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Notes on
Deformation Quantisation

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Chapter 1

Introduction

Ever since the discovery of quantum theory in the early twentieth century, the problem of finding a correspondence between classical systems and quantum systems has been a central theme of theoretical physics. The defining features of quantum mechanics include the probabilistic interpretations of measurements and the non-commutativity of multiplications of physical observables. Proposed by Bayen *et. al.* in [Bay78I] and [Bay78II], deformation quantisation originates from an attempt of understanding quantisation "*as a deformation of the structure of the classical observable, rather than as a radical change in the natural of the observables.*" In particular, the Hilbert space formalism of quantum mechanics is replaced by a deformation of classical mechanics on the phase space, which is known as a star product. In a mathematical formulation, deformation quantisation attempts to classify the star products on an associative algebra which recovers the commutative multiplication and the Poisson algebra in the formal classical limit $\hbar \to 0$.

The Weyl–Moyal product proposed by Groenewold ([Gro46]) is well-known as the first example of deformation quantisation on the flat phase space. In 1983, De Wilde and Lecomte proved the existence of star products first on the cotangent bundle of a smooth manifold ([DWL83b]) and later on a symplectic manifold ([DWL83a]). Independently, Fedosov ([Fed94]) proved the existence on a symplectic manifold in 1985 by constructing a flat connection of the Weyl bundle, which globalises the local Weyl–Moyal product in a coordinate-free manner. The classification up to equivalence on symplectic manifolds were studied by Deligne ([Del95]), Gutt ([BCG97], [GR99]), Weinstein and Xu ([WX97]) *et. al.* In contrast to the case of symplectic manifolds, where the Poisson bracket takes constant coefficients locally in the Darboux coordinates, deformation quantisation problems in the more general case of Poisson manifolds have greater complexity, since the local expression of Poisson bracket is arbitrary.

In 1997, Kontsevich ([Kon97a]) reformulated the problem as the existence of an L_∞-quasi-isomorphism between two differential graded Lie algebras, which is known as the *formality conjecture*. He proved the conjecture in the ground-breaking paper [Kon97], in which he provided an explicit formula for the star product in $\mathbb{R}^d$. The work is among one of his four accomplishments in geometry ([Tau98]) for which he won the 1998 Fields Medal Prize. The proof of globalisation in an arbitrary smooth manifold was sketched in [Kon97] and was made precise by Cattaneo, Felder & Tomassini in [CFT02] and also by Kontsevich in his subsequent work [Kon01]. Kontsevich's formality theorem has inspired numerous development in symplectic geometry, algebraic geometry, and quantum field theory.

- The underlying idea of Kontsevich's construction comes from string theory. In the paper [CF00] by Cattaneo & Felder, Kontsevich's formula for the star product is interpreted as the perturbative expansion of the functional integral of Poisson sigma model, thus providing a Feynman path integral quantisation explanation to a construction arising in the context of canonical quantisation.

- In [Tam98], Tamarkin provides another proof of the formality theorem in a pure algebraic setting using the theory of operads. He proved the formality of the differential graded Lie algebra associated to a finite dimensional vector space over a field of characteristic 0. His approach is surveyed in [Kon99] and [Hin03].

- Deformation quantisation of smooth algebraic varieties over a field of characteristic 0 is studied in [Kon01] and [Yek05].

1.1 Quantisation: From Classical to Quantum

In this section we review the physical backgrounds and motivations of quantisation. We also define all the geometric objects involved in the picture.

1.1.1 Classical Mechanics

In classical mechanics, a dynamical system is described by a smooth manifold M, whose dimension corresponds to the degree of freedom of that system. In the Hamiltonian formalism, the dynamics of the system is governed by the Hamilton's equations on the **phase space**, which is the cotangent bundle T^*M of M:

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i}, \qquad i = 1, ..., n,$$

where $(q^1, ..., q^n, p_1, ..., p_n)$ is a set of local coordinates on T^*M and $H\colon \mathrm{T}^*M \to \mathbb{R}$ is the Hamiltonian function. The cotangent bundle has a canonical symplectic form ω, given in local coordinates by $\omega = \sum_{i=1}^{\dim M} \mathrm{d}q^i \wedge \mathrm{d}p_i$. Hamiltonian mechanics can be generalised on the setting of a symplectic manifold (N, ω) without difficulty.

Definition 1.1. A **symplectic manifold** (N, ω) is a smooth manifold N with a closed non-degenerate 2-form ω, called the symplectic form. A diffeomorphism between two symplectic manifolds that preserve the symplectic form is called a symplectomorphism.

An important result in symplectic geometry in the following:

> **Lemma 1.2. Darboux's Theorem**
>
> A $2n$-dimensional symplectic manifold is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega)$, where $\omega = \sum_{i=1}^{n} \mathrm{d}q^i \wedge \mathrm{d}p_i$ is the standard symplectic form on $\mathbb{R}^{2n}$.

The symplectic form ω induces the symplectic involution $J\colon \mathrm{T}^*N \to \mathrm{T}N$ via $\omega(X, J(\xi)) = \xi(X)$ for $\xi \in \mathrm{T}^*N$ and $X \in \mathrm{T}N$, which is a linear isomorphism. For a smooth function $f \in \mathrm{C}^\infty(N)$, the Hamiltonian vector field associated to f is defined by $X_f := J(\mathrm{d}f)$. For two smooth functions f and g, we can define the **Poisson bracket** $\{-, -\}\colon \mathrm{C}^\infty(N) \times \mathrm{C}^\infty(N) \to \mathrm{C}^\infty(N)$:

$$\{f, g\} := \omega(X_f, X_g) = X_f(g).$$

The Lie bracket of the vector fields and the Poisson bracket are related by

$$[X_f, X_g] = X_{\{f,g\}}.$$

Alternatively, the Poisson bracket defines a skew-symmetric bivector field $\pi \in \Gamma(N, \bigwedge^2 \mathrm{T}N)$ such that the bracket is given by the pairing $\{f, g\} = \pi(\mathrm{d}f \wedge \mathrm{d}g)$. In local coordinates, the corresponding matrices satisfy $\pi^{ij} = (\omega^{-1})_{ij}$. Darboux's theorem demonstrates that it is always possible to find a chart such that the Poisson bivector field π has constant coefficients in the local coordinates. Furthermore, the coordinates can be chosen such that

$$\{q^i, q^j\} = 0, \qquad \{p_i, p_j\} = 0, \qquad \{q^i, p_j\} = \delta^i_j, \qquad i, j = 1, ..., n. \tag{1.1}$$

In these coordinates the Poisson bracket of two functions f, g is given by

$$\{f, g\} = \sum_{i=1}^{\dim M} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i} \right).$$

In terms of the Poisson bracket, Hamilton's equations show that the time evolution of any smooth function $f(t) \colon N \times \mathbb{R} \to N$ is given by

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\}. \tag{1.2}$$

It is straightforward to prove that the Poisson bracket satisfies the conditions for a Lie bracket on $\mathrm{C}^\infty(N)$. Moreover it is a derivation in both of its arguments. We extract these properties to make the following definition:

Definition 1.3. Let A be an associative algebra over a field k. A Poisson bracket $\{-, -\}$ on A is a k-bilinear map, satisfying:

- Skew-symmetry: $\{u, v\} = -\{v, u\}$;
- Jacobi identity: $\{u, \{v, w\}\} + \{v, \{w, u\}\} = \{w, \{u, v\}\} = 0$;
- Leibniz rule: $\{uv, w\} = u\{v, w\} + \{u, w\}v$.

where $u, v, w \in A$. An algebra A equipped with a Poisson bracket is called a **Poisson algebra**. If $A = \mathrm{C}^\infty(M)$ is the algebra of smooth functions on a smooth manifold M, then M is called a **Poisson manifold**.

Remark. A symplectic manifold must be even-dimensional due to the non-degeneracy of the symplectic form. On the other hand, any smooth manifold can be a Poisson manifold, whose Poisson bracket is allowed to be degenerate.

1.1.2 Quantum Mechanics

For convenience we consider the flat phase space $N = \mathbb{R}^{2n}$ in this part. According to the Dirac–von Neumann axiomatisation of quantum mechanics, a quantum mechanical system is described by a separable complex Hilbert space $\mathcal{H}$ with a self-adjoint operator $\widehat{H}$ acting on $\mathcal{H}$, which is called the Hamiltonian. Every physical observable f is associated with a self-adjoint operator $\widehat{f}$ on a Hilbert space $\mathcal{H}$. The spectrum of $\widehat{f}$ is the set of possible outcomes of measuring $\widehat{f}$. In the Heisenberg picture, the time evolution of the operator $\widehat{f}$ is controlled by the Hamiltonian:

$$\frac{\mathrm{d}\widehat{f}}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}[\widehat{f}, \widehat{H}] \tag{1.3}$$

where $\hbar$ is the **reduced Planck constant** and $[-, -]$ is the commutator. The coordinates q^i and p_i of the phase space $\mathbb{R}^{2n}$ are upgraded to self-operators $\widehat{q}^i$ and $\widehat{p}_i$ which satisfy the canonical commutation relation:

$$\left[\widehat{q}^i, \widehat{q}^j\right] = 0, \qquad \left[\widehat{p}_i, \widehat{p}_j\right] = 0, \qquad \left[\widehat{q}^i, \widehat{p}_j\right] = \mathrm{i}\hbar\delta^i_j \cdot \mathrm{id}_{\mathcal{H}}, \qquad i, j = 1, ..., n. \tag{1.4}$$

By Stone–von Neumann theorem, there is an irreducible unitary representation of $\mathcal{H}$ on $\mathrm{L}^2(\mathbb{R}^n)$ which is unique up to isomorphism such that

$$\widehat{q}^i \colon \psi(x) \longmapsto x_i \psi(x), \qquad \widehat{p}_i \colon \psi(x) \longmapsto -\mathrm{i}\hbar\frac{\partial\psi}{\partial x_i}. \tag{1.5}$$

which is often referred as the position representation in physics. Comparing (1.1), (1.2) with (1.3), (1.4), we recognise that a classical system is quantised to a quantum system under the naive correspondence

$$\mathrm{C}^\infty\left(\mathbb{R}^{2n}\right) \xrightarrow[\hbar \to 0]{Q} \{\text{self-adjoint operators on } \mathrm{L}^2(\mathbb{R}^n)\}$$

$$\{-,-\} \xrightarrow{\hspace{4cm}} \frac{1}{\mathrm{i}\hbar}[-,-]$$

which is called **canonical quantisation**. In particular, one may postulate that there exists a $\mathbb{R}$-linear quantisation map Q which "canonically quantises" the coordinates while mapping the Poisson brackets to commutators: $[Q(f), Q(g)] = \mathrm{i}\hbar Q(\{f,g\})$ for all $f, g \in \mathrm{C}^\infty\left(\mathbb{R}^{2n}\right)$. In 1946, Groenewold proved in his thesis [Gro46] that such quantisation map does not exist. We outline the idea of the proof following [Hal13].

Theorem 1.4. Groenewold's No-Go Theorem

Let $P_{\leqslant m}$ be the set of polynomials in $\mathbb{R}[q^1, ..., q^n; p_1, ..., p_n]$ with deg $\leqslant m$. We can identify $P_{\leqslant m}$ as a subset of $\mathrm{C}^\infty\left(\mathbb{R}^{2n}\right)$. On the other hand, let $\mathscr{D}(\mathbb{R}^n)$ be the space of differential operators with coefficients in $\mathbb{R}[x_1, ..., x_n]$. Consider $\widehat{q}^i, \widehat{p}_i \in \mathscr{D}(\mathbb{R}^n)$ as given in (1.5). There is no linear map $Q \colon P_{\leqslant 4} \to \mathscr{D}(\mathbb{R}^n)$ such that

- $Q(1) = \mathrm{id}$;
- $Q(q^i) = \widehat{q}^i$ and $Q(p_i) = \widehat{p}_i$ for all $i = 1, ..., n$;
- $[Q(f), Q(g)] = \mathrm{i}\hbar Q(\{f,g\})$ for all $f, g \in P_{\leqslant 3}$.

Proof. See Appendix 1.A. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The impossibility of finding a strict morphism taking Poisson brackets to commutators suggests that the conditions for the quantisation have to be relaxed, which leads to different quantisation schemes, the most famous among which are geometric quantisation and deforamtion quantisation. In this dissertation we only study the latter. In deformation quantisation, the condition $Q(\{f,g\}) = \mathrm{i}\hbar\,[Q(f), Q(g)]$ is replaced by an asympototic equality:

$$[Q(f), Q(g)] = \mathrm{i}\hbar Q(\{f,g\}) + \mathcal{O}(\hbar^2). \tag{1.6}$$

Furthermore, the constant $\hbar$ is treated as a formal indeterminate, so that (1.6) holds in the formal sense. We remove the Hilber space from the picture and works completely on the classical phase space. The "quantum" aspect of the theory is captured in a non-commutative associative product, which is the object we study in the next section.

1.2   Star Products

In this section we take the phase space M to be an arbitrary Poisson manifold. Let $A := \mathrm{C}^\infty(M)$ be the $\mathbb{R}$-algebra of smooth functions on M. Let Q be the quantisation map. To study the quantisation on the phase space, we assume that Q is an isomorphism and consider the pull-back of the product:

$$f \star g := Q^{-1}(Q(f)Q(g)), \qquad f, g \in A.$$

Writing this product in a formal power series:

$$f \star g = \sum_{m=0}^{\infty} \hbar^m B_m(f, g) \in A[\![\hbar]\!],$$

we infer from (1.6) that $B_0(f, g) = B_0(g, f) = fg$ and $B_1(f, g) - B_1(g, f) = \mathrm{i}\{f, g\}$. This is an example of a star product. We give the formal definition below.

Definition 1.5. A **star product** on A is an $\mathbb{R}[\![\hbar]\!]$-bilinear map $\star\colon A[\![\hbar]\!] \times A[\![\hbar]\!] \to A[\![\hbar]\!]$ such that

- Associativity: $(f \star g) \star h = f \star (g \star h)$ for $f, g \in A[\![\hbar]\!]$;
- Unit: $1 \star f = f \star 1 = f$, for $f \in A[\![\hbar]\!]$, where 1 is the constant function on M;
- $f \star g = fg + \sum_{m=1}^{\infty} \hbar^m B_m(f, g)$ for $f, g \in A$, where B_m are bi-differential operators of globally bounded order, that is, differential operators with respect to each argument in any local coordinates of M.

The star products $\star$ and $\star'$ are said to equivalent, if there exists a $\mathbb{R}$-linear isomorphism $\varphi\colon A[\![\hbar]\!] \to A[\![\hbar]\!]$ such that $\varphi(f) = f + \sum_{m=1}^{\infty} \hbar^m \varphi_m(f)$ for $f \in A$ and $\varphi(f) \star' \varphi(g) = \varphi(f \star g)$ for $f, g \in A[\![\hbar]\!]$.

Remark. If μ denotes the multiplication of functions on A, then we can write $\star = \mu + \sum_{m=1}^{\infty} \hbar^m B_m$. The notion of star products will be discuss in the broader concept of deformations of associative multiplication in Section 2.1.

A star product gives rise to a Poisson bracket on M:

> **Lemma 1.6**
>
> Suppose that $\star = \mu + \sum_{m=1}^{\infty} \hbar^m B_m$ is a star product on A. Then $\{f, g\} := B_1(f, g) - B_1(g, f)$ defines a Poisson bracket on A. Moreover, the bracket only depends on the equivalence class of $\star$.

