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Problem Sheet 4
A5: Topology

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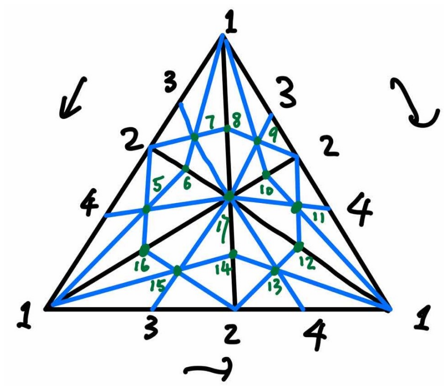
Question 1

(1) Show that the Dunce hat can be triangulated.

(2) Show that the following subspace of $\mathbb{R}^2$ cannot be triangulated:

$$\{(x,y) :\ 0 \leqslant y \leqslant 1,\ x = 0 \text{ or } 1/n \text{ for some } n \in \mathbb{N}\} \cup ([0,1] \times \{0\})$$

Proof. (1) The Dunce hat is a finite CW complex so it can be triangulated. The triangulation can be obtained by repeating barycentric subdivision until the space becomes a simplicial complex. A triangulation (which is not the simplest) of the Dunce hat is as follows:



(2) Step 1: *An infinite simplicial complex is not compact.*

Suppose that $K = (V, \Sigma)$ is an abstract simplicial complex and $|K|$ is its topological realization. For $v \in V$, by Lemma 4.18 in the notes, $\mathrm{st}_K(v)$ is an open set in K that contains a unique vertex v. If V is infinite, then $\{\mathrm{st}_K(v)\}_{v \in V}$ is an infinite open cover of K with no finite subcover. Hence K is not compact.

Step 2: *A finite simplicial complex is locally connected.*

Let $K = (V, \Sigma)$ be a finite simplicial complex. By Proposition 4.22 in the notes, there is a (continuous) embedding $\iota : |K| \hookrightarrow \mathbb{R}^n$ for $n = |V|$, in which every vertex is mapped to a unit vector in $\mathbb{R}^n$. Notice that $\{B(\boldsymbol{x}, \varepsilon) :\ \boldsymbol{x} \in \mathbb{R}^n,\ \varepsilon > 0\}$ is a topological basis of $\mathbb{R}^n$. Then the preimage $\{B(\boldsymbol{x}, \varepsilon) \cap |K| :\ \boldsymbol{x} \in \mathbb{R}^n,\ \varepsilon > 0\}$ is a basis of $|K|$ under the subset topology.

For $\boldsymbol{x} \in |K|$ and open set U of $|K|$ with $\boldsymbol{x} \in U$, there exists $\boldsymbol{x}_0 \in \mathbb{R}^n$ and $\varepsilon > 0$ such that $\boldsymbol{x} \in B(\boldsymbol{x}_0, \varepsilon) \cap |K| \subseteq U$. Let $0 < \delta < \|\boldsymbol{x} - \boldsymbol{x}_0\|$. Then $\boldsymbol{x} \in B(\boldsymbol{x}, \delta) \cap |K| \subseteq U$. Let $0 < \eta < \min\{x_1, ..., x_n, \delta\}$. We claim that for any simplex $\sigma \in \Sigma$, $|\sigma| \cap B(\boldsymbol{x}, \eta) \neq \varnothing$ if and only if $\boldsymbol{x} \in |\sigma|$.

Suppose that $\sigma = \{\boldsymbol{e}_{i_1}, ..., \boldsymbol{e}_{i_m}\}$. Then $|\sigma|$ lies in the subspace $\mathrm{span}\{\boldsymbol{e}_{i_1}, ..., \boldsymbol{e}_{i_m}\}$. If $\boldsymbol{x} \notin |\sigma|$, then $x \notin \mathrm{span}\{\boldsymbol{e}_{i_1}, ..., \boldsymbol{e}_{i_m}\}$ and $\mathrm{dist}(\boldsymbol{x}, \mathrm{span}\{\boldsymbol{e}_{i_1}, ..., \boldsymbol{e}_{i_m}\}) = \min\{x_{i_1}, ..., x_{i_m}\} > \eta$.

