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Problem Sheet 1

C3.4: Algebraic Geometry

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Throughout this problem sheet, k denotes an algebraically closed field.

Section A: Introductory

Question 1. Zariski topology

Verify that arbitrary intersections and finite unions of affine varieties are affine varieties. Deduce that the Zariski topology on an affine variety is indeed a topology.

Proof. Let $\mathbb{A}^n_{\mathsf{k}}$ be the n-dimensional affine space over k and $R := \mathsf{k}[x_1, \ldots, x_n]$.

- $\mathbb{V}(\langle 0 \rangle) = \mathbb{A}^n_{\mathsf{k}}$, $\mathbb{V}(\langle 1 \rangle) = \varnothing$.
- Let I and J be two ideals in R. By definition it is clear that $\mathbb{V}(I) \cup \mathbb{V}(J) \subseteq \mathbb{V}(IJ)$. On the other hand, for $\boldsymbol{a} \notin \mathbb{V}(I) \cup \mathbb{V}(J)$, there exist $f \in I$ and $g \in J$, such that $f(\boldsymbol{a}) \neq 0 \neq g(a) \boxed{\neq 0}$. Hence $fg(\boldsymbol{a}) = 0$. As $fg \in IJ$, we deduce that $a \notin \mathbb{V}(IJ)$. Hence $\mathbb{V}(IJ) = \mathbb{V}(I) \cup \mathbb{V}(J)$. Proceed by induction to prove it for finitely many ideals.
- Let $\{J_\alpha \colon \alpha \in I\}$ be a an arbitrary family of ideals of R. We have

$$\begin{aligned} \boldsymbol{a} \in \mathbb{V}\left(\sum_{\alpha \in I} J_\alpha\right) &\iff \forall f \in \sum_{\alpha \in I} J_\alpha \colon f(\boldsymbol{a}) = 0 \\ &\iff \forall \alpha \in I \, \forall f \in J_\alpha \colon f(\boldsymbol{a}) = 0 \\ &\iff \boldsymbol{a} \in \bigcap_{\alpha \in I} \mathbb{V}(J_\alpha) \end{aligned}$$

  Hence $\mathbb{V}\left(\sum_{\alpha \in I} J_\alpha\right) = \bigcap_{\alpha \in I} \mathbb{V}(J_\alpha)$.

We conclude that the affine varieties satisfy the axioms of the closed sets of a topology on $\mathbb{A}^n_{\mathsf{k}}$. □

Question 2. Irreducibility

(a) Show that affine n-space $\mathbb{A}^n_k$ is irreducible.

(b) Show that an affine variety $X \subseteq \mathbb{A}^n_k$ is irreducible if and only if every non-empty open subset $U \subseteq X$ is dense in the Zariski topology.

(c) Let X be an irreducible affine variety. Show that any two non-empty open sets intersect in a non-empty open dense set.

Proof. (a) Suppose that $\mathbb{A}^n_{\mathsf{k}}$ is reducible. Then $\mathbb{A}^n_{\mathsf{k}} = \mathbb{V}(I) \cup \mathbb{V}(J) = \mathbb{V}(IJ)$ for non-zero ideals $I, J \in R := \mathsf{k}[x_1, \ldots, x_n]$. By Hilbert's Nullstellensatz,

$$\sqrt{\{0\}} = \mathbb{I}(\mathbb{A}^n_{\mathsf{k}}) = \mathbb{I}(\mathbb{V}(IJ)) = \sqrt{IJ}$$

Since R is an integral domain, $\{0\}$ is a prime ideal. Hence $IJ \subseteq \sqrt{IJ} = \{0\}$. We take $f \in I \setminus \{0\}$ and $g \in J \setminus \{0\}$. So $fg \in IJ \setminus \{0\}$. This is a contradiction. Hence $\mathbb{A}^n_{\mathsf{k}}$ is irreducible.

(b) Suppose that X has a non-empty open subset U that is not dense in X. Then $\overline{U}$ (taking closure with respect to the subspace topology) is a non-trivial subvariety of X. Hence X is reducible. Conversely suppose that X is reducible. Say $X = X_1 \cup X_2$. Then $X_1 = X \setminus X_2$ is open in X and its closure is equal to itself.