Proof. The skew-symmetry and bilinearity of $\{-, -\}$ are immediate from definition. For the Jacobi identity, we can expand the bracket $\hbar^2 \{f, \{g, h\}\}$ in $\mathbb{R}[\hbar]/\langle \hbar^3 \rangle$:

$$\hbar^2 \{f, \{g, h\}\} = f \star (g \star h) - f \star (h \star g) - (g \star h) \star f + (h \star g) \star f.$$

The expansion of two other brackets are similar. By associativity of the star product, we note that the sum vanishes:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

The Leibniz rule for $\{-, -\}$ follows from the Leibniz rule of B_1. Therefore $\{-, -\}$ is a Poisson bracket. Suppose that $\star$ is equivalent to $\star'$ via the isomorphism φ. Then at order $\hbar$ we find that

$$g\varphi_1(f) + f\varphi_1(g) + B_1'(f, g) = \varphi_1(fg) + B_1(f, g).$$

Note that $B_1' - B_1$ is symmetric in the two arguments and hence does not contribute to the Poisson bracket. The star product $\star'$ induces the same Poisson bracket as $\star$. $\qquad\square$

Definition 1.7. Let A be a Poisson algebra. We say that $(A[\![\hbar]\!], \star)$ is a **deformation quantisation** of A, if $\star = \mu + \sum_{m=1}^{\infty} \hbar^m B_m$ is a star product on $A[\![\hbar]\!]$ such that $\{f, g\} = B_1(f, g) - B_1(g, f)$ coincides with the Poisson bracket on A.

Remark. Note that the definition has a factor i different from our picture in quantum mechanics. Since $\hbar$ is considered as a formal symbol, it is possible to absorb i into the $\hbar$ and to work entirely over $\mathbb{R}$.

We are more interested in the converse of Lemma 1.6, that is, to find a deformation quantisation for a given Poisson bracket. This seems more difficult and requires some additional structure on the algebra A as we want to constructing $B_2, B_3, \ldots$ from B_1. The classification theorem of deformation quantisation, stated in its greatest generality, is the following bijective correspondence on a smooth manifold M:

$$\frac{\{\text{Formal Poisson structures}\}}{\text{Equivalence}} \quad \longleftrightarrow \quad \frac{\{\text{Star products}\}}{\text{Equivalence}}. \tag{1.7}$$

By the end of Chapter 2, we shall elucidate this matter as a consequence of Kontsevich's formality theorem 2.56.

1.2.1   Weyl–Moyal Product

In this part we study a simple construction of deformation quantisation on $\mathbb{R}^{2n}$. The idea of Weyl quantisation and Moyal bracket originates from the works by Weyl ([Wey27]), Moyal ([Moy49]), Wigner ([Wig32]), and Groenewold ([Gro46]) which had been studied before deformation quantisation appeared. The story begins with the ordering ambiguity in canonical quantisation of monomials like $qp^2 \in \mathrm{C}^\infty(\mathbb{R}^2)$. The naive strategy of "replacing q by $\widehat{q}$ and p by $\widehat{p}$" does not work because $\widehat{q}\widehat{p}^2$ fails to be self-adjoint on $\mathrm{L}^2(\mathbb{R})$. A more serious approach is to take the total symmetrisation of the expression:

$$Q(qp^2) = \frac{1}{3}\left(\widehat{q}\widehat{p}^2 + \widehat{p}\widehat{q}\widehat{p} + \widehat{p}^2\widehat{q}\right) = \frac{1}{2}\left(\widehat{q}\widehat{p}^2 + \widehat{p}^2\widehat{q}\right).$$

This leads to the definition of Weyl quantisation:

Definition 1.8. For a monomial $f = q^I p^J \in \mathbb{R}[q^1, \ldots, q^n; p_1, \ldots, p_n]$, we define the **Weyl quantisation** $Q_{\mathrm{W}}(f)$ of f to be the total symmetrisation of the monomial $\widehat{q}^I \widehat{p}^J$, where I, J are multi-indices. Then we extend Q_{W} linearly to $\mathbb{R}[q^1, \ldots, q^n; p_1, \ldots, p_n]$. Furthermore, Weyl quantisation can be extended to $\mathrm{L}^2(\mathbb{R}^{2n})$ functions as follows:

$$Q_{\mathrm{W}}(f) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{F}f(\boldsymbol{a}, \boldsymbol{b})\, \mathrm{e}^{\mathrm{i}(\boldsymbol{a}\cdot\widehat{\boldsymbol{q}}+\boldsymbol{b}\cdot\widehat{\boldsymbol{p}})}\, \mathrm{d}^n\boldsymbol{a}\, \mathrm{d}^n\boldsymbol{b},$$

where $\mathcal{F}f(\boldsymbol{a}, \boldsymbol{b})$ is the Fourier transform of $f(\boldsymbol{q}, \boldsymbol{p})$, and the exponential of operators is formally understood by the *Baker–Campbell–Hausdorff formula* (*cf.* (2.4)). If we identify a suitable codomain of Q_{W}, it becomes invertible with inverse Q_{W}^{-1} known as the *Wigner transform*.

Definition 1.9. The **Weyl–Moyal product** of $f, g \in \mathrm{C}^\infty(\mathbb{R}^{2n})$ is defined as $f \star g := Q_{\mathrm{W}}^{-1}(Q_{\mathrm{W}}(f)Q_{\mathrm{W}}(g))$.

In this definition, $\hbar$ is treated as a real parameter. It can be proven (*cf.* [Hal13]) that $f \star g \to fg$ uniformly as $\hbar \to 0$, so that $f \star g$ is an *actual* deformation of fg, not just a formal one. However, it is sufficient to consider the formal aspect of the the Weyl–Moyal product. The closed form of the Weyl–Moyal product is given by

$$(f \star g)(x) = \exp\left(\frac{\mathrm{i}\hbar}{2} \sum_{i,j=1}^{2n} \pi^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j}\right) f(x)g(y) \Bigg|_{x=y}.$$

$$= (fg)(x) + \sum_{m=1}^{\infty} \frac{(\mathrm{i}\hbar)^m}{2^m m!} \prod_{k=1}^{m}\left(\sum_{i_k,j_k=1}^{2n} \pi^{i_k,j_k} \frac{\partial}{\partial x^{i_k}} \frac{\partial}{\partial y^{j_k}}\right) f(x)g(y) \Bigg|_{x=y} \tag{1.8}$$

where π is the Poisson bivector field associated to the standard symplectic structure of $\mathbb{R}^{2n}$. Now we consider $\hbar$ as a formal symbol, so that $f \star g \in \mathrm{C}^\infty(\mathbb{R}^{2n})[\![\hbar]\!]$. Next we replace $\mathbb{R}^{2n}$ by $\mathbb{R}^d$ and allow the constant matrix $(\pi^{ij})_{i,j}$ to have arbitrary form.

> **Proposition 1.10**
>
> By extending the Weyl–Moyal product $\mathbb{R}[\![\hbar]\!]$-bilinearly, we obtain a star product on $C^\infty(\mathbb{R}^d)[\![\hbar]\!]$, which is called the **Moyal star product**.

Proof. The only non-trivial part is checking associativity. For $f, g, h \in C^\infty(\mathbb{R}^d)$,

$$
\begin{aligned}
((f \star g) \star h)(x) &= \exp\left(\frac{i\hbar}{2}\pi^{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial z_j}\right)(f \star g)(x)g(z)\Big|_{x=z} \\
&= \exp\left(\frac{i\hbar}{2}\pi^{ij}\left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i}\right)\frac{\partial}{\partial z^j}\right)\exp\left(\frac{i\hbar}{2}\pi^{k\ell}\frac{\partial}{\partial x^k}\frac{\partial}{\partial y^\ell}\right)f(x)g(y)g(z)\Big|_{x=y=z} \\
&= \exp\left(\frac{i\hbar}{2}\pi^{ij}\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j} + \frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j} + \frac{\partial}{\partial x^i}\frac{\partial}{\partial z^j}\right)\right)f(x)g(y)g(z)\Big|_{x=y=z} \\
&= (f \star (g \star h))(x). \qquad \square
\end{aligned}
$$

Remark. A natural question that arises is the possibility to extend this construction to include Poisson bivector field π with coefficients depending on the coordinates. This proves to be the most difficult part of Kontsevich's quantisation theorem and is a consequence of the formality theorem. There is a canonical star product associated to π, whose formula is given in (3.9).

1.A Proof of Groenewold's No-Go Theorem

> **Lemma 1.11**
>
> For $f \in P_{\leqslant 2}$ and $g \in P$, we have $[Q_W(f), Q_W(g)] = i\hbar Q_W(\{f, g\})$.

Proof. The result is trivially true for $f \in P_0$. Let $f \in P_1$. Suppose that $f = x^i$. We claim that

$$
Q_W(x^i g) = Q_W(x^i)Q_W(g) - \frac{i\hbar}{2}Q_W\left(\frac{\partial g}{\partial p_i}\right) = Q_W(g)Q_W(x^i) + \frac{i\hbar}{2}Q_W\left(\frac{\partial g}{\partial p_i}\right).
$$

We know that every $g \in P_m$ can be expressed as a sum of the monomials $(\boldsymbol{a} \cdot \boldsymbol{x} + \boldsymbol{b} \cdot \boldsymbol{p})^m$ for some $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^n$. So it suffices to prove the equality when $g = (\boldsymbol{a} \cdot \boldsymbol{x} + \boldsymbol{b} \cdot \boldsymbol{p})^m$.

Let $h := \boldsymbol{a} \cdot \boldsymbol{x} + \boldsymbol{b} \cdot \boldsymbol{p}$, $\widehat{h} := \boldsymbol{a} \cdot \widehat{\boldsymbol{x}} + \boldsymbol{b} \cdot \widehat{\boldsymbol{p}} = Q_W(h)$. Note that $[\widehat{x}^i, \widehat{h}] = a\,\mathrm{id}$ for some $a \in \mathbb{C}$. We have, for $0 \leqslant k \leqslant m$,

$$
\widehat{x}^i \widehat{h}^m = \widehat{h}^k \widehat{x}^i \widehat{h}^{m-k} + k[\widehat{x}^i, \widehat{h}]\widehat{h}^{m-1}
$$

Summing over k:

$$
\widehat{x}^i \widehat{h}^m = \frac{1}{m+1}\sum_{k=0}^m \widehat{h}^k \widehat{x}^i \widehat{h}^{m-k} + \frac{m}{2}[\widehat{x}^i, \widehat{h}]\widehat{h}^{m-1}
$$

Note that $\dfrac{1}{m+1}\displaystyle\sum_{k=0}^m \widehat{h}^k \widehat{x}^i \widehat{h}^{m-k} = Q_W(x^i h^m) = Q_W(g)$. The above equation is

$$
Q_W(x^i)Q_W(g) = Q_W(x^i g) + \frac{m}{2}i\hbar b_i Q_W(h^{m-1}) = Q_W(x^i g) + \frac{i\hbar}{2}Q_W\left(\frac{\partial g}{\partial p_i}\right)
$$

On the other hand, we have

$$\widehat{h}^m \widehat{x}^i = \widehat{h}^k \widehat{x}^i \widehat{h}^{m-k} - (m-k)[\widehat{x}^i, \widehat{h}]\widehat{h}^{m-1}$$

Summing over k gives

$$Q_{\mathrm{W}}(g)Q_{\mathrm{W}}(x^i) = Q_{\mathrm{W}}(x^i g) - \frac{i\hbar}{2}\, Q_{\mathrm{W}}\left(\frac{\partial g}{\partial p_i}\right)$$

which completes the proof of the claim.

Therefore we have

$$\left[Q_{\mathrm{W}}(x^i), Q_{\mathrm{W}}(g)\right] = i\hbar Q_{\mathrm{W}}\left(\frac{\partial g}{\partial p_i}\right) = i\hbar Q_{\mathrm{W}}(\{x^i, g\}).$$

For $f = p_i$, similarly we can prove that

$$[Q_{\mathrm{W}}(p_i), Q_{\mathrm{W}}(g)] = -i\hbar Q_{\mathrm{W}}\left(\frac{\partial g}{\partial x^i}\right) = i\hbar Q_{\mathrm{W}}(\{p_i, g\}).$$

We deduce that the result is true for all $f \in P_1$. Now let $f \in P_2$. By linearity it suffices to consider the case $f = f_1 f_2$, where $f_1, f_2 \in P_1$. The above argument shows that

$$Q_{\mathrm{W}}(f_i h) = \frac{1}{2}\left(Q_{\mathrm{W}}(f_i)Q_{\mathrm{W}}(h) + Q_{\mathrm{W}}(h)Q_{\mathrm{W}}(f_i)\right)$$

for all $h \in P$. Therefore we have

$$\begin{aligned}
[Q_{\mathrm{W}}(f_1 f_2), Q_{\mathrm{W}}(g)] &= \frac{1}{2}\left[Q_{\mathrm{W}}(f_1)Q_{\mathrm{W}}(f_2) + Q_{\mathrm{W}}(f_2)Q_{\mathrm{W}}(f_1), Q_{\mathrm{W}}(g)\right] \\
&= \frac{1}{2}(Q_{\mathrm{W}}(f_1)\left[Q_{\mathrm{W}}(f_2), Q_{\mathrm{W}}(g)\right] + Q_{\mathrm{W}}(f_2)\left[Q_{\mathrm{W}}(f_1), Q_{\mathrm{W}}(g)\right] \\
&\quad + \left[Q_{\mathrm{W}}(f_1), Q_{\mathrm{W}}(g)\right]Q_{\mathrm{W}}(f_2) + \left[Q_{\mathrm{W}}(f_2), Q_{\mathrm{W}}(g)\right]Q_{\mathrm{W}}(f_1)) \\
&= \frac{1}{2}(Q_{\mathrm{W}}(f_1)Q_{\mathrm{W}}(\{f_2, g\}) + Q_{\mathrm{W}}(f_2)Q_{\mathrm{W}}(\{f_1, g\}) \\
&\quad + Q_{\mathrm{W}}(\{f_1, g\})Q_{\mathrm{W}}(f_2) + Q_{\mathrm{W}}(\{f_2, g\})Q_{\mathrm{W}}(f_1)) \\
&= Q_{\mathrm{W}}(f_1\{f_2, g\}) + Q_{\mathrm{W}}(f_2\{f_1, g\}) \\
&= Q_{\mathrm{W}}(\{f_1 f_2, g\})
\end{aligned}$$

which establishes the result for all $f \in P_2$. $\square$

Lemma 1.12

Let $A \in D(\mathbb{R}^n)$. If $[A, \widehat{x}^i] = 0$ and $[A, \widehat{p}_i] = 0$ for all $i \in \{1, ..., n\}$, then $A = c\,\mathrm{id}$ for some $c \in \mathbb{C}$.

Proof. We adopt the notion of multi-index $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$, where $\alpha_1, ..., \alpha_n \in \mathbb{N}$. Let $|\boldsymbol{\alpha}| = \sum_{k=1}^n \alpha_k$, and $\partial^{\boldsymbol{\alpha}} := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. Let $\boldsymbol{e}_i$ be the multi-index with 1 in the i-th entry and 0 otherwise. For $\boldsymbol{\alpha}, \boldsymbol{\beta}$, we define $\boldsymbol{\alpha} - \boldsymbol{\beta} = (\alpha_1 - \beta_1, ..., \alpha_n - \beta_n)$ if $\alpha_i \geqslant \beta_i$ for all i, and $\boldsymbol{\alpha} - \boldsymbol{\beta} = \boldsymbol{0}$ otherwise.