For $\sigma \in \Sigma$ with $\boldsymbol{x} \in |\sigma|$, $|\sigma|$ and $B(\boldsymbol{x}, \varepsilon)$ are both convex. Hence $B(\boldsymbol{x}, \eta) \cap |\sigma|$ is convex and is connected. In particular $B(\boldsymbol{x}, \eta) \cap |K|$ has a unique connected component. $B(\boldsymbol{x}, \eta) \cap |K| \subseteq U$ is connected.

Step 3: *The given subspace is compact and is not locally connected.*

Let S be the given subspace in the question. Notice that the sequence $\{1/n\}_{n \in \mathbb{Z}_+}$ has a unique limit point 0. Hence $\{0\} \cup \{1/n : n \in \mathbb{Z}_+\}$ is closed. The product set $(\{0\} \cup \{1/n : n \in \mathbb{Z}_+\}) \times [0,1]$ is also closed. S is the union of it with $0,1] \times \{0\}$ so S is closed. Obviously S is bounded. By Heine-Borel Theorem S is compact.

Consider the open ball B centred at $(0,0)$ with radius $1/2$. For any open subset B' of B, $B' \cap S$ is disconnected, Hence S is not locally connected.

The three steps together lead to the conclusion that S cannot be triangulated. $\square$

Question 2

Let K be a simplicial complex (that need not be finite). Prove that $|K|$ is Hausdorff.

Proof. For each $\sigma \in K$, we consider the standard n-simplex Δ_σ^n. Let L be the disjoint union of all Δ_σ^n. Then $|K|$ is a quotient space of L. Let $\pi : L \twoheadrightarrow |K|$ be the canonical projection. By Proposition 3.21 in the notes, $|K|$ is Hausdorff if and only if any two distinct equivalence classes of L are contained in two disjoint open saturated sets.

By Lemma 4.12, any point in $|K|$ is contained in the inside of a unique simplex. For $[x], [y] \in |K|$, there exists $x, y \in L$ and unique simplices $\Delta_x, \Delta_y \subseteq L$ such that $x \in \operatorname{int}(\Delta_x^m)$ and $y \in \operatorname{int}(\Delta_y^n)$. ($\pi(x) = [x], \pi(y) = [y]$) Without loss of generality suppose that $m \leqslant n$. We shall construct an ascending chain of open neighbourhoods of x and of y which are disjoint and saturated. We use induction on the dimension.

Let Σ_r be the disjoint union of all simplices in K with dimension $\leqslant r$. For $r = 0$, $[x] \cap \Sigma_0 = \{x\}$ if $m = 0$ or $[x] \cap \Sigma_0 = \varnothing$ if $m > 0$. Clearly $[x] \cap \Sigma_0$ and $[y] \cap \Sigma_0$ are disjoint because x and y are distinct.

Suppose that we have constructed disjoint saturated open sets U_r, V_r such that $[x] \cap \Sigma_r \subseteq U_r$ and $[y] \cap \Sigma_r \subseteq V_r$. And $\pi(U_i) \subseteq \pi(U_{i+1}), \pi(V_i) \subseteq \pi(V_{i+1})$ for $0 \leqslant i < r$. For each Δ_σ^{r+1}, let

$$I_\sigma := \{z \in \Delta_\sigma^{r+1} : \pi(z) \in \pi(U_r)\} \qquad J_\sigma := \{z \in \Delta_\sigma^{r+1} : \pi(z) \in \pi(V_r)\}$$

A careful choice of U_r and V_r may let us assume that $\overline{U_r} \cap \overline{V_r} = \varnothing$. Then $\overline{I_\sigma}$ and $\overline{J_\sigma}$ are separated. Since Δ_σ^{r+1} is T4, there exists disjoint open sets M_σ, N_σ such that $I_\sigma \subseteq M_\sigma$ and $J_\sigma \subseteq N_\sigma$. Let $U_{r+1} = \bigcup_\sigma M_\sigma \cup U_r$ and $V_{r+1} = \bigcup_\sigma N_\sigma \cup V_r$. Clearly U_{r+1} and V_{r+1} are disjoint, which completes the induction.

Finally, let $U = \bigcup_{r=0}^\infty U_r$ and $V = \bigcup_{r=0}^\infty V_r$. U and V are disjoint saturated open neighbourhoods of $[x]$ and $[y]$ respectively. Hence $|K|$ is Hausdorff. $\square$

Question 3

Let X_1, X_2 be disjoint copies of $\mathbb{R}^2$. We define an equivalence relation $\sim$ on $Y = X_1 \sqcup X_2$ by:

$$(x_1, y_1) \in X_1 \sim (x_2, y_2) \in X_2 \iff x_1 = x_2,\ y_1 = y_2,\ (x_1, y_1), (x_2, y_2) \neq (0, 0)$$

Show that every point in $Y/\sim$ is contained in an open set homeomorphic to an open subset of $\mathbb{R}^2$ but $Y/\sim$ is not a surface.