(c) Let U_1, U_2 be two non-empty open subsets of X. Then $X \setminus (U_1 \cup U_2) = X \setminus U_1 \cup X \setminus U_2$ is a union of two non-empty closed subsets of X. Since X is irreducible, $X \neq X \setminus U_1 \cup X \setminus U_2$. Hence $U_1 \cup X_2 = X \setminus (X \setminus U_1 \cup X \setminus U_2)$ is non-empty. By the previous result it is dense in X.
Which one is X_2? □

Section B: Core

Question 3. The Zariski topology in low dimensions

(a) List the open and closed subsets of $\mathbb{A}^1_k$ in the Zariski topology.

(b) Describe carefully the Zariski closed subsets of $\mathbb{A}^2_k$, proving your statements.

(c) Show that the Zariski topology on $\mathbb{A}^2_k$ is not the product topology on $\mathbb{A}^1_k \times \mathbb{A}^1_k$.

Proof. (a) Let X be an algebraic variety of $\mathbb{A}^1_{\mathsf{k}}$. Then $X = \mathbb{V}(\langle f \rangle)$ for some $f \in \mathsf{k}[x]$, as $\mathsf{k}[x]$ is a principal ideal domain. Hence $X = \{x_1, \dots, x_n\}$ is exactly the set of roots of f (if $f \neq 0$). We deduce that the closed subsets of $\mathbb{A}^1_{\mathsf{k}}$ are all finite subsets of $\mathbb{A}^1_{\mathsf{k}}$, and $\mathbb{A}^1_{\mathsf{k}}$ itself. Correspondingly, the open subsets of $\mathbb{A}^1_{\mathsf{k}}$ are all co-finite subsets of $\mathbb{A}^1_{\mathsf{k}}$, and $\varnothing$.

(b) We say that $C \subseteq \mathbb{A}^2_{\mathsf{k}}$ is an **irreducible affine plane curve**, if $C = \mathbb{V}(\langle f \rangle)$, where $f \in \mathsf{k}[x, y]$ is irreducible. We claim that the closed subsets of $\mathbb{A}^2_{\mathsf{k}}$ are generated by irreducible affine curves. More specifically, the closed subsets are given by

(1) $\mathbb{A}^2_{\mathsf{k}}$;

(2) finite subsets of $\mathbb{A}^2_{\mathsf{k}}$;

(3) finite unions of finitely many irreducible affine plane curves;

(4) finite unions of the sets in (2) and (3).

Let X be a proper affine variety in $\mathbb{A}^2_{\mathsf{k}}$. Let $X = \mathbb{V}(I)$. Since $\mathsf{k}[x, y]$ is Noetherian, $I = \langle f_1, \dots, f_n \rangle$ for $f_1, \dots, f_n \in \mathsf{k}[x, y]$ and hence

$$X = \mathbb{V}(\langle f_1, \dots, f_n \rangle) = \mathbb{V}\left(\sum_{i=1}^{n} \langle f_i \rangle\right) = \bigcap_{i=1}^{n} \mathbb{V}(\langle f_i \rangle)$$

Since $\mathsf{k}[x, y]$ is a unique factorisation domain, $f_i = \prod_{j=1}^{m_i} g_j$ for irreducible polynomials $g_1, \dots, g_{m_i} \in \mathsf{k}[x, y]$. Then

$$X = \bigcap_{i=1}^{n} \mathbb{V}(\langle f_i \rangle) = \bigcap_{i=1}^{n} \mathbb{V}\left(\prod_{j=1}^{m_i} \langle g_j \rangle\right) = \bigcap_{i=1}^{n} \bigcup_{j=1}^{m_i} \mathbb{V}(\langle g_j \rangle)$$

where each $\mathbb{V}(\langle g_j \rangle)$ is an irreducible affine plane curve by definition.

Finally, if C and D are distinct irreducible affine plane curves, by the (weak form of) Bezóut's Theorem, $C \cap D$ is a finite set. Therefore we have proven that any closed subset of $\mathbb{A}^2_{\mathsf{k}}$ must have the form as claimed above.

Conversely, it is clear that all irreducible affine plane curves (and their finite unions) are closed in Zariski topology. In addition, we take $\{(x_0, y_0)\} = \mathbb{V}(\langle x - x_0, y - y_0 \rangle)$ and take finite unions to obtain all finite subsets of $\mathbb{A}^2_{\mathsf{k}}$. This proves the other direction of the claim.