Note by induction that

$$\frac{\partial^k}{\partial(x^i)^k}(x^i g(\boldsymbol{x})) = k\frac{\partial^{k-1}}{\partial(x^i)^{k-1}}g(\boldsymbol{x}) + x^i\frac{\partial^k}{\partial(x^i)^k}g(\boldsymbol{x}).$$

Then for any multi-index $\boldsymbol{\alpha}$, we have

$$\left[f(\boldsymbol{x})\partial^{\boldsymbol{\alpha}}, \widehat{x}^i\right] = \alpha_i f(\boldsymbol{x})\partial^{\boldsymbol{\alpha}-\boldsymbol{e}_i}.$$

Suppose that $A \in D(\mathbb{R}^n) \setminus \{0\}$ such that A commutes with all $\widehat{x}^i$ and $\widehat{p}_i$. If $\deg A > 0$, then A has a term of the form $f_{\boldsymbol{\beta}}(\boldsymbol{x})\partial^{\boldsymbol{\beta}}$ such that $f_{\boldsymbol{\beta}}(\boldsymbol{x}) \neq 0$ and $\beta_i > 0$ for some j. Then

$$[A, \widehat{x}_i] = \beta_i f_{\boldsymbol{\beta}}(\boldsymbol{x})\partial^{\boldsymbol{\beta}-\boldsymbol{e}_i} + \cdots$$

Hence $[A, \widehat{x}_i] \neq 0$, which is a contradiction. Therefore $\deg A = 0$. $A = f(\boldsymbol{x})\,\mathrm{id}$ for some polynomial f. But we also have

$$0 = [A, \widehat{p}_i] = \mathrm{i}\hbar \frac{\partial f}{\partial x_j}\,\mathrm{id}$$

Hence $f = c$ for some $c \in \mathbb{C}$. $A = c\,\mathrm{id}$. $\qquad\square$

Lemma 1.13

1. For $f \in P_2$, there exist $g_1, ..., g_k, h_1, ..., h_k \in P_2$ such that $f = \sum_{i=1}^{k} \{g_i, h_i\}$;

2. For $f \in P_3$, there exist $g_1, ..., g_k \in P_3$ and $h_1, ..., h_k \in P_2$ such that $f = \sum_{i=1}^{k} \{g_i, h_i\}$.

Proof. We adopt the notion of multi-index $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$, where $\alpha_1, ..., \alpha_n \in \mathbb{N}$. Let $\boldsymbol{x}^{\boldsymbol{\alpha}} := (x^1)^{\alpha_1} \cdots (x^n)^{\alpha_n}$ and similar for $\boldsymbol{p}^{\boldsymbol{\alpha}}$.

Let $g_1 := \sum_{i=1}^{n} x^i p_i \in P_2$. Note that $\{g_1, \boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{p}^{\boldsymbol{\beta}}\} = (|\boldsymbol{\beta}| - |\boldsymbol{\alpha}|)\boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{p}^{\boldsymbol{\beta}}$.

1. Let $f := \boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{p}^{\boldsymbol{\beta}} \in P_2$. If $|\boldsymbol{\alpha}| \neq |\boldsymbol{\beta}|$, then $f = \left\{g_1, \frac{1}{|\boldsymbol{\beta}| - |\boldsymbol{\alpha}|}\boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{p}^{\boldsymbol{\beta}}\right\}$. If $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$, then $f = x^j p_k$ for some j, k. We have $f = \left\{\frac{1}{2}(x^j)^2, \frac{1}{2}(p_k)^2\right\}$. The result follows from taking linear combinations of such f.

2. Let $f := \boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{p}^{\boldsymbol{\beta}} \in P_3$. Then $|\boldsymbol{\alpha}| \neq |\boldsymbol{\beta}|$ and we have $f = \left\{g_1, \frac{1}{|\boldsymbol{\beta}| - |\boldsymbol{\alpha}|}\boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{p}^{\boldsymbol{\beta}}\right\}$. The result follows from taking linear combinations of such f. $\qquad\square$

Proposition 1.14

Let Q satisfies the conditions in the Groenewold's Theorem 1.4. Then $Q = Q_{\mathrm{W}}$ on $P_{\leqslant 3}$.

Proof. It is clear that $Q = Q_{\mathrm{W}}$ on $P_{\leqslant 1}$. Let $f \in P_2$. Let $A_f := Q(f) - Q_{\mathrm{W}}(f)$. For any $g \in P_1$,

$$Q[\{f, g\}] = \frac{1}{\mathrm{i}\hbar}[Q(f), Q(g)] = \frac{1}{\mathrm{i}\hbar}([Q_{\mathrm{W}}(f), Q_{\mathrm{W}}(g)] + [A_f, Q_{\mathrm{W}}(g)])$$

$$= Q_{\mathrm{W}}(\{f, g\}) + \frac{1}{\mathrm{i}\hbar}[A_f, Q_{\mathrm{W}}(g)] = Q(\{f, g\}) + \frac{1}{\mathrm{i}\hbar}[A_f, Q_{\mathrm{W}}(g)].$$

We used the fact that $\{f, g\} \in P_1$ in the last equality. Hence $[A_f, Q_{\mathrm{W}}(g)] = 0$ for all $g \in P_1$. By Lemma 1.12, $A_f = a_f\,\mathrm{id}$ for $a_f \in \mathbb{C}$. Now for $f, h \in P_2$,

$$\mathrm{i}\hbar Q[\{f, h\}] = [Q_{\mathrm{W}}(f) + a_f\,\mathrm{id}, Q_{\mathrm{W}}(h) + a_h\,\mathrm{id}] = [Q_{\mathrm{W}}(f), Q_{\mathrm{W}}(h)] = \mathrm{i}\hbar Q_{\mathrm{W}}(\{f, h\}).$$

By Lemma 1.13.1, we deduce that $Q(f) = Q_\mathrm{W}(f)$ for all $f \in P_2$.

Let $f \in P_3$. Let $B_f := Q(f) - Q_\mathrm{W}(f)$. For $g \in P_1$, we have $\{f, g\} \in P_2$. Then the result we just proved shows that $Q(\{f, g\}) = Q_\mathrm{W}(\{f, g\})$. The same method as above shows that $B_f = b_f \, \mathrm{id}$ for $b_f \in \mathbb{C}$. Then for $h \in P_2$,

$$\mathrm{i}\hbar Q\left[\{f, h\}\right] = [Q_\mathrm{W}(f) + b_f \, \mathrm{id}, Q_\mathrm{W}(h)] = [Q_\mathrm{W}(f), Q_\mathrm{W}(h)] = \mathrm{i}\hbar Q_\mathrm{W}\left(\{f, h\}\right).$$

By Lemma 1.13.2, we deduce that $Q(f) = Q_\mathrm{W}(f)$ for all $f \in P_3$. This finishes the proof. $\qquad\square$

Proof of Theorem 1.4. cf. §13.4 of [Hal13].

Suppose that Q exists. It suffices to consider the case $n = 1$. Let $x := x^1$ and $p := p_1$. Note that we have the identity of Poisson brackets:

$$\left\{x^3, p^3\right\} = 9x^2 p^2 = 3\left\{x^2 p, x p^2\right\} = \frac{1}{12}\left\{\left\{x^3, p^2\right\}, \left\{x^2, p^3\right\}\right\}. \tag{1.9}$$

Now we apply the quantisation map Q to both sides of the equation. Note that by Proposition 1.14, $Q = Q_\mathrm{W}$ when acting on these polynomials. We have

$$\begin{aligned}
0 &= Q\left(\left\{x^3, p^3\right\}\right) - \frac{1}{12}Q\left(\left\{\left\{x^3, p^2\right\}, \left\{x^2, p^3\right\}\right\}\right) \\
&= \frac{1}{\mathrm{i}\hbar}[Q(x^3), Q(p^3)] - \frac{1}{12(\mathrm{i}\hbar)^3}\left[\left[Q(x^3), Q(p^2)\right], \left[Q(x^2), Q(p^3)\right]\right] \\
&= \frac{1}{\mathrm{i}\hbar}[Q_\mathrm{W}(x^3), Q_\mathrm{W}(p^3)] - \frac{1}{12(\mathrm{i}\hbar)^3}\left[\left[Q_\mathrm{W}(x^3), Q_\mathrm{W}(p^2)\right], \left[Q_\mathrm{W}(x^2), Q_\mathrm{W}(p^3)\right]\right] \\
&= \frac{1}{\mathrm{i}\hbar}[\widehat{x}^3, \widehat{p}^3] - \frac{1}{12(\mathrm{i}\hbar)^2}\left[\left[\widehat{x}^3, \widehat{p}^2\right], \left[\widehat{x}^2, \widehat{p}^3\right]\right] \\
&= 3(\widehat{x}^2\widehat{p}^2 + \widehat{p}\widehat{x}^2\widehat{p} + \widehat{p}^2\widehat{x}^2) - \frac{1}{3\mathrm{i}\hbar}\left[(\widehat{x}^2\widehat{p} + \widehat{x}\widehat{p}\widehat{x} + \widehat{p}\widehat{x}^2), (\widehat{p}^2\widehat{x} + \widehat{p}\widehat{x}\widehat{p} + \widehat{x}\widehat{p}^2)\right] \\
&= 3\left(3\widehat{p}^2\widehat{x}^2 + 6\mathrm{i}\hbar\widehat{p}\widehat{x} + 2(\mathrm{i}\hbar)^2\,\mathrm{id}\right) - \frac{1}{3}\left(27\widehat{p}^2\widehat{x}^2 + 54\mathrm{i}\hbar\widehat{p}\widehat{x} + 9(\mathrm{i}\hbar)^2\,\mathrm{id}\right) \\
&= -3(\mathrm{i}\hbar)^2\,\mathrm{id}
\end{aligned}$$

which is a contradiction. $\qquad\square$

Chapter 2

Deformation Theory

2.1 Deformations of Associative Algebras

In this section, we consider the deformation problems in a more general setting. From now on, we denote by k a fixed ground field of characteristic 0. Let A be an associative k-algebra with the multiplication map $\mu\colon A \otimes_k A \to A$. We would like to deform μ into another product $\star\colon A_R \otimes_k A_R \to A_R$ using a test algebra R.

2.1.1 Deformation Functor

Definition 2.1. A **test algebra** R is a commutative local Artinian k-algebra with the maximal ideal $\mathfrak{m}_R$ and the residue field $R/\mathfrak{m}_R \cong k$. The category of test algebras is denoted by $\mathscr{R}$, whose morphisms are local k-algebra homomorphisms, that is, the k-algebra homomorphisms $f\colon R \to S$ satisfying $f(\mathfrak{m}_R) \subseteq \mathfrak{m}_S$.

Remark. Note that R is commutative local Artinian implies that $\mathfrak{m}_R$ is nilpotent. We will use this fact in the proof of Lemma 2.21.

Definition 2.2. Let R be a test algebra. Let $A_R := A \otimes_k R$. An **R-deformation** of the multiplication μ on A is an associative R-linear map $\star\colon A_R \otimes_R A_R \to A_R$ such that $\mu = \star$ on $A_R/\mathfrak{m}_R \otimes_R A_R/\mathfrak{m}_R$. This can be represented in the commutative diagram:

$$
\begin{array}{ccc}
A_R \otimes_R A_R & \xrightarrow{\ \mathrm{mod}\,\mathfrak{m}_R\ } & A \otimes_k A \\
{\scriptstyle \star}\downarrow & & \downarrow{\scriptstyle \mu} \\
A_R & \xrightarrow{\ \mathrm{mod}\,\mathfrak{m}_R\ } & A
\end{array}
$$

Note that an R-deformation is uniquely determined by its restriction on $A \otimes_k A$. The R-deformations $\star$ and $\star'$ are said to be equivalent, if there exists an R-module automorphism $\varphi\colon A_R \to A_R$ such that $\varphi(u \star v) = \varphi(u) \star' \varphi(v)$ and $\varphi = \mathrm{id}$ on $A_R/\mathfrak{m}_R$.

Definition 2.3. For a test algebra R, let $\mathsf{Defo}_A(R)$ be the set of equivalence classes of R-deformations of A. Then we obtain a functor $\mathsf{Defo}_A\colon \mathscr{R} \to \mathsf{Set}$, called the **deformation functor**.

Example 2.4. An *infinitesimal deformation* of A is a R-deformation with $R = k[\hbar]/\langle \hbar^2 \rangle$.

Example 2.5. A *formal deformation* is a R-deformation with $R = k[\![\hbar]\!]$. Although $k[\![\hbar]\!]$ is not Artinian, it can be expressed as a completion in the $\hbar$-adic topology: $k[\![\hbar]\!] = \varprojlim_m k[\hbar]/\langle \hbar^m \rangle$. In fact, we have an natural isomorphism of functors:

$$
\mathsf{Defo}_A(k[\![\hbar]\!]) \simeq \varprojlim_m \mathsf{Defo}_A(k[\hbar]/\langle \hbar^m \rangle).
$$

So we can study the formal deformations by studying the deformation functor.

Example 2.6. Let $A := \mathrm{C}^\infty(M)$. A *star product* $\star$ on A is an associative $\mathbb{R}[\![\hbar]\!]$-deformation such that $u \star v = uv + \sum_{n=1}^{\infty} \hbar^n B_n(u, v)$ for $u, v \in A$, where B_n are bi-differential operators in the local coordinates of M.

2.1.2   Hochschild Cohomology

The information of deformations are encoded in the Hochschild complex defined below. This part consists of standard material which can be found in any homological algebra textbook. We include the details here for completeness.

Definition 2.7. For each $p \in \mathbb{Z}_{>0}$, let $C^p(A, A) := \mathrm{Hom}_k(A^{\otimes p}, A)$. We define the **Hochschild differential** $\delta_{\mathsf{H}} \colon C^p(A, A) \to C^{p+1}(A, A)$ as

$$(\delta_{\mathsf{H}} f)(u_0, ..., u_p) = u_0 f(u_1, ..., u_p) + \sum_{r=1}^{p} (-1)^r f(u_0, ..., u_{r-1} u_r, ..., u_p) + (-1)^{p+1} f(u_0, ..., u_{p-1}) u_p.$$

The chain can be augmented to $p = 0$ by defining $\delta_{\mathsf{H}} \colon C^0(A, A) \cong A \to C^1(A, A)$ as

$$(\delta_{\mathsf{H}} a)(u_0) = u_0 a - a u_0.$$

> **Lemma 2.8. Hochschild Complex**
>
> The Hochschild differential δ_{H} satisfies $\delta_{\mathsf{H}}^2 = 0$. Therefore $(C^p(A, A))_{p \geqslant 0}$ is a cochain complex, called the Hochschild complex.