Proof. For simplicity we write $Y = \mathbb{R}^2 \times \{0, 1\}$. For $[(x, y)] \in Y/\sim$ with $(x, y) \neq (0, 0)$, there exists an open neighbourhood U of (x, y) such that $(0, 0) \notin U$. Let $\pi : Y \twoheadrightarrow Y/\sim$ be the canonical projection and $\pi'(x, y) = \pi(x, y, 0)$. $\pi'(U)$ is an open neighbourhood of $[(x, y)]$ because $\pi^{-1}(\pi'(U)) = U \times \{0, 1\}$ is open in Y. We claim that $\pi'(U)$ is homeomorphic to U. Clearly $\pi'|_U : U \to \pi(U)$ is a continuous bijection. For an open subset $V \subseteq U$, $\pi'(V)$ is open for the similar reason that $\pi'(U)$ is open. In particular, $\pi'|_U$ is an open mapping and is a homeomorphism.

The only singleton equivalence classes in $Y/\sim$ are $\{(0, 0, 0)\}$ and $\{(0, 0, 1)\}$. It suffices to consider one of them. Let U be an open neighbourhood of $(0, 0)$ in $\mathbb{R}^2$. $\pi'(U)$ is an open neighbourhood of $\{(0, 0, 0)\}$ because

$$\pi^{-1}(\pi'(U)) = (U \times \{0\}) \cup ((U \backslash \{(0, 0)\}) \times \{1\})$$

is the union of two open sets. Similarly we have $\pi'(U)$ is homeomorphic to U. Hence we have proven the first half of the statement in the question.

Since $\mathbb{R}^2 \backslash \{(0, 0)\}$ is not closed, from Question 5.(1).(b) we know that the quotient space $Y/\sim$ is not Hausdorff. Hence $Y/\sim$ is not a surface. $\square$

Question 4

Find an example of a connected, finite, simplicial complex K that is not a closed combinatorial surface, but that satisfies the following three conditions:

(1) It contains only 0-simplices, 1-simplices and 2-simplices.

(2) Every 1-simplex is a face of precisely two 2-simplices.

(3) Every point of $|K|$ lies in a 2-simplex.

Proof. We define an abstract simplicial complex $K = (V, \Sigma)$ by:

$$\begin{aligned} V &= \{1,2,3,4,5,6,7\} \\ \Sigma &= \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\},\{4,5\},\{4,6\},\{4,7\},\{5,6\},\{5,7\},\{6,7\}, \\ &\qquad \{1,2,3\},\{1,3,4\},\{1,2,4\},\{2,3,4\},\{4,5,6\},\{4,5,7\},\{4,6,7\},\{5,6,7\}\} \end{aligned}$$

In other words, K is two tetrahedrons gluing together at one vertex.

It is immediate that all three conditions are satisfied. However, $|K|$ is not a closed combinatorial surface. We observe that the link $\operatorname{link}_K(4) = (V', \Sigma')$, where:

$$\begin{aligned} V' &= \{1,2,3,5,6,7\} \\ \Sigma' &= \{\{1,2\},\{1,3\},\{2,3\},\{5,6\},\{5,7\},\{6,7\}\} \end{aligned}$$

Σ' is the union of two disjoint simplicial circles. Then $|K|$ cannot be a closed combinatorial surface. □

Question 5

A simple closed curve C in a space X is the image of a continuous injection $S^1 \to X$. Find simple closed curves C_1, C_2 and C_3 in the Klein bottle K such that

(1) $K \backslash C_1$ has one component, which is homeomorphic to an open annulus $S^1 \times (0,1)$.

(2) $K \backslash C_2$ has one component, which is homeomorphic to an open Möbius band.

(3) $K \backslash C_3$ has two components, each of which is homeomorphic to an open Möbius band.

Proof. The Klein bottle K is defined by the unit square $A = [0,1]^2$ with side identifications $(0,y) \sim (1,y)$ and $(x,0) \sim (1-x,1)$. Let $\pi : A \twoheadrightarrow K$ be the projection.