(c) We consider the affine variety $X = \mathbb{V}(\langle x - y \rangle) \subseteq \mathbb{A}^2_{\mathsf{k}}$. Suppose that it is closed under the product Zariski topology of $\mathbb{A}^1_{\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}}$. Then

$$\mathbb{A}^2_{\mathsf{k}} \setminus X = \bigcup_{i=1}^{n} (Y_i \times Z_i)$$

where $Y_1, \dots, Y_n$ and $Z_1, \dots, Z_n$ are co-finite subsets of $\mathbb{A}^1_{\mathsf{k}}$ by the result in (a). As $\bigcup_{i=1}^{n} (\mathbb{A}^1_{\mathsf{k}} \setminus Y_i \cup \mathbb{A}^1_{\mathsf{k}} \setminus Z_i)$ is a finite set, we can choose $a \in \mathbb{A}^1_{\mathsf{k}}$ such that $a \notin \bigcup_{i=1}^{n} (\mathbb{A}^1_{\mathsf{k}} \setminus Y_i \cup \mathbb{A}^1_{\mathsf{k}} \setminus Z_i)$. Then $(a, a) \in \bigcup_{i=1}^{n} (Y_i \times Z_i)$. But $(a, a) \in X$ by definition. This is a contradiction. Hence $\mathbb{A}^2_{\mathsf{k}} \setminus X$ is open in the Zariski topology of $\mathbb{A}^2_{\mathsf{k}}$ but not in the product Zariski topology of $\mathbb{A}^1_{\mathsf{k}} \times \mathbb{A}^1_{\mathsf{k}}$. The two topologies on $\mathbb{A}^2_{\mathsf{k}}$ are distinct. $\square$

Question 4. Reduced algebras as coordinate rings

(a) Show that $\sqrt{I\cap J}=\sqrt{I}\cap\sqrt{J}$ for ideals I, J of a finitely generated k-algebra R.

(b) Show that the ideal $(xy, xz)\subseteq k[x,y,z]$ is radical but not prime. Sketch the variety it defines in $\mathbb{A}^3_k$.

(c) Let $X\subseteq\mathbb{A}^n_k$ be an affine variety. Show that a radical ideal in $k[X]$ is the intersection of all the maximal ideals containing it.

(Hint: using methods of this course, it is easier to first translate this into a geometrical statement, and prove that. For an algebraic proof, you might find helpful the following theorem due to Krull: the nilradical $\mathrm{nil}(A)=\{x : x^m=0 \text{ some } m\}$ of a ring A equals the intersection of all its prime ideals.)

Proof. (a) This is true for any CRI (commutative ring with identity) R.

For $f\in\sqrt{I\cap J}$, $f^n\in I\cap J$ for some $n\in\mathbb{N}$. Hence $f\in\sqrt{I}$ and $f\in\sqrt{J}$. We deduce that $\sqrt{I\cap J}\subseteq\sqrt{I}\cap\sqrt{J}$.

Conversely, for $f\in\sqrt{I}\cap\sqrt{J}$, we have $f^n\in I$ and $f^m\in J$ for some $n,m\in\mathbb{N}$. Then $f^{n+m}\in I\cap J$. Hence $f\in\sqrt{I\cap J}$. We deduce that $\sqrt{I}\cap\sqrt{J}\subseteq\sqrt{I\cap J}$.

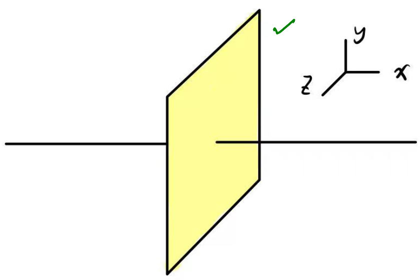
or nm or max(n,m)

(b) Note that $\langle xy, xz\rangle=\langle x\rangle\cap\langle y,z\rangle$, where both $\langle x\rangle$ and $\langle y,z\rangle$ are prime, and hence and radical. We have

$$\sqrt{\langle xy,xz\rangle}=\sqrt{\langle x\rangle\cap\langle y,z\rangle}=\sqrt{\langle x\rangle}\cap\sqrt{\langle y,z\rangle}=\langle x\rangle\cap\langle y,z\rangle=\langle xy,xz\rangle$$

Hence $\langle xy,xz\rangle$ is radical. But it is not prime, as $x,y\notin\langle xy,xz\rangle$ and $xy\in\langle xy,xz\rangle$.