Proof. Let $A^e := A \otimes_k A^{\mathrm{op}}$. We define the *bar complex* $(\mathsf{B}_n A, b_n)_{n \geqslant 0}$ as follows. Let $\mathsf{B}_n A := A \otimes_k A^{\otimes n} \otimes_k A$, with the left A^e-module structure

$$(u \otimes u')(u_1 \otimes \cdots u_n) := u u_1 \otimes \cdots \otimes u_n u'.$$

We define $b_n \colon \mathsf{B}_n A \to \mathsf{B}_{n-1} A$ as the A^e-module homomorphism given by

$$b_n(u_0 \otimes \cdots \otimes u_{n+1}) := \sum_{k=0}^{n} (-1)^k \cdots \otimes u_k u_{k+1} \otimes \cdots .$$

We claim that $b_{n-1} \circ b_n = 0$. Indeed, $b_{n-1} \circ b_n(u_0 \otimes \cdots \otimes u_{n+1})$ is a linear combination of the elements of form $\cdots \otimes u_i u_{i+1} \otimes \cdots \otimes u_j u_{j+1} \otimes \cdots$ and $\cdots \otimes u_{i-1} u_i u_{i+1} \otimes \cdots$. For each of these elements, there are exactly two ways of contractions from $u_0 \otimes \cdots \otimes u_{n+1}$, with opposite signs, which cancels each other. Hence $(\mathsf{B}_n A, b_n)_{n \geqslant 0}$ is indeed a chain complex.

Next we have an A^e-module isomorphism $\varphi \colon C^n(A, A) \xrightarrow{\sim} \mathrm{Hom}_{A^e}(\mathsf{B}_n A, A)$ given by $f \mapsto (u_0 \otimes \cdots \otimes u_{n+1} \mapsto u_0 f(u_1, ..., u_n) u_{n+1})$. Moreover, the following diagram commutes:

$$
\begin{array}{ccccc}
C^{n+1}(A, A) & \xrightarrow{\varphi} & \mathrm{Hom}_{A^e}(\mathsf{B}_{n+1} A, A) & \xrightarrow{\mathrm{dual}} & \mathsf{B}_{n+1} A \\
{\scriptstyle \delta_{\mathsf{H}}} \uparrow & & {\scriptstyle b^\vee} \uparrow & & \downarrow {\scriptstyle b} \\
C^n(A, A) & \xrightarrow{\varphi} & \mathrm{Hom}_{A^e}(\mathsf{B}_n A, A) & \xrightarrow{\mathrm{dual}} & \mathsf{B}_{n+1} A
\end{array}
$$

This implies that $\delta_{\mathsf{H}}^2 = 0$. $\square$

The cohomology of the Hochschild complex $(C^\bullet(A, A), \delta_{\mathsf{H}})$ is called **Hochschild cohomology**, and is denoted by $\mathrm{HH}^\bullet(A, A)$. We shall see in Section 2.3.3 that, when A is the algebra of smooth functions on a smooth manifold M, the Hochschild cohomology computes the polyvector fields on M.

Remark. The n-th Hochschild cohomology $\mathrm{HH}^n(A, A)$ can also be identified with the Ext group $\mathrm{Ext}^n_{A^e}(A, A)$ in the category of A^e-modules.

2.1.3 Gerstenhaber Bracket

We define a graded Lie bracket on the shifted[1] Hochschild complex $C^\bullet(A, A)[1]$.

Definition 2.9. For $f \in C^{p+1}(A, A)$ and $g \in C^{q+1}(A, A)$, we define the **Gerstenhaber product** $f \bullet g \in C^{p+q+1}(A, A)$ as

$$(f \bullet g)(u_0, ..., u_{p+q}) := \sum_{i=0}^{p} (-1)^{iq} (f \bullet_i g)(u_0, ..., u_{p+q})$$

$$:= \sum_{i=0}^{p} (-1)^{iq} f(u_0, ..., u_{i-1}, g(u_i, ..., u_{i+q}), u_{i+q+1}, ..., u_{p+q}).$$

Then we define the **Gerstenhaber bracket** of f and g as

$$[f, g]_{\mathsf{G}} := f \bullet g - (-1)^{pq} g \bullet f$$

and extend it k-bilinearly to $C^\bullet(A, A)$.

Lemma 2.10. Graded Jacobi identity for Gerstenhaber bracket

For $f \in C^{p+1}(A, A)$, $g \in C^{q+1}(A, A)$, and $h \in C^\bullet(A, A)$, the Gerstenhaber bracket satisfies
$$[f, [g, h]_{\mathsf{G}}]_{\mathsf{G}} = [[f, g]_{\mathsf{G}}, h]_{\mathsf{G}} + (-1)^{pq} [g, [f, h]_{\mathsf{G}}]_{\mathsf{G}}.$$

Proof. By linearity, we may assume that $h \in C^{r+1}(A, A)$. Consider the *associator* of the Gerstenhaber product:

$$A(f, g, h) := (f \bullet g) \bullet h - f \bullet (g \bullet h).$$

A direct computation shows that

$$[[f, g]_{\mathsf{G}}, h]_{\mathsf{G}} + (-1)^{|f| \cdot |g|} [g, [f, h]_{\mathsf{G}}]_{\mathsf{G}} - [f, [g, h]_{\mathsf{G}}]_{\mathsf{G}}$$
$$= -(-1)^{pq} A(g, f, h) + A(f, g, h) - (-1)^{qr} A(f, h, g)$$
$$+ (-1)^{pq+pr} A(g, h, f) + (-1)^{pr+qr} A(h, f, g) - (-1)^{pq+pr+qr} A(h, g, f).$$

The right-hand side of the above equation would vanish if the associator satisfies the graded symmetry:

$$A(f, g, h) = (-1)^{qr} A(f, h, g).$$

By definition we have:

$$A(f, g, h) - (-1)^{qr} A(f, h, g) = \sum_{i=0}^{p} \sum_{j=0}^{p+q} (-1)^{i(q+r)} (f \bullet_i g) \bullet_j h - \sum_{i=0}^{p} \sum_{j=0}^{q} (-1)^{iq} f \bullet_i (g \bullet_j h)$$
$$- (-1)^{qr} \left(\sum_{i=0}^{p} \sum_{j=0}^{p+r} (-1)^{i(q+r)} (f \bullet_i h) \bullet_j g - \sum_{i=0}^{p} \sum_{j=0}^{r} (-1)^{ir} f \bullet_i (h \bullet_j g) \right) \tag{2.1}$$

The composition of partial products satisfies:

---
[1] For a graded vector space $V := \bigoplus V^i$, the shifted space $V[k]$ has the grading $V[k]^i := V^{i+k}$.

$$(f \bullet_i g) \bullet_j h = \begin{cases} (f \bullet_j h) \bullet_i g, & i > j, \\ f \bullet_i (g \bullet_{j-i} h), & i \leqslant j \leqslant i + q, \\ (f \bullet_{j-q} h) \bullet_i g, & i + q < j. \end{cases}$$

Substituting this into (2.1) we see that the right-hand side is cancelled. Hence $A(f, g, h) = (-1)^{qr} A(f, h, g)$.
□

Lemma 2.11. Hochschild Differential via Gerstenhaber Bracket

For $f \in C^p(A, A)$, we have

$$\delta_{\mathsf{H}} f = (-1)^{p+1} [\mu, f]_{\mathsf{G}},$$

where δ_{H} is the Hochschild differential and $\mu \in C^2(A, A)$ is the multiplication on A.

Proof. This is straightforward by definition.

$$[\mu, f]_{\mathsf{G}} (u_0, ..., u_p) = (\mu \bullet f)(u_0, ..., u_p) - (-1)^{p-1}(f \bullet \mu)(u_0, ..., u_p)$$

$$= f(u_0, ..., u_{p-1})u_p + (-1)^{p-1}u_0 f(u_1, ..., u_p) + \sum_{r=1}^{p-1}(-1)^{r+p} f(u_0, ..., u_{r-1}u_r, ..., u_p)$$

$$= (-1)^{p+1}(\delta_{\mathsf{H}} f)(u_0, ..., u_{p+1}).$$
□

Remark. For reasons which will be clear shortly, we define a **modified differential** d on the Hochschild complex by $d_{\mathsf{H}} f = (-1)^{p+1} \delta_{\mathsf{H}} f$ for $f \in C^p(A, A)$. Then the new differential satisfies $d_{\mathsf{H}} f = [\mu, f]_{\mathsf{G}}$ for any $f \in C^\bullet(A, A)$. We will be using d_{H} instead of δ_{H} from now on.

Corollary 2.12

The multiplication μ on A has vanishing Gerstenhaber bracket: $[\mu, \mu]_{\mathsf{G}} = 0$.

Proof. This follows from associativity of μ:

$$[\mu, \mu]_{\mathsf{G}}(u, v, w) = 2u(vw) - 2(uv)w = 0.$$
□

Remark. The same proof shows that the associativity of any $\eta \in C^2(A, A)$ is equivalent to $[\eta, \eta]_{\mathsf{G}} = 0$.

Lemma 2.13. Graded Leibniz Rule

For $f \in C^{p+1}(A, A)$ and $g \in C^{q+1}(A, A)$, we have $d_{\mathsf{H}} [f, g]_{\mathsf{G}} = [d_{\mathsf{H}} f, g]_{\mathsf{G}} + (-1)^p [f, d_{\mathsf{H}} g]_{\mathsf{G}}$.

Proof. By Lemma 2.11, this is the the graded Jacobi identity, which is Lemma 2.10.
□

Remark. The compatible structure of a differential and a graded Lie bracket on the Hochschild complex gives rise to a structure of differential graded Lie algebra, which is introduced in the next section.

2.2 Differential Graded Lie Algebras

Definition 2.14. A **graded Lie algebra** $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}^i$ is a $\mathbb{Z}$-graded vector space over a field k with a bilinear map $[-,-] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, satisfying that

- $[x, y] \in \mathfrak{g}^{|x|+|y|}$;

- Graded skew-symmetry: $[x, y] = -(-1)^{|x| \cdot |y|}[y, x]$;

- Graded Jacobi identity: $(-1)^{|x| \cdot |y|}[[x, y], z] + (-1)^{|y| \cdot |z|}[[y, z], x] + (-1)^{|z| \cdot |x|}[[z, x], y] = 0$.

where $x, y, z \in \mathfrak{g}$ are homogeneous elements with grading $|x|, |y|, |z|$ respectively.

A **differential graded Lie algebra** (DGLA) is a graded Lie algebra $\mathfrak{g}$ with a differential $\mathrm{d} \colon \mathfrak{g}^i \to \mathfrak{g}^{i+1}$ which is k-linear of degree 1 and satisfies that

- Graded Leibniz rule: $\mathrm{d}[x, y] = [\mathrm{d}x, y] + (-1)^{|x|}[x, \mathrm{d}y]$;

- Sqaure-zero: $\mathrm{d}^2 = 0$.

Remark. Let $\mathfrak{g}$ be a graded Lie algebra. Then the zeroth grading $\mathfrak{g}^0$ is a Lie algebra in the usual sense.

Example 2.15. Following the results established in the previous section, the shifted Hochschild complex $C^\bullet(A, A)[1]$ with the Gerstenhaber bracket $[-,-]_{\mathsf{G}}$ and the modified Hochschild differential d is a DGLA.

Example 2.16. Let $\mathfrak{g}$ be a DGLA over k, and S be a graded commutative k-algebra, that is, $ab = (-1)^{|a| \cdot |b|}ba$ for homogeneous $a, b \in S$. We can define a DGLA structure on $\mathfrak{g} \otimes_k S$ as follows:

- $\mathfrak{g} \otimes_k S$ is graded by $(\mathfrak{g} \otimes_k S)^i = \bigoplus_{p+q=i} (\mathfrak{g}^p \otimes_k S^q)$;

- The differential is given by $\mathrm{d}(x \otimes a) = \mathrm{d}x \otimes a$ for $x \in \mathfrak{g}$ and $a \in S$;

- The graded Lie bracket is given by $[x \otimes a, y \otimes b] = (-1)^{|a| \cdot |y|}[x, y] \otimes ab$ for homogeneous $x, y \in \mathfrak{g}$ and $a, b \in S$.

A DGLA $(\mathfrak{g}, \mathrm{d})$ is naturally a cochain complex. The corresponding cohomology $\mathrm{H}^\bullet(\mathfrak{g}) := \bigoplus_{i \in \mathbb{Z}} \mathrm{H}^i(\mathfrak{g})$ is a graded vector space with an induced graded Lie bracket. $\mathrm{H}^\bullet(\mathfrak{g})$ with the zero differential is again a DGLA.

Definition 2.17. A **morphism of DGLAs** $f \colon (\mathfrak{g}, [,]_{\mathfrak{g}}, \mathrm{d}_{\mathfrak{g}}) \to (\mathfrak{h}, [,]_{\mathfrak{h}}, \mathrm{d}_{\mathfrak{h}})$ is a k-linear map $f \colon \mathfrak{g} \to \mathfrak{h}$ homogeneous of degree 0 such that $f \circ \mathrm{d}_{\mathfrak{g}} = \mathrm{d}_{\mathfrak{h}} \circ f$ and $f([x, y]_{\mathfrak{g}}) = [f(x), f(y)]_{\mathfrak{h}}$.

$$
\begin{array}{ccc}
\mathfrak{g}^i & \xrightarrow{\mathrm{d}_{\mathfrak{g}}} & \mathfrak{g}^{i+1} \\
f \downarrow & & \downarrow f \\
\mathfrak{h}^i & \xrightarrow{\mathrm{d}_{\mathfrak{h}}} & \mathfrak{h}^{i+1}
\end{array}
$$

If f induces the isomorphism of the cohomologies $\mathrm{H}^\bullet(f) \colon \mathrm{H}^\bullet(\mathfrak{g}) \to \mathrm{H}^\bullet(\mathfrak{h})$, then f is called a **quasi-isomorphism of DGLAs**.

2.2.1 Maurer–Cartan Equation

In general deformation theory, the philosophy due to Deligne ([Del87]) is that deformation problems in characteristic 0 are controlled by some differential graded Lie algebra. For deformation of algebras, the DGLA is the

shifted Hochschild complex. We show that the deformation functor we defined in the previous section can be identified with a quotient of the set of zeroes of the Maurer–Cartan equation.

Definition 2.18. Let $\mathfrak{g}$ be a DGLA. For $x \in \mathfrak{g}^1$, the **Maurer–Cartan equation** is given by

$$\mathrm{d}x + \frac{1}{2}[x, x] = 0. \tag{2.2}$$

The set of solutions is denoted by $\mathsf{MC}(\mathfrak{g})$. It is clear from definition that the solutions of the Maurer–Cartan equation is preserved under a morphism of DGLAs.

$\mathsf{MC}(\mathfrak{g})$ is also preserved under the gauge actions of $\mathfrak{g}^0$, which will be defined and studied in the following. For this part we follow the approach outlined in [Man05].

We would like to adjoint an element δ of degree 1 to $\mathfrak{g}$ such that $\mathrm{d}x = [\delta, x]$. To do this, we construct a new DGLA $\mathfrak{g}_\delta$ given by $\mathfrak{g}_\delta^1 = \mathfrak{g}^1 \oplus k\delta$ and $\mathfrak{g}_\delta^i = \mathfrak{g}^i$ for $i \neq 1$. The graded Lie bracket and the differential are defined for homogeneous elements by

$$[x + v\delta, y + w\delta]_\delta = [x, y] + v\mathrm{d}y + (-1)^{|x|}w\mathrm{d}x, \qquad \mathrm{d}_\delta(x + v\delta) = \mathrm{d}x = [x, v\delta]_\delta.$$

The Maurer–Cartan equation can be expressed as

$$\mathrm{d}x + \frac{1}{2}[x, x] = 0 \quad \iff \quad [x + \delta, x + \delta]_\delta = 0. \tag{2.3}$$

We define the right adjoint action of $\mathfrak{g}^0$ on $\mathfrak{g}_\delta^1$, given by $\overline{\mathrm{ad}}(x_0)(y + v\delta) = [y + v\delta, x_0]_\delta$. Suppose that $\mathfrak{g}^0$ is ad-nilpotent. The following lemma shows that the Lie algebra action can be exponentiated to a group action:

Lemma 2.19. Exponentiation of Nilpotent Lie Algebras

There exists a functor exp, which attaches a group G to each nilpotent Lie algebra $\mathfrak{h}$, such that there is a surjection $\exp \colon \mathfrak{h} \to G$ satisfying $\exp(x)\exp(y) = \exp(x \star y)$, where $x \star y \in \mathfrak{h}$ is given by the Baker–Campbell–Hausdorff formula:

$$x \star y := x + y + \frac{1}{2}[x, y] + \frac{1}{12}[x, [x, y]] - \frac{1}{12}[y, [x, y]] + \cdots . \tag{2.4}$$

The infinite sum truncates at some finite term because of the nilpotency of $\mathfrak{h}$.