(1) Define $\gamma_1 : [0,1] \to A$ by $\gamma_1(t) = (t, 1/2)$. It is easy to see that $C_1 := \pi \circ \gamma_1([0,1])$ is a simple closed curve on K as $(0,1/2)$ and $(1,1/2)$ are identified on K.

$K \backslash C_1$ is path-connected, since for any $x \in A \backslash \gamma_1([0,1])$, there is a straight line in A from x to one of $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$, which are identified on K.

Let $\sigma_1 : A \backslash \gamma_1([0,1]) \to S^1 \times (0,1)$ be given by

$$\sigma_1(x,y) = \begin{cases} \left(\mathrm{e}^{2\pi \mathrm{i} x}, y + 1/2\right) & 0 \leqslant y < 1/2 \\ \left(\mathrm{e}^{2\pi \mathrm{i}(1-x)}, y - 1/2\right) & 1/2 < y \leqslant 1 \end{cases}$$

Clearly A is compact and $S^1 \times (0,1)$ is Hausdorff. The equivalence classes on $A \backslash \gamma_1([0,1])$ coincides with the partition $\{\sigma_1^{-1}(\alpha) : \alpha \in S^1 \times (0,1)\}$. By Proposition 3.11, $K \backslash C_1$ and $S^1 \times (0,1)$ are homeomorphic.

(2) Define $\gamma_2 : [0,1] \to A$ by $\gamma_1(t) = (1/2, t)$. It is easy to see that $C_2 := \pi \circ \gamma_2([0,1])$ is a simple closed curve on K as $(1/2, 0)$ and $(1/2, 1)$ are identified on K.

$K \backslash C_2$ is path-connected for the same reason as above.

Let $\sigma_2 : A \backslash \gamma_2([0,1]) \to (0,1) \times [0,1]$ be given by

$$\sigma_2(x,y) = \begin{cases} (x + 1/2, y) & 0 \leqslant x < 1/2 \\ (x - 1/2, y) & 1/2 < x \leqslant 1 \end{cases}$$

The equivalence classes on $A \backslash \gamma_2([0,1])$ coincides with the equivalence classes on $(0,1) \times [0,1]$ induced by the Möbius band. By Proposition 3.11, $K \backslash C_2$ is homeomorphic to an open Möbius band.

(3) Define $\gamma_3 : [0,1] \to A$ by

$$\gamma_3(t) = \begin{cases} (1/3, 2t) & 0 \leqslant t \leqslant 1/2 \\ (2/3, 2t - 1) & 1/2 \leqslant t \leqslant 1 \end{cases}$$

$C_3 := \pi \circ \gamma_3([0,1])$ is a simple closed curve on K because $(1/3, 1)$ is identified with $(2/3, 0)$ and $(2/3, 1)$ is identified with $(1/3, 0)$.

$K \backslash C_3$ has two connected components. This is obvious in intuition but very hard to argue rigorously.

Define $\sigma_3 : A \backslash \gamma_3([0,1]) \to (0,1) \times [0,1]$

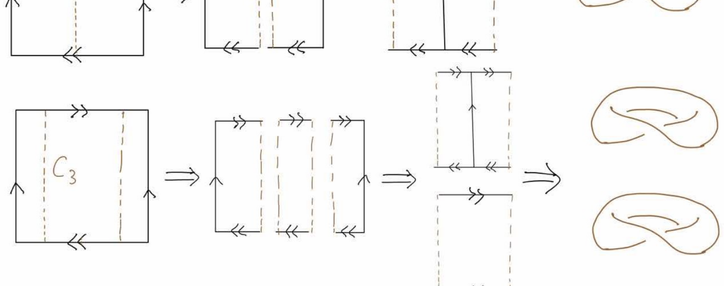
$$\sigma_3(x,y) = \begin{cases} \left(\frac{3}{2}x + \frac{1}{2}, y\right) & 0 \leqslant x < 1/3 \\ \left(\frac{3}{2}x - 1, y\right) & 2/3 < x \leqslant 1 \end{cases}$$

and $\theta_3 : A \backslash \gamma_3([0,1]) \to (0,1) \times [0,1]$

$$\theta_3(x,y) = (3x - 1, y) \quad 1/3 < x < 2/3$$

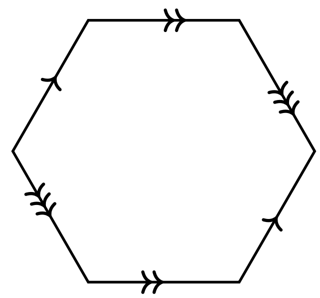
By universal property of quotient space, σ_3 and θ_3 induces continuous maps $\tilde{\sigma}_3$ and $\tilde{\theta}_3$ from components of $K \backslash C_3$ to $(0,1) \times [0,1]$. Let $p : (0,1) \times [0,1] \to S$ be the projection of the square onto a Möbius band. $p \circ \tilde{\sigma}_3$ and $p \circ \tilde{\theta}_3$ maps two components of $K \backslash C_3$ to two open Möbius bands respectively. □