The variety $\mathbb{V}(\langle xy,xz\rangle)$ is just $\{x=0\}\cup\{y=z=0\}$.



(c) First we prove this when k is algebraically closed.

Let $\pi:\mathsf{k}[x_1,\dots,x_n]\to\mathsf{k}[X]:=\mathsf{k}[x_1,\dots,x_n]/I$ be the canonical projection. Let $\widetilde{J}$ be a radical ideal on $k[X]$ and $J:=\pi^{-1}(\widetilde{J})$ be its preimage in $\mathsf{k}[x_1,\dots,x_n]$. Then $\mathsf{k}[X]/J=\mathsf{k}[x_1,\dots,x_n]/(I+J)$. By an immediate corollary of Hilbert's weak Nullstellensatz, the maximal ideals of $\mathsf{k}[X]/J$ are of the form $\langle x_1-a_1,\dots,x_n-a_n\rangle+I+J$, $(a_1,\dots,a_n)\in\mathbb{A}^n_{\mathsf{k}}$. Hence the Jacobson radical of $\mathsf{k}[X]/J$ is $\{0\}$. This implies that J is the intersection of all maximal ideals of $\mathsf{k}[X]$ containing J.

Next we prove this for any general field k. We say that a CRI R is a **Jacobson ring**, if the radical and Jacobson radical of any ideal $I\lhd R$ coincide. The result of (c) follows immediately from the following (in fact stronger) lemma:

Lemma 1

Any finitely generated k-algebra is a Jacobson ring.

(*See also Corollary 9.4 of B2.2 Commutative Algebra (2020-2021)*.)

Let R be a finitely generated k-algebra. It suffices to show that the nilradical $N(R)$ and Jacobson radical $J(R)$

of R coincide. Let $f \notin N(R)$. Consider the localisation R_f on $\{f^n : n \in \mathbb{N}\}$, which is non-zero. Let M be a maximal ideal of R_f. Consider the composition of canonical homomorphisms:

$$R \xrightarrow{\varphi} R_f \xrightarrow{\pi} R_f/M$$

Let $\psi = \pi \circ \varphi$. Since R_f is finitely generated k-algebra, so is R_f/M. But R_f/M is also a field. Then R_f/M is a finite field extension of k, by Hilbert's weak Nullstellensatz[1], and hence is integral over k. Then $\operatorname{im}\psi$ is also integral over k. Hence $\operatorname{im}\psi$ is also a field. By first isomorphism theorem, $\ker\psi$ is a maximal ideal of R. Note that $\varphi(f) \neq 0$ because $f/1$ is a unit in R_f. Hence $f \notin \ker\psi \supseteq J(R)$. We conclude that $J(R) = N(R)$. □

Question 5. The pull-back map between coordinate rings

Suppose that $F : X \to Y$ is a morphism of affine varieties over a field k, associated to a map $F^* : k[Y] \to k[X]$ between their coordinate rings.

(a) Show that F^* is injective if and only if F is dominant, i.e. the image set $F(X)$ is dense in Y.

(b) Show that F^* is surjective if and only if F defines an isomorphism between X and some algebraic subvariety of Y.

(c) Find an example where F is injective but F^* is not surjective.

Proof. (a) Suppose that $F(X)$ is not dense in Y. Then $\overline{F(X)} = Z \subsetneq Y$, where $Z = \mathbb{V}(I)$ is a proper subvariety of $Y = \mathbb{V}(J)$. Take $f \in I \setminus J$ and let $\overline{f}$ be the image of f in $k[Y]$. Then $F^*(\overline{f}) = 0$ and $\overline{f} \neq 0$. Hence F^* is injective.

Conversely, suppose that F^* is not injective. Let $f \in \ker F^* \setminus \{0\}$. We note that $U := \{\boldsymbol{b} \in Y : f(\boldsymbol{b}) \neq 0\}$ is an open set of Y. Moreover, for $\boldsymbol{b} = F(\boldsymbol{a}) \in F(X)$, $f(\boldsymbol{b}) = f \circ F(\boldsymbol{a}) = F^*(f)(\boldsymbol{a}) = 0$. Hence $F(X) \cap U = \varnothing$. We deduce that $F(X)$ is not dense in Y.