Proof. A discussion of this fact can be found in e.g. [Get09]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Remark. Moreover, for any representation $\rho \colon \mathfrak{h} \to \mathfrak{gl}(V)$ such that $\rho(\mathfrak{h})$ is a nilpotent subalgebra of $\mathfrak{gl}(V)$, the corresponding exponential representation $\exp(\rho) \colon \exp(\mathfrak{h}) \to \mathrm{GL}(V)$ is given by $(\exp\rho)\mathrm{e}^a = \mathrm{e}^{\rho(a)}$, where e^a denotes the usual exponential of an endomorphism.

Therefore $\overline{\mathrm{ad}}$ induces the action of $\exp(\mathfrak{g}^0)$ on $\mathfrak{g}_\delta^1$. We embed $\mathfrak{g}^1$ into $\mathfrak{g}_\delta^1$ as a hyperplane $\{x + \delta \colon x \in \mathfrak{g}^1\}$, which is preserved under this group action. Explicitly, we obtain a right action of $\exp(\mathfrak{g}^0)$ on $\mathfrak{g}^1$, given by

$$\exp(x_0) \longmapsto \left(y \longmapsto \mathrm{e}^{\overline{\mathrm{ad}}(x_0)}(y + \delta) - \delta = y + \mathrm{d}x_0 + [y, x_0] + \frac{1}{2}\left([\mathrm{d}x_0, x_0] + [[y, x_0], x_0]\right) + \cdots \right)$$

The group $\exp(\mathfrak{g}^0)$ is called the **gauge group**. On the other hand, $\overline{\mathrm{ad}}$ induces the right action of $\mathfrak{g}^0$ on $\mathfrak{g}^1$ given by

$$x_0 \longmapsto (y \longmapsto \mathrm{d}x_0 + [y, x_0]) .$$

The key observation is that:

> **Lemma 2.20**
>
> The gauge action of $\exp(\mathfrak{g}^0)$ on $\mathfrak{g}^1$ preserves the set $\mathsf{MC}(\mathfrak{g})$ of the solutions of the Maurer–Cartan equation.

Proof. We have $[x + \delta, x + \delta]_\delta = 0$ if and only if $x \in \mathsf{MC}(\mathfrak{g})$ by (2.3). For convenience we denote by $x \cdot \exp(x_0)$ the right action of $\exp(x_0)$ on $x \in \mathfrak{g}^1$. Let $x \in \mathsf{MC}(\mathfrak{g})$. We have

$$\mathrm{d}(x \cdot \exp(x_0)) + \frac{1}{2}[x \cdot \exp(x_0), x \cdot \exp(x_0)]$$

$$= [\delta, \mathrm{e}^{\overline{\mathrm{ad}}(x_0)}(x + \delta)]_\delta + \frac{1}{2}\left([\mathrm{e}^{\overline{\mathrm{ad}}(x_0)}(x + \delta), \mathrm{e}^{\overline{\mathrm{ad}}(x_0)}(x + \delta)]_\delta - 2[\delta, \mathrm{e}^{\overline{\mathrm{ad}}(x_0)}(x + \delta)]_\delta\right)$$

$$= \mathrm{e}^{\overline{\mathrm{ad}}(x_0)}[x + \delta, x + \delta]_\delta = 0.$$

This shows that $x \cdot \exp(x_0) \in \mathsf{MC}(\mathfrak{g})$. $\qquad\square$

> **Lemma 2.21**
>
> Suppose that R is a test k-algebra (*cf.* Definition 2.1) and $\mathfrak{g}$ is any DGLA. Then the DGLA $\mathfrak{g} \otimes_k \mathfrak{m}_R$ is ad-nilpotent in its zeroth grading.

Proof. The grading of $\mathfrak{m}_R$ is centred at zero, so $(\mathfrak{g} \otimes_k \mathfrak{m}_R)^0 = \mathfrak{g}^0 \otimes_k \mathfrak{m}_R$. The graded Lie bracket is given by $[x \otimes a, y \otimes b] = [x, y] \otimes ab$, where $x, y \in \mathfrak{g}$ are homogeneous and $a, b \in \mathfrak{m}_R$. Since R is local Artinian, $\mathfrak{m}_R$ is nilpotent. For sufficiently large n, we have

$$\mathrm{ad}(x \otimes a)^n(y \otimes b) = (\mathrm{ad}\,x)^n(y) \otimes a^n b = 0.$$

By taking linear combinations we deduce that $(\mathfrak{g} \otimes_k \mathfrak{m}_R)^0$ is ad-nilpotent. $\qquad\square$

Following 2.21, for a test algebra R, we have a well-defined action of $\exp(\mathfrak{g}^0 \otimes_k \mathfrak{m}_R)$ on $\mathsf{MC}(\mathfrak{g} \otimes_k \mathfrak{m}_R)$. The set of orbits is denoted by

$$\overline{\mathsf{MC}}_{\mathfrak{g}}(R) := \mathsf{MC}(\mathfrak{g} \otimes_k \mathfrak{m}_R)/\exp(\mathfrak{g}^0 \otimes_k \mathfrak{m}_R). \tag{2.5}$$

This is referred as the Maurer–Cartan moduli set. We can also check that the guage actions are preserved under a morphism of DGLAs. In this way we obtain a functor $\overline{\mathsf{MC}}_{\mathfrak{g}} : \mathscr{R} \to \mathsf{Set}$, called the **Maurer–Cartan functor** associated to $\mathfrak{g}$. This is related to the deformation functor in Definition 2.3 by the following proposition.

> **Proposition 2.22. Deformation via Maurer–Cartan Functor**
>
> Let A be an associative k-algebra, and $C^\bullet(A, A)[1]$ be the corresponding shifted Hochschild complex. Then we have an isomorphism of functors:
> $$\mathsf{Defo}_A \simeq \overline{\mathsf{MC}}_{C^\bullet(A,A)[1]}.$$

Proof. Let R be a test algebra. Let μ be the multiplication on A, We consider $\star \in \mathrm{Hom}_k(A \otimes_k A, A \otimes_k R)$, and let $B(u, v) := u \star v - uv$. So $B = \star - \mu$ is of degree 1 in the DGLA $\mathfrak{g} := C^\bullet(A, A)[1] \otimes_k \mathfrak{m}_R$. We can compute the Maurer–Cartan equation for B:

$$\mathrm{d}_{\mathsf{H}}B + \frac{1}{2}[B, B]_{\mathsf{G}} = \mathrm{d}\star - \mathrm{d}\mu + \frac{1}{2}[\mu, \mu]_{\mathsf{G}} + \frac{1}{2}[\star, \star]_{\mathsf{G}} - [\mu, \star]_{\mathsf{G}} = \frac{1}{2}[\star, \star]_{\mathsf{G}}.$$

The vanishing of $[\star, \star]_{\mathsf{G}}$ is exactly the associativity condition for $\star$. We deduce that $\star$ is a R-deformation if and only if $B \in \mathsf{MC}(\mathfrak{g})$.

Next, we need to check that the equivalence of R-deformations in Definition 2.2 is generated by the gauge group $\exp(\mathfrak{g}^0)$. Before that we need to know how this action works. Since we have $\mu \in \mathfrak{g}^1$ such that $\mathrm{d}_{\mathsf{H}} x = [\mu, x]_{\mathsf{G}}$ for $x \in \mathfrak{g}$, the adjunction of δ is unnecessary. $\exp(\mathfrak{g}^0)$ acts on $\mathfrak{g}^1$ by

$$\exp(x_0) \longmapsto \left(y \longmapsto \mathrm{e}^{[-, x_0]_{\mathsf{G}}}(y + \mu) - \mu\right).$$

Suppose that $\star$ and $\star'$ are two R-deformations. Note that $B = \star - \mu$ and $B' = \star' - \mu$ are related by a gauge action $\exp(f)$ for some $f \in \mathfrak{g}^0$ if and only if $\star' = \mathrm{e}^{[-, f]_{\mathsf{G}}} \star$. This is equivalent to saying that f is an R-equivalence of $\star$ and $\star'$ in the sense of Definition 2.2.

In summary, we obtain a bijective correspondence:

$$\mathsf{Defo}_A(R) \simeq \overline{\mathsf{MC}}_{C^\bullet(A,A)[1]}(R).$$

It is straightforward to check that the correspondence is functorial in R. $\square$

Remark. It is more common in the literature (*cf.* [Kon97], [Man05]) to call $\overline{\mathsf{MC}}_{\mathfrak{g}}$ the deformation functor. Nevertheless we have shown that this is equivalent to our Definition 2.3.

Remark. Following Example 2.5, it is straightforward to extend the previous discussion to the formal deformation $R = k[\![\hbar]\!]$ as a completion:

$$\mathfrak{g} \,\widehat{\otimes}\, \mathfrak{m}_R := \varprojlim_n \mathfrak{g} \otimes \hbar \cdot k[\hbar]/\langle \hbar^n \rangle = \hbar \mathfrak{g}[\![\hbar]\!].$$

For $x \in (\mathfrak{g} \,\widehat{\otimes}\, \mathfrak{m}_R)^1 = \hbar \mathfrak{g}^1[\![\hbar]\!]$, the associated Maurer–Cartan equation is called the **formal Maurer–Cartan equation**.

Before closing the section, we mention a classical result in deformation theory. We will present and prove a generalised version of it in Section 2.4.4.

Theorem 2.23. Quasi-Isomorphism Theorem for DGLAs

Let $f \colon \mathfrak{g} \to \mathfrak{g}'$ be a morphism of DGLAs. If f is a quasi-isomorphism, then f induces an natural isomorphism of the Maurer–Cartan functors:

$$\overline{\mathsf{MC}}_{\mathfrak{g}} \simeq \overline{\mathsf{MC}}_{\mathfrak{g}'}.$$

Proof. See Theorem 3.1 of [Man05]. $\square$

2.3   Polydifferential Operators and Polyvector Fields

We have seen how the differential graded Lie algebra controls the deformation problem of a general associative algebra. In this section we return to the settings in differential geometry. Let M be a smooth manifold. Denote by $A := \mathrm{C}^\infty(M)$ the smooth functions on M.

Definition 2.24. An ***n*-differential operator** $f \in \mathrm{Hom}_{\mathbb{R}}(A^{\otimes n}, A)$ is a n-linear map such that f is a differential operator in each argument. The set of all n-differential operators is denoted by $\mathscr{D}_{\mathsf{poly}}^{n-1}(M)$. Note that $\mathscr{D}_{\mathsf{poly}}^n(M) \subseteq C^{n+1}(A, A)$ and $\delta(\mathscr{D}_{\mathsf{poly}}^n(M)) \subseteq \mathscr{D}_{\mathsf{poly}}^{n+1}(M)$, where δ is the Hochschild differential. Therefore $\mathscr{D}_{\mathsf{poly}}^\bullet(M)$ is a subcomplex of the shifted Hochschild complex $C^\bullet(A, A)[1]$, and is called the **differential Hochschild complex**.

Just like $C^\bullet(A, A)[1]$, $\mathscr{D}_{\mathsf{poly}}^\bullet(M)$ with the modified Hochschild differential $\mathsf{d_H}$ and the Gerstenhaber bracket $[-, -]_\mathsf{G}$ is a DGLA.

With $R = \mathbb{R}[\![\hbar]\!]$ and $\mathfrak{m}_R = \hbar\mathbb{R}[\![\hbar]\!]$, an analogy of Proposition 2.22 is the following:

> **Proposition 2.25. Star Products via Maurer–Cartan Equation**
>
> The set of equivalence classes of star products on M is in bijective correspondence with the set of solutions of the *formal* Maurer–Cartan equation for $\mathscr{D}_{\mathsf{poly}}(M)$ modulo gauge actions:
>
> $$\overline{\mathsf{MC}}_{\mathscr{D}_{\mathsf{poly}}(M)}(\mathbb{R}[\![\hbar]\!]) = \mathsf{MC}\big(\hbar\mathscr{D}_{\mathsf{poly}}(M)[\![\hbar]\!]\big)/\exp\big(\hbar\mathscr{D}_{\mathsf{poly}}^0(M)[\![\hbar]\!]\big).$$

2.3.1 Polyvector Fields

To study the deformations of the Poisson structure on M, we need to introduce another differential graded Lie algebra.

Definition 2.26. Let $\mathrm{T}M$ be the tangent bundle of M. An **n-vector field** is a section of the n-th exterior product $\bigwedge^n \mathrm{T}M$. The set of n-vector fields is denoted by $\mathscr{T}_{\mathsf{poly}}^{n-1}(M) := \Gamma(M, \bigwedge^n \mathrm{T}M)$. The set of polyvector fields is the graded vector space given by

$$\mathscr{T}_{\mathsf{poly}}(M) = \bigoplus_{i=-1}^{\infty} \mathscr{T}_{\mathsf{poly}}^i(M) = \Gamma(M, \textstyle\bigwedge^\bullet \mathrm{T}M)[1],$$

where $\bigwedge^0 \mathrm{T}M = M$ by convention. Also notice that we have shifted the degree by 1, as in the case of $\mathscr{D}_{\mathsf{poly}}(M)$.

Recall that $\mathscr{T}_{\mathsf{poly}}^0(M) = \Gamma(M, \mathrm{T}M)$ has a Lie algebra structure given by the Lie derivatives:

$$[X, Y] := \mathcal{L}_X Y.$$

It has a unique extension to the graded Lie algebra structure on $\mathscr{T}_{\mathsf{poly}}(M)$:

Definition 2.27. For $X := X_1 \wedge \cdots \wedge X_p \in \mathscr{T}_{\mathsf{poly}}^{p-1}(M)$ and $Y := Y_1 \wedge \cdots \wedge Y_q \in \mathscr{T}_{\mathsf{poly}}^{q-1}(M)$, we define the **Schouten–Nijenhuis bracket** $[X, Y]_{\mathsf{SN}} \in \mathscr{T}_{\mathsf{poly}}^{p+q-2}(M)$ as

$$[X, Y]_{\mathsf{SN}} := \sum_{r=1}^{p}\sum_{s=1}^{q}(-1)^{r+s}[X_r, Y_s] \wedge X_1 \wedge \cdots \wedge \widehat{X}_r \wedge \cdots \wedge X_p \wedge Y_1 \wedge \cdots \wedge \widehat{Y}_s \wedge \cdots \wedge Y_q.$$

where $\widehat{X}_i$ denotes the omission of X_i in the expression. If $f \in \mathrm{C}^\infty(M) = \mathscr{T}_{\mathsf{poly}}^{-1}(M)$, we define:

$$[X, f]_{\mathsf{SN}} := \sum_{r=1}^{p}(-1)^{p-r}X_r(f) \cdot X_1 \wedge \cdots \wedge \widehat{X}_r \wedge \cdots \wedge X_p.$$

Finally we extend the bracket bilinearly to $\mathscr{T}_{\mathsf{poly}}^\bullet(M)$.