Remark. The geometric visualization of the constructions above is as follows:



Question 6

The following polygon with side identifications is homeomorphic to which surface?

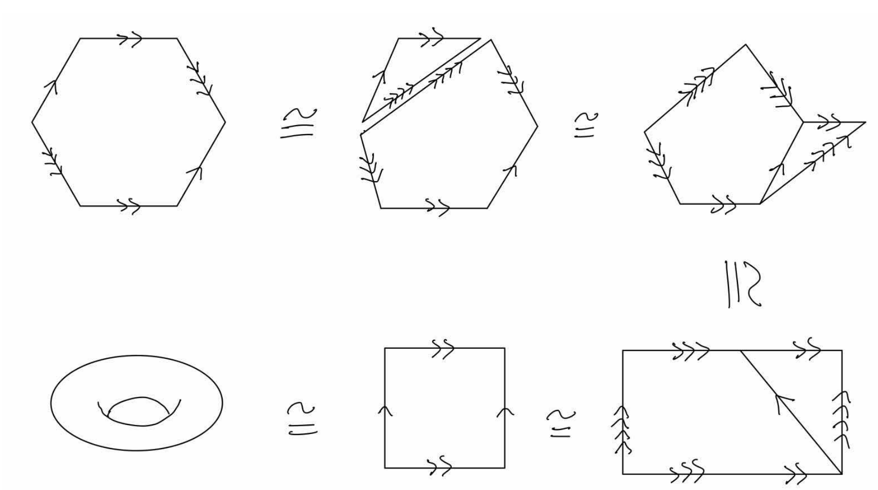


Proof. The surface S is obtained by first attaching three 1-cells onto two 0-cells, then one 2-cells onto it by sending S^1 to the path $xyzx^{-1}y^{-1}z^{-1}$. Hence the fundamental group of the surface is given by $\pi_1(S) = \langle x, y, z \mid xyzx^{-1}y^{-1}z^{-1}\rangle$. We can perform a sequence of Tietze transformations:

$$\begin{aligned}
\langle x, y, z \mid xyzx^{-1}y^{-1}z^{-1}\rangle &\cong \langle x, y, z, w \mid xyzx^{-1}y^{-1}z^{-1},\ w^{-1}xy\rangle && (T5)\\
&\cong \langle x, y, z, w \mid xyzx^{-1}y^{-1}z^{-1},\ xyw^{-1}\rangle && (T2+T3+T4)\\
&\cong \langle x, y, z, w \mid xyzx^{-1}y^{-1}z^{-1},\ xyw^{-1},\ x^{-1}wy^{-1}\rangle && (T2)\\
&\cong \langle x, y, z, w \mid xx^{-1}wy^{-1}yzx^{-1}xyw^{-1}y^{-1}z^{-1},\ xyw^{-1},\ x^{-1}y^{-1}w\rangle && (T4)\\
&\cong \langle x, y, z, w \mid wzyw^{-1}y^{-1}z^{-1},\ xyw^{-1},\ x^{-1}y^{-1}w\rangle && (T3)\\
&\cong \langle x, y, z, w, u \mid wzyw^{-1}y^{-1}z^{-1},\ xyw^{-1},\ x^{-1}y^{-1}w,\ u^{-1}zy\rangle && (T5)\\
&\cong \langle x, y, z, w, u \mid wuw^{-1}u^{-1},\ xyw^{-1},\ x^{-1}y^{-1}w,\ u^{-1}zy\rangle && (T2+T3+T4)\\
&\cong \langle w, u \mid wuw^{-1}u^{-1}\rangle && (T5)\\
&\cong \mathbb{Z}\oplus\mathbb{Z}
\end{aligned}$$

Hence $\pi_1(S) = \pi_1(M_1)$. By Proposition 5.17, S is a closed combinatorial surface. By the classification theorem of closed surfaces, S is homeomorphic to one of M_g or N_h. As no two of M_g or N_h have the same fundamental group, we conclude that $S \cong M_1$. S is homeomorphic to the torus. □

Remark. Geometrically we can visualize the transformations above as follows:



Question 7

Suppose that the sphere S^2 is given the structure of a closed combinatorial surface. Let C be a sub-complex that is a simplicial circle. Suppose that $S^2\backslash C$ has two components. Indeed, suppose that this is true for every simplicial circle in S^2. Let E be one of these components.

