclusion $\partial D \hookrightarrow X_1$ and is~~

$\pi_1(N) \cong \pi_1(S^1) = \mathbb{Z}$.

Then $\pi_1(S) \cong F_2 *_{\mathbb{Z}} F_2$.

i_{1*} is induced by the inclusion $\partial D \hookrightarrow X_1$, hence is the
? canonical monomorphism $i_{1*}: \mathbb{Z} \to \mathbb{Z} * \mathbb{Z}$. In particular it is injective. And i_{2*} is similar. Hence $\pi_1(S)$ is an amalgamated free product.

AB

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