> **Lemma 2.28. Properties of Schouten–Nijenhuis Bracket**
>
> The Schouten–Nijenhuis bracket is the unique graded Lie bracket on $\mathscr{T}_{\mathsf{poly}}(M)$ such that
>
> - $[X, f]_{\mathsf{SN}} = \mathcal{L}_X f = X(f)$ for $X \in \Gamma(M, \mathrm{T}M)$ and $f \in \mathrm{C}^\infty(M)$;
>
> - $[X, Y]_{\mathsf{SN}} = \mathcal{L}_X Y$ for $X, Y \in \Gamma(M, \mathrm{T}M)$;
>
> - $[X, Y \wedge Z]_{\mathsf{SN}} = [X, Y]_{\mathsf{SN}} \wedge Z + (-1)^{(p-1)q} Y \wedge [X, Z]_{\mathsf{SN}}$ $\qquad\qquad$ $(*)$
>
>   for $X, Y, Z \in \mathscr{T}_{\mathsf{poly}}(M)$ homogeneous of degree p, q, r respectively.

Proof. We can verify all properties (graded skew-symmetry, graded Jacobi identity, equation $(*)$) of the Schouten–Nijenhuis bracket directly by definition, though some brute force computations are inevitable. We omit the details, which can be found in, e.g. [CI04]. The uniqueness is proven by inductions on the degree of X and Y consecutively. $\qquad\qquad\square$

Remark. Now $\mathscr{T}_{\mathsf{poly}}(M)$ with the Schouten–Nijenhuis bracket $[-, -]_{\mathsf{SN}}$ and the *zero differential* is a DGLA.

In local coordinates, a polyvector field $X \in \mathscr{T}_{\mathsf{poly}}^{p+1}(M)$ has the expansion in terms of the frame vector fields:

$$X = \sum_{i_1, \dots, i_p = 1}^{\dim M} X^{i_1, \dots, i_p} \partial_{i_1} \wedge \cdots \wedge \partial_{i_k}.$$

Analogous to the Gerstenhaber product, for $X \in\in \mathscr{T}_{\mathsf{poly}}^{p-1}(M)$ and $Y \in\in \mathscr{T}_{\mathsf{poly}}^{q-1}(M)$ we can define the product $X \diamond Y \in \mathscr{T}_{\mathsf{poly}}^{p+q-2}(M)$ by:

$$X \diamond Y := \sum_{r=1}^{p} \sum_{i_1, \dots, i_p, j_1, \dots, j_q = 1}^{\dim M} (-1)^{r+1} X^{i_1, \dots, i_p} \partial_r Y^{j_1, \dots, j_q} \partial_{i_1} \wedge \cdots \wedge \widehat{\partial_{i_r}} \wedge \cdots \wedge \partial_{i_p} \wedge \partial_{j_1} \wedge \cdots \wedge \partial_{j_q} \qquad (2.6)$$

> **Lemma 2.29**
>
> The Schouten–Nijenhuis bracket can be expressed in terms of the product defined in (2.6) as:
>
> $$[X, Y]_{\mathsf{SN}} = (-1)^p X \diamond Y - (-1)^{(p-1)q} Y \diamond X$$
>
> where $X \in \mathscr{T}_{\mathsf{poly}}^{p}(M)$ and $Y \in \mathscr{T}_{\mathsf{poly}}^{q}(M)$.

Proof. See Lemma IV.2.1 of [AMM02]. $\qquad\qquad\square$

Remark. Let $V := \Gamma(M, \mathrm{T}M)$. Then $\mathscr{T}_{\mathsf{poly}}^{p-1}(M) = \bigwedge^p V$. Using the décalage isomorphism given in (2.9), it is isomorphic to $\mathrm{Sym}^p(V[1])[-p]$ via $\mathrm{déc}_p$. In this way the product (2.6) has the following compact form:

$$X \diamond Y = \mathrm{déc}_{p+q-1} \left(\sum_{r=1}^{\dim M} \frac{\partial \mathrm{déc}_p^{-1}(X)}{\partial \psi_i} \frac{\partial \mathrm{déc}_p^{-1}(Y)}{\partial x_i} \right), \qquad (2.7)$$

where $X \in \mathscr{T}_{\mathsf{poly}}^{p-1}(M)$, $Y \in \mathscr{T}_{\mathsf{poly}}^{q-1}(M)$, and $\psi_i := \mathrm{déc}_1^{-1}(\partial_i)$ is an odd variable of degree 1.

2.3.2   Poisson Deformations

Now we would like to show how the DGLA $\mathscr{T}_{\mathsf{poly}}(M)$ controls the deformations of the Poisson bracket on M. First we shall make the definition precise. For a general Poisson algebra, the Poisson deformations can be formulated analogously to Definition 2.2. As before, We denote by k a ground field of characteristic 0.

Definition 2.30. Let R be a test algebra. Let $A_R := A \otimes_k R$. A **Poisson R-deformation** of the Poisson bracket π_0 on A is a Poisson bracket $\pi \colon A_R \otimes_k A_R \to A_R$ such that $\pi = \pi_0$ on $A_R/\mathfrak{m}_R \otimes_k A_R/\mathfrak{m}_R$. The Poisson R-deformations π and π' are said to be equivalent, if there exists an R-module automorphism $\varphi \colon A_R \to A_R$ such that $\varphi(\pi(u,v)) = \pi'(\varphi(u), \varphi(v))$ and $\varphi = \mathrm{id}$ on $A_R/\mathfrak{m}_R$.

For $A = \mathrm{C}^\infty(M)$ and $R = \mathbb{R}[\![\hbar]\!]$, a Poisson bracket on A_R is an element of $\mathscr{T}^1_{\mathsf{poly}}(M)[\![\hbar]\!]$. We say that it defines a **formal Poisson structure** on M in contrast to an actual Poisson structure on M, which is an element of $\mathscr{T}^1_{\mathsf{poly}}(M)$.

Lemma 2.31

The bivector field π on M has vanishing Schouten–Nijenhuis bracket $[\pi, \pi]_{\mathsf{SN}} = 0$ if and only if π is a Poisson bracket.

Proof. In local coordinates, $\pi = \sum\limits_{i,j=1}^{\dim M} \pi^{ij} \partial_i \wedge \partial_j$. The equation $[\pi, \pi]_{\mathsf{SN}} = 0$ becomes:

$$\sum_{i,j,k,\ell=1}^{\dim M} \pi^{ij} \partial_j \pi^{k\ell} \partial_i \wedge \partial_k \wedge \partial_\ell = 0.$$

On the other hand, π defines a Poisson structure if and only if it satisfies the Jacobi identity:

$$\pi(\pi(f,g), h) + \pi(\pi(g,h), f) + \pi(\pi(h,f), g) = 0, \qquad f, g, h \in \mathrm{C}^\infty(M).$$

In local coordinates, this becomes

$$\sum_{i,j,k,\ell=1}^{\dim M} \pi^{ij} \partial_j \pi^{k\ell} \left(\partial_i f \partial_k g \partial_\ell h + \partial_i g \partial_k h \partial_\ell f + \partial_i h \partial_k f \partial_\ell g \right).$$

So we have the equivalence as claimed. $\square$

Proposition 2.32. Poisson Structures via Maurer–Cartan Equation

The set of equivalence classes of formal Poisson structures on M is in bijective correspondence with the set of solutions of the *formal* Maurer–Cartan equation for $\mathscr{T}_{\mathsf{poly}}(M)$ modulo gauge actions:

$$\overline{\mathsf{MC}}_{\mathscr{T}_{\mathsf{poly}}(M)}(\mathbb{R}[\![\hbar]\!]) = \mathsf{MC}(\hbar\mathscr{T}_{\mathsf{poly}}(M)[\![\hbar]\!]) / \exp\left(\hbar\mathscr{T}^0_{\mathsf{poly}}(M)[\![\hbar]\!]\right).$$

Proof. The formal Maurer–Cartan equation for $\mathscr{T}_{\mathsf{poly}}(M)$ is given by

$$\mathrm{d}\pi + \frac{1}{2} [\pi, \pi]_{\mathsf{SN}} = \frac{1}{2} [\pi, \pi]_{\mathsf{SN}} = 0, \qquad \pi \in \hbar\mathscr{T}^1_{\mathsf{poly}}(M)[\![\hbar]\!].$$

The previous lemma shows that the set of solutions corresponds to the set of formal Poisson structures. By a similar proof as Proposition 2.22, it can be shown that the equivalence relation on formal Poisson structures is exactly generated by the gauge actions $\exp\left(\hbar\mathscr{T}^0_{\mathsf{poly}}(M)[\![\hbar]\!]\right)$. $\square$

Remark. Combining Theorem 2.23, Proposition 2.25, and Proposition 2.32, it is tempting to look for a quasi-isomorphism of DGLAs between $\mathscr{T}_{\mathsf{poly}}(M)$ and $\mathscr{D}_{\mathsf{poly}}(M)$, in which case we would obtain the claimed bijective correspondence (1.7). Unfortunately such quasi-isomorphism does not exist. In the next part we will construct a quasi-isomorphism of graded vector spaces, which does not preserve the graded Lie brackets. This is the first step towards the construction of a weaker notion of quasi-isomorphism, which will be introduced in the Section 2.4.

2.3.3   Hochschild–Kostant–Rosenberg Theorem

For a polyvector field $X = X_1 \wedge \cdots \wedge X_n \in \mathscr{T}_{\mathsf{poly}}^{n-1}(M)$, X acts on $f_1, ..., f_n \in \mathrm{C}^\infty(M)$ as

$$(X_1 \wedge \cdots \wedge X_n)(f_1, ..., f_n) = \frac{1}{n!} \det(X_i(f_j))_{i,j}.$$

This identifies X as a polydifferential operator which is skew-symmetric and 1-differential in each argument. Therefore we obtain a morphism of graded vector spaces,

$$U_1^{(0)} \colon \mathscr{T}_{\mathsf{poly}}(M) \to \mathscr{D}_{\mathsf{poly}}(M) \tag{2.8}$$

which is equal to the identity map at degree -1. We called it the **HKR map**.

Vey proved the following version of the Hochschild–Kostant–Rosenberg Theorem [2] ([Vey75]), which says that $U_1^{(0)}$ is a quasi-isomorphism of graded vector spaces.

Theorem 2.33. Smooth Hochschild–Kostant–Rosenberg Theorem

The HKR map $U_1^{(0)} \colon \mathscr{T}_{\mathsf{poly}}(M) \to \mathscr{D}_{\mathsf{poly}}(M)$ induces isomorphisms in the cohomology spaces

$$\mathrm{H}^n(U_1^{(0)}) \colon \mathscr{T}_{\mathsf{poly}}^n(M) = \mathrm{H}^n(\mathscr{T}_{\mathsf{poly}}(M)) \xrightarrow{\sim} \mathrm{H}^n(\mathscr{D}_{\mathsf{poly}}(M)).$$

Proof. First we check that $U_1^{(0)}$ is a chain map. Since $\mathscr{T}_{\mathsf{poly}}(M)$ has zero differential, we must have $\mathrm{d}_{\mathsf{H}} \circ U_1^{(0)} = 0$. Indeed, for $X = X_0 \wedge \cdots \wedge X_p \in \mathscr{T}_{\mathsf{poly}}^p(M)$ and $f_0, ..., f_n \in \mathrm{C}^\infty(M)$, we have

$$(-1)^p \mathrm{d}_{\mathsf{H}}(U_1^{(0)}(X))(f_0, ..., f_{p+1}) = f_0 U_1^{(0)}(X)(f_1, ..., f_{n+1})$$

$$+ \sum_{r=1}^{n+1} (-1)^r U_1^{(0)}(X)(f_0, ..., f_{r-1}f_r, ..., f_{n+1}) + (-1)^p U_1^{(0)}(X)(f_0, ..., f_n)f_{n+1}$$

$$= f_0 U_1^{(0)}(X)(f_1, ..., f_{n+1}) + \sum_{r=1}^{n+1} (-1)^r f_r U_1^{(0)}(X)(f_0, ..., \widehat{f_r}, ..., f_{n+1})$$

$$+ \sum_{r=1}^{n+1} (-1)^r f_{r-1} U_1^{(0)}(X)(f_0, ..., \widehat{f_{r-1}}, ..., f_{n+1}) + (-1)^p f_{n+1} U_1^{(0)}(X)(f_0, ..., f_n)$$

$$= 0,$$

where we used the fact that $U_1^{(0)}(X)$ acts on each argument as a derivation. Let $\mathrm{H}^n(U_1^{(0)}) \colon \mathscr{T}_{\mathsf{poly}}^n(M) \to \mathrm{H}^n(\mathscr{D}_{\mathsf{poly}}(M))$ be the induced map in the n-th cohomology. Note that $U_1^{(0)}$ embed $\mathscr{T}_{\mathsf{poly}}(M)$ into $\mathscr{D}_{\mathsf{poly}}(M)$ as a sub-complex of skew-symmetric 1-differential operators. So $\mathrm{H}^n(U_1^{(0)})$ is injective. To show that it is surjective, we shall show that every $\alpha \in \mathscr{D}_{\mathsf{poly}}^p(M)$ is of the form:

---

[2]Originally, the HKR theorem was purposed by Hochschild, Kostant, and Rosenberg ([HKR62]) for smooth affine varieties.

$$\alpha = d_H \beta + \gamma$$

for some $\beta \in \mathscr{D}_{\mathsf{poly}}^{p-1}(M)$ and $\gamma \in U_1^{(0)}(\mathscr{T}_{\mathsf{poly}}^p(M))$. By a standard argument by the partition of unity, it suffices to prove this in a local chart (*cf.* Proposition 2.13–15 of [GR99]). So we may assume that $M = \mathbb{R}^d$. We sketch the proof in this case following [GR99].