(b) Suppose that Z is a subvariety of Y such that $F : X \to Z \subseteq Y$ is an isomorphism. Then we know that $k[X] \cong k[Z]$. Then F^* factors through $k[Z]$ via:

$$F^* :\ k[Y] \xrightarrow{\iota^*} k[Z] \xrightarrow{\simeq} k[X]$$

($Z \overset{i}{\hookrightarrow} Y$ closed embedding)

Hence F^* is surjective.

Conversely, suppose that F^* is surjective. By first isomorphism theorem, $k[X] \cong k[Y]/\ker F^*$. Let J be the preimage of $\ker F^*$ in $k[y_1,\ldots,y_n]$. We claim that F defines an isomorphism from X to $Z := Y \cap \mathbb{V}(J) \subseteq Y$. For $f \in J$, $F^*(f) = f \circ F = 0$. Hence $F(X) \subseteq \mathbb{V}(J)$. So F indeed maps into $Y \cap \mathbb{V}(J)$. Moreover, the pull-back $\widetilde{F}^* : k[Z] \cong k[Y]/\ker F^* \to k[X]$ is an isomorphism of rings. Hence $\widetilde{F} : X \to Z$ is an isomorphism of varieties.

(c) Let $X = \mathbb{V}(\langle xy - 1\rangle) \subseteq \mathbb{A}^2_k$ and $Y = \mathbb{A}^1_k$. Let $F : X \to Y$ be a morphism given by $(x, y) \mapsto x$. F is injective because every point on X is of the form (x, x^{-1}) for $x \notin \{0\}$. But $F(X) = \mathbb{A}^1_k \setminus \{0\}$ is not a subvariety of Y. By (b) F^* is not surjective. □

Question 6. The affine normal curve

Consider the homomorphism of rings

$$F^* : k[x_0, \ldots, x_{n-1}] \to k[t]$$

given by $x_i \mapsto t^i$.

(a) Show that the corresponding morphism of affine varieties $F : \mathbb{A}^1_k \to \mathbb{A}^n_k$ defines an isomorphism between $\mathbb{A}^1_k$ and its image under F.

[1] The version of weak Nullstellensatz we are using states that, if R is a finitely generated k-algebra and also a field, then R is finite over k.

(b) Find generators for the ideal defining the image of F in $\mathbb{A}^n_k$.

Proof. (a) It is clear that F^* is surjective because

$$F^*\left(\sum_{k=0}^{n} a_k x_1^k x_0^{n-k}\right) = \sum_{k=0}^{n} a_k t^k$$

By Question 5.(b), F is an isomorphism between $\mathbb{A}^1_k$ and $F(\mathbb{A}^1_k)$. $=\{-\}$

(b) We claim that $\mathbb{I}(F(\mathbb{A}^1_k)) = \langle x_0 - 1, x_2 - x_1^2, ..., x_{n-1} - x_1^{n-1}\rangle = \ker F^*$. I believe that it is self-evident and there is nothing non-trivial that needs to prove here... □

Question 7. A reducible variety

Consider the ideal

$$J = \langle uw - v^2, u^3 - vw\rangle$$

in the ring $k[u, v, w]$, and the corresponding affine variety $X = \mathbb{V}(J) \subseteq \mathbb{A}^3_k$.

(a) By taking suitable combinations of the generators, show that J is not prime.

(b) Show that X is a reducible variety, which decomposes as

$$X = X_1 \cup X_2$$

with one component, say X_1 isomorphic to the affine line $\mathbb{A}^1_k$.

(c) Show that the other piece X_2 is the image of a map $\mathbb{A}^1_k \to \mathbb{A}^3_k$ defined by $t \mapsto (t^a, t^b, t^c)$ for some positive integers a, b, c. Deduce that X_2 is irreducible.

Proof. (a) We have $v(w^2 - u^2 v) = u^2(uw - v^2) - w(u^3 - vw) \in J$. It is clear that $v \notin J$, as any polynomial in J cannot contain linear terms. Similarly, $w^2 - u^2 v \notin J$, as w^2 cannot appear as a term in a polynomial in J. We deduce that J is not a prime ideal.