Our aim is to show that $\overline{E}$ is homeomorphic to a disc. This is a version of the *Jordan curve theorem*.

We will prove this by induction on the number of 2-simplices in $\overline{E}$. Our actual inductive hypothesis is: *there is a homeomorphism from $\overline{E}$ to D^2, which takes C to the boundary circle ∂D^2.*

(1) Let σ_1 be a 1-simplex in C. Since S^2 is a closed combinatorial surface, σ_1 is adjacent to two 2-simplices. Show that precisely one of these 2-simplices lies in $\overline{E}$. Call this 2-simplex σ_2.

(2) Start the induction by showing that if $\overline{E}$ contains at most one 2-simplex, then $\overline{E} = \sigma_2$.

(3) Let v be the vertex of σ_2 not lying in σ_1. Suppose that v does not lie in C. Show how to construct a sub-complex C' of S^2, that is a simplicial circle, and that has the following properties:

- $S^2\backslash C'$ has two components;
- one of these components F is a subset of E;
- $\overline{F}$ contains fewer 2-simplices than $\overline{E}$.

Show in this case that there is a homeomorphism from $\overline{E}$ to D^2, which takes C to the boundary circle ∂D^2.

(4) Suppose now that v lies in C. How do we complete the proof in this case?

Proof. (1) We shall first prove that every component of $S^2\backslash C$ contains the inside of at least one 2-simplex. We choose $x \in E$. By Lemma 5.16.(3), x is in some 2-complex ρ, whose inside is in E. The same thing holds for the other connected component.

Let $\rho_1, \rho_2, ..., \rho_k$ be the 1-simplices (in order) on C. Let η_i, λ_i be the 2-simplices adjacent to ρ_i. Suppose for contradiction that $\eta_1, \lambda_1 \in \overline{E}$. We claim that $\eta_1, .., \eta_n, \lambda_1, ..., \lambda_n \in \overline{E}$. This is because the vertices of every η_i is connected to those of η_1 by an edge path by connectivity of $\overline{E}$, and similar for λ_i. Let G be the other connected component. We know that $\overline{G}$ contains a 2-simplex α. The vertices of α is connected to the vertices on C by an edge path by connectivity of $\overline{G}$. Then there is a 2-simplex adjacent to some ρ_i and is contained in $\overline{G}$. Contradiction. Similarly we can prove that it cannot be the case that $\eta_1, \lambda_1 \in \overline{G}$.

(2) Trivial. A 2-simplex is homeomorphic to a disc.

(3) We can add v into the simplicial circle C and remove the 1-simplex σ_1. By doing this we obtain a component F such that $\overline{F}$ contains one 2-simplex fewer than $\overline{E}$, because we have taken σ_2 into the other component when modifying C. By induction hypothesis there is a homeomorphism $\varphi : \overline{F} \to D^2$ such that $\varphi(C') = \partial D^2$. Note that D^2 is homeomorphic to a standard 2-simplex. If we attach a 2-simplex onto another 2-simplex by identifying two faces of them, then the result simplicial complex is still homeomorphic to D^2. Then we can say that $\overline{E}$ is homeomorphic to D^2 with boundary being sent to ∂D^2.

(4) If C contains only 3 vertices, then there is nothing to prove. Suppose that C have more then 3 vertices. Let $\sigma_1 = a_1, a_2$ and $\sigma_2 = \{a_1, a_2, v\}$. If a_1, a_2, v is the order of these vertices on C, then we add $\{a_1, v\}$ to the simplicial circle and remove all vertices between a_1 and v. By doing this we obtain a component F such that $\overline{F}$ contains fewer 2-simplices than $\overline{E}$. The situation is similar to part (3) except that we will attach a 2-simplex onto another 2-simplex by identifying one face of them. The result simplicial complex is also a disc. □