We use induction on the degree p of $\alpha \in \mathscr{D}_{\mathsf{poly}}^p(M)$. For $p = 0$, cocycles are vector fields so the result is true trivially. Suppose the result holds for all $r < p$. For $\alpha \in \mathscr{D}_{\mathsf{poly}}^p(M)$, consider the term of highest order q in f_0:

$$\alpha(f_0, ..., f_n) = \sum_{|I|=q} \partial_I f_0 \cdot \eta_I(f_1, ..., f_n) + \cdots = \sum_{|I|=q} (\partial_I \otimes \eta_I)(f_0 \otimes \cdots \otimes f_n) + \cdots,$$

where $I = (i_1, ..., i_q)$ is a multi-index, and $\eta \in \mathscr{D}_{\mathsf{poly}}^{p-1}(M)$. Taking the differential, we note that α is a p-cocycle implies that η_I are $(p-1)$-cocycles. In such case, by induction hypothesis, $\eta_I = d_H \xi_I + \zeta_I$ for some $\xi_I \in \mathscr{D}_{\mathsf{poly}}^{p-1}(M)$ and $\zeta_I \in U_1^{(0)}(\mathscr{T}_{\mathsf{poly}}^p(M))$. We claim that α is cohomologous to $\alpha' \in \mathscr{D}_{\mathsf{poly}}^p(M)$ with the differential term highest order q' in f_0 such that $q' < q$. This follows from a careful analysis of the expansions of the terms ζ_I and ψ arising from

$$\alpha + d_H \left(\sum_{|I|=q} \partial_I \otimes \xi_I \right) = \sum_{|I|=q} \partial_I \otimes \zeta_I + \psi.$$

Refer to [GR99] for details. By induction, we reduce to the case $q = 1$, where

$$\alpha = \sum_{i=0}^{p} \partial_i \otimes \eta_i.$$

With $\eta_i = d_H \xi_i + \zeta_i$ as before, we have

$$\alpha = -d_H \left(\sum_{i=0}^{p} \partial_i \otimes \xi_i \right) + \sum_{i=0}^{p} \partial_i \otimes \zeta_i,$$

where $\sum_{i=0}^{p} \partial_i \otimes \zeta_i$ is 1-differential in all arguments. Finally, we can show that every 1-differential operator is cohomologous to its total skew-symmetrisation. Therefore $\alpha = d_H \beta + \gamma$ for some $\beta \in \mathscr{D}_{\mathsf{poly}}^{p-1}(M)$ and $\gamma \in U_1^{(0)}(\mathscr{T}_{\mathsf{poly}}^p(M))$ as claimed. $\square$

The HKR map $U_1^{(0)}$ is not a morphism of DGLAs, because the Schouten–Nijenhuis bracket is not mapped to the Gerstenhaber bracket. However, the failure of $U_1^{(0)}$ being a graded Lie algebra homomorphism can be resolved by finding a morphism U whose first-order approximation $U_1 = U_1^{(0)}$. This is the motivation of introducing L∞-algebras and L∞-morphisms in the next section.

2.4 L∞-Algebras

In this section we discuss the L∞-algebras. We first present an abstract definition using graded coalgebras. This allows us to define L∞-morphisms in a clean way. Then we show its equivalence with another definition using the Taylor coefficients, which presents L∞-algebras as a generalisation of DGLAs in a natural way.

2.4.0   Sign Conventions for Tensor Algebras

Before going into the main topic, we quickly review the constructions and sign conventions which will be useful for the rest of the dissertation.

Convention. We adopt the **Koszul sign convention**. Let V and W be $\mathbb{Z}$-graded vector spaces. Let $f\colon V \to V$ and $g\colon W \to W$ be linear maps of degree $|f|$ and $|g|$ respectively. Then $f \otimes g \in \mathrm{End}(V \otimes W)$ is defined as

$$(f \otimes g)(x \otimes y) := (-1)^{|g|\cdot|x|} f(x) \otimes g(y),$$

where $x \in V$ and $y \in W$ are homogeneous. Colloquially, we pick up a sign $(-1)^{ab}$ whenever we commute two graded objects of degree a and b.

Definition 2.34. Let $V = \bigoplus_{i\in\mathbb{Z}} V^i$ be a graded k-vector space. Let $V^{\otimes n}$ be the n-th tensor product of V with itself. For homogeneous elements x, y, we define the **twisting map** τ by

$$\tau(x \otimes y) = (-1)^{|x|\cdot|y|}(y \otimes x).$$

The symmetric product and the exterior product are identified as quotient spaces of $V^{\otimes n}$:

$$\mathrm{Sym}^n V := V^{\otimes n}/\langle x \otimes y - \tau(x \otimes y)\rangle, \qquad \textstyle\bigwedge^n V := V^{\otimes n}/\langle x \otimes y + \tau(x \otimes y)\rangle.$$

The image of $x_1 \otimes \cdots \otimes x_n$ under the quotients are denoted by $x_1 \cdots x_n$ and $x_1 \wedge \cdots \wedge x_n$ respectively. These constructions give rise to the following algebras:

- Tensor algebra: $\bigotimes(V) := \bigoplus_{n=0}^{\infty} V^{\otimes n};$
- Reduced tensor algebra: $\bigotimes{}^{+}(V) := \bigoplus_{n=1}^{\infty} V^{\otimes n};$
- Symmetric algebra: $\mathrm{Sym}(V) := \bigoplus_{n=0}^{\infty} V^{\otimes n};$
- Reduced symmetric algebra: $\mathrm{Sym}^{+}(V) := \bigoplus_{n=1}^{\infty} \mathrm{Sym}^n V;$
- Exterior algebra: $\bigwedge(V) := \bigoplus_{n=0}^{\infty} \bigwedge^n V;$
- Reduced exterior algebra: $\bigwedge{}^{+}(V) := \bigoplus_{n=1}^{\infty} \bigwedge^n V.$

Remark. These algebras are graded by the induced grading from V:

$$|x_1 \otimes \cdots \otimes x_n| = \sum_{i=1}^{n} |x_i|.$$

Convention. Let $x := x_1 \otimes \cdots \otimes x_n \in V^{\otimes n}$. We identify x with its projection in $\mathrm{Sym}^n V$ (*resp.* in $\bigwedge^n V$). Let $\sigma \in S_n$ be a permutation. The **Koszul sign** $\varepsilon_x(\sigma)$ (*resp.* $\chi_x(\sigma)$) is defined as the change of sign when permuting the product x in the symmetric (*resp.* exterior) algebra:

$$x_1 \cdots x_n = \varepsilon_x(\sigma) x_{\sigma(1)} \cdots x_{\sigma(n)}, \qquad x_1 \wedge \cdots \wedge x_n = \chi_x(\sigma) x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)}.$$

Unlike the ungraded case, the definition of twisting map in a graded space allows the exchange of symmetric and exterior product by a shift in the degree. This is encoded in the following lemma.

Lemma 2.35. Décalage Isomorphism

There exists a canoncial isomorphism of degree 0 between the symmetric algebra and the shifted exterior algebra, namely the **décalage isomorphism**, given by

$$\text{déc}_n: \quad \text{Sym}^n(V[1]) \xrightarrow{\hspace{3cm}} {\textstyle\bigwedge}^n(V)[n] \tag{2.9}$$
$$x_1 \cdots x_n \longmapsto (-1)^{\sum_{i=1}^{n}(n-i)(|x_i|-1)} x_1 \wedge \cdots \wedge x_n$$

2.4.1 L∞-Algebras via Coalgebras

First of all, we need to introduce the coalgebras over a field k, which is the category-theoretic dual of the associative algebras. The notions of co-associativity, co-commutativity, co-freeness, and so on can be defined as for their counterparts for algebras, but with all arrows reversed in the corresponding commutative diagrams.

Definition 2.36. A **co-associative coalgebra** C over a field k is a k-vector space equipped with a k-linear co-multiplication $\Delta: C \to C \otimes_k C$ satisfying the co-associativity:

$$(\text{id}_C \otimes \Delta) \circ \Delta = (\Delta \otimes \text{id}_C) \circ \Delta.$$

A co-unit of C is a k-linear map $e: C \to k$ satisfying

$$(\text{id}_C \otimes e) \circ \Delta = (e \otimes \text{id}_C) \circ \Delta = \text{id}_C.$$

The commutative diagrams are shown below.

$$
\begin{array}{ccc}
C & \xrightarrow{\ \Delta\ } & C \otimes_k C \\
{\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle \text{id}_C \otimes \Delta} \\
C \otimes_k C & \xrightarrow{\ \Delta \otimes \text{id}_C\ } & C \otimes_k C \otimes_k C
\end{array}
\qquad
\begin{array}{ccc}
C & \xrightarrow{\ \Delta\ } & C \otimes_k C \\
{\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle \text{id}_C \otimes e} \\
C \otimes_k C & \xrightarrow{\ e \otimes \text{id}_C\ } & C
\end{array}
$$

A **graded coalgebra** $C = \bigoplus_{i \in \mathbb{Z}} C^i$ is both a graded vector space and a coalgebra, such that the co-multiplication is compatible with the grading:

$$\Delta(C^i) \subseteq \bigoplus_{p+q=i} (C^p \otimes_k C^q).$$

A morphism $F: (C, \Delta) \to (C', \Delta')$ of graded coalgebras is a linear map $F: C \to C'$ of degree 0 such that $\Delta' \circ F = (F \otimes F) \circ \Delta$. In [Kon97], such morphism is called a *pre-L∞-morphism*.

Definition 2.37. We introduce some more properties. A graded coalgebra C is said to be

- **graded co-commutative**, if $\Delta = \tau \circ \Delta$, where τ is the twisting map;

- **co-nilpotent**, if for each $x \in C$ there exists $n \in \mathbb{N}$ such that $\Delta^n(x) = 0$, where Δ^n is defined recursively by $\Delta^{n+1} = (\Delta \otimes \text{id}_{C^{\otimes n}}) \circ \Delta^n$.

Example 2.38. Let $V = \bigoplus_{i \in \mathbb{Z}} V^i$ be a graded k-vector space. Recall that the **reduced symmetric algebra** is given by

$$\text{Sym}^+(V) := \bigoplus_{n=1}^{\infty} \text{Sym}^n(V) = \bigoplus_{n=1}^{\infty} V^{\otimes n} \Big/ \Big\langle x \otimes y - (-1)^{|x| \cdot |y|} y \otimes x \Big\rangle,$$

We can put a graded coalgebra structure on $\text{Sym}^+(V)$. The co-multiplication is defined as

$$\Delta(x_1 \cdots x_n) := \sum_{i=1}^{n-1} \sum_{\sigma \in S(i,n-i)} \varepsilon_x(\sigma)(x_{\sigma(1)} \cdots x_{\sigma(i)}) \otimes (x_{\sigma(i+1)} \cdots x_{\sigma(n)}),$$

where the $(i, n-i)$-**shuffles** $S(i, n-i)$ is the set of $\sigma \in S_n$ such that $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(n)$, and the Koszul sign $\varepsilon_x(\sigma)$ is defined in the previous part.

The reduced symmetric space $\mathrm{Sym}^+(V)$ is the **co-free** [3] co-nilpotent co-commutative coalgebra without co-unit co-generated by V. By co-freeness we mean that Sym^+ is the right adjoint functor of the forgetful functor to the category of differential graded vector spaces. In this case, it has the universal property that any homomorphism of co-nilpotent co-commutative coalgebras $A \to V$ uniquely lifts to a homomorphism $A \to \mathrm{Sym}^+(V)$.

$$\begin{array}{ccc}
 & & \mathrm{Sym}^+(V) \\
 & \overset{\exists!}{\nearrow} & \downarrow \\
A & \longrightarrow & V
\end{array}$$

Definition 2.39. A **co-derivation** δ of degree d on a graded coalgebra C is a graded k-linear map $\delta \colon C^i \to C^{i+d}$ satisfying the co-Leibniz rule:

$$\Delta \circ \delta = (\delta \otimes \mathrm{id}_C + \mathrm{id}_C \otimes \delta) \circ \Delta$$

Definition 2.40. An **L_∞-algebra** $(\mathfrak{g}, Q)$ is a graded k-vector space $\mathfrak{g}$ with a co-derivation Q of degree 1 on the reduced symmetric space $\mathrm{Sym}^+(\mathfrak{g}[1])$ such that $Q^2 = 0$. A **L_∞-morphism** between L_∞-algebras, $F \colon (\mathfrak{g}, Q) \to (\mathfrak{g}', Q')$, is a morphism of graded coalgebras $F \colon \mathrm{Sym}^+(\mathfrak{g}[1]) \to \mathrm{Sym}^+(\mathfrak{g}'[1])$ such that $F \circ Q = Q' \circ F$.

2.4.2 L_∞-Algebras via Taylor Coefficients

Definition 2.41. Let $F \colon (\mathfrak{g}, Q) \to (\mathfrak{g}', Q')$ be a morphism of L_∞-algebras. The **Taylor coefficients** of F (*resp.* of Q) are the sequence of maps $F_n \colon \mathrm{Sym}^n(\mathfrak{g}[1]) \to \mathfrak{g}'[1]$ (*resp.* $Q_n \colon \mathrm{Sym}^n(\mathfrak{g}[1]) \to \mathfrak{g}[1]$). By the universal property of $\mathrm{Sym}^+(\mathfrak{g}[1])$, the morphism F (*resp.* the codifferential Q) is uniquely determined by its Taylor coefficients.

Under the décalage isomorphisms (2.9), a Taylor coefficient Q_n corresponds to a multi-bracket $\ell_n = Q_n \circ \mathrm{déc}_n^{-1} \colon \bigwedge^n(\mathfrak{g})[n] \to \mathfrak{g}$ of degree $2 - n$.

Lemma 2.42. Reconstruction from Taylor Coefficients

Let $F \colon (\mathfrak{g}, Q) \to (\mathfrak{g}', Q')$ be a morphism of L_∞-algebras. Then Q and F are given in terms of the Taylor coefficients by

$$Q(x_1 \cdots x_n) = \sum_{I,J}^{\{1,\ldots,n\}} \varepsilon_x(I, J) Q_{|I|}(x_I) \cdot x_J;$$

$$F(x_1 \cdots x_n) = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{I_1,\ldots,I_k}^{\{1,\ldots,n\}} \varepsilon_x(I_1, \ldots, I_k) F_{|I_1|}(x_{I_1}) \cdots F_{|I_k|}(x_{I_k}).$$

Proof. See Theorem III.2.1 of [AMM02]. $\square$

---

[3]The authors of [DMZ07] point out that, contrary to the common belief (e.g. in [Kon97]), $\mathrm{Sym}^+(V)$ is *not* co-free in the category of co-commutative coalgebras without co-unit. The subtlety is discussed in Section II.3.7 of [MSS02]. Also see [Gra99] for a complete proof of the claims.

Remark. In the lemma $I_1, ..., I_k$ are multi-indices such that $(I_1, ..., I_k)$ is a partition of $\{1, ..., n\}$. The sum is over all such possible partitions. For each partition $(I_1, ..., I_k)$, it defines an ordering of $\{1, ..., n\}$ and hence a permutation $\sigma \in S_n$ in the following way: all elements of I_i precedes those of I_j if $i < j$; in each I_i the order is the usual ordering of natural numbers. $\varepsilon_x(I_1, ..., I_k)$ is the Koszul sign associated to this permutation. The convention for (I, J) is identical.

The condition $Q^2 = 0$ can be expanded into an infinite sequence of constraints imposing on the Taylor coefficients Q_n. For convenience, we denote by $Q_n^m \colon \mathrm{Sym}^n(\mathfrak{g}[1]) \to \mathrm{Sym}^m(\mathfrak{g}[1])$ the map induced from Q by restriction on the domain and projection on the codomain. In particular $Q_n^1 = Q_n$ is the n-th Taylor coefficient. Then $Q^2 = 0$ is equivalent to

$$\sum_{i=1}^n Q_i^1 \circ Q_n^i = 0, \qquad \forall\, n \in \mathbb{Z}_{>0}. \tag{2.10}$$

- For $n = 1$, (2.10) gives $Q_1 \circ Q_1 = 0$, which implies that Q_1 is a differential of degree 1 on $\mathfrak{g}$. Let $\mathrm{d}x := (-1)^{|x|}Q_1(x)$ for homogeneous $x \in \mathfrak{g}$.

- For $n = 2$, (2.10) gives $Q_1(Q_2(x \cdot y)) + Q_2(Q_1(x) \cdot y + (-1)^{|x|-1}x \cdot Q_1(y)) = 0$. If $[x, y] := \ell_2(x \wedge y)$, then this becomes the graded Leibniz rule of $[-, -]$.