(b) For $(a, b, c) \in X = \mathbb{V}(J)$, we have $ac - b^2 = 0$ and $a^3 - bc = 0$. If $a = 0$, then $a = b = 0$. If $a \neq 0$, then $b \neq 0$. And we have $b^6 = a^3c^3 = bc^4$ and hence $b^5 = c^4$. Then $b^8 = a^4c^4 = a^4b^5$ and hence $a^4 = b^3$. In summary, we have $X = X_1 \cup X_2$, where

$$X_1 = \{a = b = 0\}, \qquad X_2 = \{(t^3, t^4, t^5) : t \in k\}$$

We note that $\mathbb{A}^1_k \cong X_1$, with the isomorphism given by $t \mapsto (0, 0, t)$.

(c) We have found X_2 in (b). The map $t \mapsto (t^3, t^4, t^5) \in X_2$ is an isomorphism. Hence $X_2 \cong \mathbb{A}^1_k$ is irreducible. □

Section C: Optional

Keep up the excellent work. Seek clarifications if any.

Question 8. The disjoint union of affine varieties

Show that a variety $X \subseteq \mathbb{A}^n_k$ is a union of two disjoint closed subvarieties if and only if its coordinate ring $k[X]$ may be written as the product of two non-trivial finitely generated reduced k-algebras.

(*Hint: recall the algebraic form of the Chinese Remainder Theorem: if I_1, I_2 are coprime ideals in a ring R, meaning $I_1 + I_2 = R$, then $I_1 \cap I_2 = I_1 \cdot I_2$ and there is a ring isomorphism $R/(I_1 \cap I_2) \to R/I_1 \times R/I_2$ given by $f \mapsto (f + I_1, f + I_2)$.*)

Proof. Let $R := k[x_1, .., x_n]$. Suppose that $X = \mathbb{V}(I) \cup \mathbb{V}(J)$ such that $\mathbb{V}(I) \cap \mathbb{V}(J) = \varnothing$. Then $\mathbb{V}(I + J) = \varnothing$ and hence $I + J = R$. The ideals I and J are coprime. Since $I + J \subseteq \sqrt{I} + \sqrt{J}$, $\sqrt{I}$ and $\sqrt{J}$ are also coprime. By Chinese Remainder

Theorem,

$$k[X] = k[\mathbb{V}(I) \cup \mathbb{V}(J)] = k[\mathbb{V}(IJ)] = k[\mathbb{V}(I \cap J)] = \frac{R}{\sqrt{I \cap J}} = \frac{R}{\sqrt{I} \cap \sqrt{J}} \cong \frac{R}{\sqrt{I}} \times \frac{R}{\sqrt{J}} = k[\mathbb{V}(I)] \times k[\mathbb{V}(J)]$$

We deduce that $k[X]$ is isomorphic to the product $R/\sqrt{I} \times R/\sqrt{J}$, and both $R/\sqrt{I}$ and $R/\sqrt{J}$ are reduced by the definition of the radical ideal.

Conversely, suppose that $k[X] \cong S \times T$, where S, T are finitely generated reduced k-algebra. The inclusion $S \hookrightarrow k[X] = k[x_1, ..., x_n]/\mathbb{I}(X)$ implies that S can be generated by n elements, and hence can be realised as a quotient of $k[x_1, ..., x_n]$. That is, $S \cong k[x_1, ..., x_n]/I$. Similarly, $T \cong k[x_1, ..., x_n]/J$. Since S, T are reduced, I, J are radical ideals. So S and T are in fact coordinate rings of $\mathbb{V}(I)$ and $\mathbb{V}(J)$. We claim that $\mathbb{V}(I)$ and $\mathbb{V}(J)$ are disjoint. Suppose that $\boldsymbol{a} \in \mathbb{V}(I) \cap \mathbb{V}(J)$. For $f = (0, 1) \in S \times T$, we have $f(\boldsymbol{a}) = 0$ as $a \in \mathbb{V}(I)$. Similarly, $g = (1, 0) \in S \times T$ satisfies that $g(\boldsymbol{a}) = 0$. But $f(\boldsymbol{a}) + g(\boldsymbol{a}) = 1(\boldsymbol{a}) = 1 \neq 0$. This is a contradiction, which proves our claim. Now $\mathbb{V}(I) \cap \mathbb{V}(J) = \varnothing$ implies that $I + J = R$. By Chinese Remainder Theorem again, we have

$$k[X] \cong S \times T \cong \frac{R}{I \cap J} = k[\mathbb{V}(I) \cup \mathbb{V}(J)]$$

Hence $X \cong \mathbb{V}(I) \cup \mathbb{V}(J)$ is a disjoint union of two non-empty subvarieties. □

Question 9. The variety of nilpotent matrices

We work in the affine space $\mathbb{A}^4$ parametrising 2×2 matrices over k, with variables being the matrix entries x_{ij}.

(a) Prove that the following conditions are equivalent for a 2×2 matrix A over a field k :

(1) A is nilpotent: there exists an $n \geqslant 1$ such that $A^n = 0$;

(2) $A^2 = 0$;

(3) $\det A = \operatorname{tr} A = 0$.

Let $I \lhd R = k[x_{11}, x_{12}, x_{21}, x_{22}]$ be the ideal formed by the polynomials $d = \det A$, $t = \operatorname{tr} A$, viewed as polynomials in the matrix entries. Let $J \lhd R$ be the ideal formed by the entries of A^2, as polynomials in the matrix entries. Show the following.

(b) The ideal J is not radical: it contains a power of t but not t itself.

(c) The ideal I is radical.

(*Hint: aim to show that I is prime and therefore radical. Show this by mapping R/I to an isomorphic ring using the linear generator in I.*)

(d) Deduce that $X = \mathbb{V}(I) = \mathbb{V}(J) \subseteq \mathbb{A}^4$ with $\sqrt{J} = I$, and conversely $\mathbb{I}(X) = I$.

Proof. (a) We embed k into its algebraic closure $\overline{k}$ and make the identification $A \in M_{2\times2}(\overline{k})$. Note that any non-zero nilpotent matrix in $M_{2\times2}(\overline{k})$ is similar to its Jordan normal form:

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

So we have (1) $\iff$ (2) $\implies$ (3).

Suppose that $\det A = \operatorname{tr} A = 0$. Then the characteristic polynomial of A is given by

$$\chi_A(x) = x^2 - (\operatorname{tr} A)x + \det A = x^2$$

Hence $A^2 = 0$ by Cayley-Hamilton theorem. We have (3) $\implies$ (2).

(b) We write $x = x_{11}.y = x_{12}, z = x_{21}, w = x_{22}$ for simplicity. Then

$$A^2 = \begin{pmatrix} x & y \\ z & w \end{pmatrix}^2 = \begin{pmatrix} x^2 + yz & y(x+w) \\ z(x+w) & w^2 + yz \end{pmatrix}$$

So $J = \langle x^2 + yz, w^2 + yz, y(x+w), z(x+w) \rangle$. Note that

$$t^2 = (x+w)^2 = (x^2 + yz) + (w^2 + yz) + 2(xw - yz) = \operatorname{tr} A^2 + 2 \det A$$

Hence

$$t^4 = (\operatorname{tr} A^2)^2 + 4 \operatorname{tr} A^2 \det A + 4(\det A)^2 = (\operatorname{tr} A^2)^2 + 4 \operatorname{tr} A^2 \det A + 4 \det A^2$$

As $\operatorname{tr} A^2, \det A^2 \in J$, we deduce that $t^4 \in J$. It is clear that $t \notin J$, as the generators of J are homogeneous of degree 2. Hence J is not a radical ideal.

(c) We have $I = \langle d, t \rangle = \langle xw - yz, x + w \rangle$. We have the ring isomorphism

$$R/I \cong k[s,t,u]/\langle st - u^2 \rangle$$

given by the map $x \mapsto u$, $y \mapsto s$, $z \mapsto t$, $w \mapsto -u$. Note that $k[s,t,u]$ is a unique factorisation domain, and $st - u^2$ is an irreducible polynomial. Hence $\langle st - u^2 \rangle$ is a prime ideal of $k[s,t,u]$. Then R/I is an integral domain. Hence I is a prime ideal of R. We deduce that I is radical.

(d) We have $d^2 = \det A^2 \in J$ and $t^4 \in J$. Hence $d, t \in \sqrt{J}$ and then $I \subseteq \sqrt{J}$. But by (c) I is radical. Therefore $I = \sqrt{J}$. We have $\mathbb{V}(I) = \mathbb{V}(J)$. Finally, if k is algebraically closed, then by Hilbert's strong Nullstellensatz, we have

$$\mathbb{I}(X) = \mathbb{I}(\mathbb{V}(()I)) = \sqrt{I} = I$$

□