- For $n = 3$, (2.10) gives

$$(-1)^{|x|\cdot|z|}[[x, y], z] + (-1)^{|y|\cdot|z|}[[z, x], y] + (-1)^{|x|\cdot|y|}[[y, z], x]$$
$$= (-1)^{|x|\cdot|z|}\left(\mathrm{d}(\ell_3(x \wedge y \wedge z)) + \ell_3(\mathrm{d}x \wedge y \wedge z) + (-1)^{|x|}\ell_3(x \wedge \mathrm{d}y \wedge z) + (-1)^{|x|+|y|}\ell_3(x \wedge y \wedge \mathrm{d}z)\right).$$

for homogeneous $x, y, z \in \mathfrak{g}$. This reduces to the graded Jacobi identity for $[-, -]$ if $\ell_3 = 0$.

Therefore, the L∞-algebras can be considered as the generalisation of DGLAs where the graded Jacobi identity for the graded Lie bracket ℓ_2 holds only up to homotopy of a higher bracket ℓ_3. In particular, we have the following result:

Proposition 2.43. From L∞-Algebras to DGLAs

An L∞-algebra $(\mathfrak{g}, Q)$ defines a DGLA structure on $\mathfrak{g}$ if and only if $Q_n = 0$ for $n \geqslant 3$.

Remark. In general, it can be shown (*cf.* [DMZ07], [Jur19]) that the constraint $Q \circ Q = 0$ is equivalent to the *homotopy Jacobi identities* satisfied by the multi-brackets:

$$\sum_{i=1}^n (-1)^i \sum_{\sigma \in S(i,n-i)} \chi_x(\sigma)\ell_{n+1-i}\big(\ell_i\big(x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(i)}\big) \wedge x_{\sigma(i+1)} \wedge \cdots \wedge x_{\sigma(n)}\big) = 0.$$

This provides an equivalent alternative definition for L∞-algebras, which justifies some other names for L∞-algebras (Lie-∞ algebras or strongly homotopy Lie algebras).

For the L∞-morphism $F \colon (\mathfrak{g}, Q) \to (\mathfrak{g}', Q')$, an similar analysis for the condition $F \circ Q = Q' \circ F$ shows that:

- For $n = 1$, $F_1 \circ Q_1 = Q_1' \circ F_1$, which implies that F_1 is a morphism between the differrential graded vector spaces $\mathfrak{g}$ and $\mathfrak{g}'$;

- For $n = 2$, $F_1 \circ Q_2 - Q_2' \circ F_1 = $ (terms involving F_2). This suggests the following result:

Proposition 2.44. From L_∞-Morphisms to DGLA Morphisms

An L_∞-morphism $F\colon (\mathfrak{g}, Q) \to (\mathfrak{g}', Q')$ between the DGLAs $\mathfrak{g}$ and $\mathfrak{g}'$ defines a morphism of DGLAs if and only if $F_n = 0$ for $n \geqslant 2$.

Remark. A morphism of L_∞-algebras $F\colon (\mathfrak{g}, Q) \to (\mathfrak{g}', Q')$ is called an L_∞**-quasi-isomorphism**, if the first Taylor coefficient $F_1\colon \mathfrak{g} \to \mathfrak{g}'$ is a quasi-isomorphism, that is, F_1 induces isomorphisms of the cohomology spaces $H^n(F_1)\colon H^n(\mathfrak{g}) \to H^n(\mathfrak{g}')$. In this case we say that $\mathfrak{g}$ is **homotopy equivalent** to $\mathfrak{g}'$. It is not immediate that homotopy equivalence defines an equivalence relation in the category of L_∞-algebras. It is a non-trivial result that an L_∞-quasi-isomorphism *admits an L_∞-quasi-inverse*, which will be shown in the next part.

2.4.3   Minimal Models

Definition 2.45. Let $(\mathfrak{g}, Q)$ and $(\mathfrak{g}', Q')$ be two L_∞-algebras. We define the direct sum $\mathfrak{g} \oplus \mathfrak{g}'$ to be an L_∞-algebra with the co-differential given in terms of the Taylor coefficients by

$$(Q \oplus Q')_n((x_1 + x_1') \cdots (x_n + x_n')) = Q_n(x_1 \cdots x_n) + Q_n'(x_1' \cdots x_n')$$

where $x_1, ..., x_n \in \mathfrak{g}$ and $x_1', ..., x_n' \in \mathfrak{g}'$.

Definition 2.46. Let $(\mathfrak{g}, Q)$ be an L_∞-algebra. It is called

- **minimal**, if the first Taylor coefficient $Q_1 = 0$;

- **linear contractible**, if the Taylor coefficients $Q_n = 0$ for $n \geqslant 2$ and $H^\bullet(\mathfrak{g}, Q_1) = 0$.

Proposition 2.47. Decomposition Theorem

Every L_∞-algebra is isomorphic to a direct sum of a minimal L_∞-algebra and a linear contractible L_∞-algebra.

Proof. (*Adapted from [AMM02] and [Jur19].*) The first step of the proof is a general fact in linear algebra that any cochain complex of vector spaces is a direct sum of a complex with zero differential and a complex with zero cohomology. For this we consider a cochain complex $(C^\bullet, \mathrm{d})$ of vector spaces. Note that the two short exact sequences

$$0 \longrightarrow \ker \mathrm{d}^n \longrightarrow C^n \longrightarrow \operatorname{im} \mathrm{d}^n \longrightarrow 0$$
$$0 \longrightarrow \operatorname{im} \mathrm{d}^{n-1} \longrightarrow \ker \mathrm{d}^n \longrightarrow H^n(C^\bullet) \longrightarrow 0$$

split. Therefore we have a decomposition $C^n = Z^n \oplus Z_c^n = B^n \oplus H^n \oplus Z_c^n$, where $Z^n = \ker \mathrm{d}^n$, $B^n = \operatorname{im} \mathrm{d}^{n-1} \cong Z_c^{n-1}$, and $H^n \cong H^n(C^\bullet)$. We define a linear map $h^n\colon C^n \to C^{n-1}$ by the composition:

$$C^n \longrightarrow\!\!\!\!\!\rightarrow B^n \overset{\sim}{\longrightarrow} Z_c^{n-1} \lhook\joinrel\longrightarrow C^{n-1} \ .$$

h^n is called the *splitting map*. It follows that $B^n = \operatorname{im}(\mathrm{d}^{n-1} \circ h^n)$ and $Z_c^n = \operatorname{im}(h^{n+1} \circ \mathrm{d}^n)$. Therefore we have a decomposition of the identity map on C^n:

$$\mathrm{id} = p^n + \mathrm{d}^{n-1} \circ h^n + h^{n+1} \circ \mathrm{d}^n,$$

where $p^n\colon C^n \to H^n$ is the projection map. This shows that h is a chain homotopy between id and p. Therefore the cohomology of $H^\bullet \cong H^\bullet(C^\bullet)$ induced by the projection p is trivial. On the other hand,

the projection $1 - p^n \colon C^n \to B^n \oplus Z_c^n$ is chain-homotopic to the zero map. Hence the induced differential on $B^\bullet \oplus Z_c^\bullet$ is zero.

Now let $(\mathfrak{g}, Q)$ be an L_∞-algebra. According to the above construction we can decompose $(\mathfrak{g}, Q_1)$ into $\mathfrak{g}_m \oplus \mathfrak{g}_c$, such that $\mathfrak{g}_m$ has zero differential: $M_1 = 0$; and $\mathfrak{g}_c = B \oplus Z_c$ has zero cohomology: $H^\bullet(\mathfrak{g}_c, L_1) = 0$. Correspondingly, the reduced symmetric algebra is decomposed as

$$\mathrm{Sym}^+(\mathfrak{g}[1]) \cong \mathrm{Sym}^+(\mathfrak{g}_m[1]) \oplus \mathrm{Sym}^+(\mathfrak{g}_c[1]) \oplus \mathrm{Sym}^+(\mathfrak{g}_m[1]) \otimes \mathrm{Sym}^+(\mathfrak{g}_c[1]).$$

We would like to find an L_∞-isomorphism $F \colon (\mathfrak{g}, Q) \to (\mathfrak{g}, \overline{Q})$ such that $(\mathfrak{g}, \overline{Q}) = (\mathfrak{g}_m, M) \oplus (\mathfrak{g}_m, L)$. In other words, the codifferential $\overline{Q}$ satisfies

$$\overline{Q}\big|_{\mathrm{Sym}^+(\mathfrak{g}_m[1])} = M, \quad \overline{Q}\big|_{\mathrm{Sym}^+(\mathfrak{g}_m[1])} = L, \quad \overline{Q}\big|_{\mathrm{Sym}^+(\mathfrak{g}_m[1]) \otimes \mathrm{Sym}^+(\mathfrak{g}_c[1])} = M \otimes \mathrm{id} + \mathrm{id} \otimes L.$$

The process is equivalent to finding an infinite chain of L_∞-isomorphisms:

$$(\mathfrak{g}, Q) \xrightarrow{F^{(1)} = \mathrm{id}} (\mathfrak{g}, Q^{(1)}) \xrightarrow{F^{(2)}} (\mathfrak{g}, Q^{(2)}) \xrightarrow{F^{(3)}} (\mathfrak{g}, Q^{(3)}) \longrightarrow \cdots \longrightarrow (\mathfrak{g}, \overline{Q})$$

with the following properties $(*)$: for each $k \in \mathbb{Z}_{>0}$,

$(*.1)$ $F^{(k)}$ is an L_∞-isomorphism such that $F_j^{(k)} = 0$ for all $j \neq k$;

$(*.2)$ $Q_j^{(k-1)} = Q_j^{(k)}$ for all $j \neq k$;

$(*.3)$ $Q_j^{(k)}(\mathrm{Sym}^+(\mathfrak{g}_m[1])) \subseteq \mathfrak{g}_m$ for $1 \leqslant j \leqslant k$;

$(*.4)$ $Q_j^{(k)}\big|_{\mathrm{Sym}^+(\mathfrak{g}_c[1])} = 0$ for $2 \leqslant j \leqslant k$;

$(*.5)$ $Q_j^{(k)}\big|_{\mathrm{Sym}^+(\mathfrak{g}_m[1]) \otimes \mathrm{Sym}^+(\mathfrak{g}_c[1])} = 0$ for $1 \leqslant j \leqslant k$.

Assuming these, we will obtain an L_∞-isomorphism $F \colon (\mathfrak{g}, Q) \to (\mathfrak{g}, \overline{Q})$ such that the Taylor coefficients of F and $\overline{Q}$ are given by $F_k = F_k^{(k)}$ and $\overline{Q}_k = Q_k^{(k)}$ for all $k \in \mathbb{Z}_{>0}$ respectively. Moreover, $(\mathfrak{g}, \overline{Q})$ is the claimed direct sum. The strategy is using induction on k to construct $\overline{Q}_k$ and F_k and to prove the properties $(*)$ for k. The base case $k = 1$ is proven in the first part of this proof. Now we assume that $Q^{(k-1)}$ is given and construct $Q_k^{(k)}$ and $F_k^{(k)}$. To ease the notations, we write $Q_1 = Q_1^{(k-1)} = Q_1^{(k)}$, $Q_k = Q_k^{(k-1)}$, $\overline{Q}_k = Q_k^{(k)}$, and $F_k = F_k^{(k)}$.

The condition $F^{(k)} \circ Q^{(k-1)} = Q^{(k)} \circ F^{(k)}$ of $F^{(k)}$ being an L_∞-morphism is expanded as follows (*neglecting any signs*):

$$F_k Q_1(x_1 \cdots x_k) + Q_k(x_1 \cdots x_k) = Q_1 F_k(x_1 \cdots x_k) + \overline{Q}_k(x_1 \cdots x_k). \tag{2.11}$$

For $x \in \mathrm{Sym}^k(\mathfrak{g}[1])$, $k \geqslant 2$, we say that x is of *type r* for $0 \leqslant r \leqslant k$, if $x = x_1 \cdots x_k$, where $x_1, ..., x_r \in Z_c$ and $x_{r+1}, ..., x_k \in Z$. We write $\mathsf{Type}(x) = r$. We define $\overline{Q}_k(x)$ to be the projection of $Q_k(x)$ onto $\mathfrak{g}_m \oplus Z_c$ if $\mathsf{Type}(x) = 0$, and $\overline{Q}_k(x) = 0$ if $\mathsf{Type}(x) > 0$. Then $(*.4)$ and $(*.5)$ holds immediately. $F_k(x)$ will be defined recursively for the type of x.

First, we consider $\mathsf{Type}(x) = 0$, i.e. $x = x_1 \cdots x_k$ for $x_1, ..., x_k \in Z$. An easy induction shows that $Q_k(x) \in Z$. Hence $\overline{Q}_k(x) \in \mathfrak{g}_m$ by definition, which proves $(*.3)$. On the other hand, the equation (2.11) becomes

$$Q_1 F_k(x_1 \cdots x_k) = \overline{Q}_k(x_1 \cdots x_k) - Q_k(x_1 \cdots x_k).$$

This defines $F_k(x)$ up to an element in Z. Furthermore, suppose that $x_1 = Q_1 y_1 \in B$. The expansion of

the equation $Q^{(k-1)} \circ Q^{(k-1)} = 0$ takes the form (*neglecting any signs*):

$$Q_k(Q_1 y_1 \cdot x_2 \cdots x_k) + \sum_{1 \leqslant i,j < k} Q_i(\cdots Q_j(\cdots) \cdots) + Q_1 Q_k(y_1 \cdot x_2 \cdots x_k) = 0. \qquad (2.12)$$

Since Q_i, Q_j satisfies the property (∗.4) and (∗.5), all intermediate terms vanish. We are left with

$$Q_k(Q_1 y_1 \cdot x_2 \cdots x_k) = -Q_1 Q_k(y_1 \cdot x_2 \cdots x_k) \in B,$$

which implies that $\overline{Q}_k(Q_1 y_1 \cdot x_2 \cdots x_k) = 0$. Let $\widetilde{Q}_1 \colon \mathrm{Sym}^k(\mathfrak{g}[1]) \to \mathrm{Sym}^k(\mathfrak{g}[1])$ be the coalgebra morphism induced by Q_1. Then $\overline{Q}_k \widetilde{Q}_1(x) = 0$.

Next, we define $F_k(x)$ recursively by the following construction. Let $r \geqslant 1$. Suppose that $F_k(y)$ is determined for all y with $\mathsf{Type}(y) \leqslant r - 2$ and for all z with $\mathsf{Type}(z) = r - 1$ *up to an element in* Z. Let $x \in \mathrm{Sym}^k(\mathfrak{g}[1])$ with $\mathsf{Type}(x) = r$. We would like to determine $F_k(x)$ up to an element in Z, and specify $F_k(y)$ for all y with $\mathsf{Type}(y) = r - 1$.

The following sub-lemma (*cf.* Lemma V.3 of [AMM02]) will be useful in the subsequent proof:

> Let $x \in \mathrm{Sym}^k(\mathfrak{g}[1])$ *with* $\mathsf{Type}(x) = r > 0$. *If* $Q_1 x = 0$, *then there exists* y *of type* $r + 1$ *such that* $x = \widetilde{Q}_1 y$.

Applying (2.11) to $\widetilde{Q}_1(x)$, we obtain that

$$Q_1 F_k \widetilde{Q}_1(x) = Q_k \widetilde{Q}_1(x),$$

where we used the facts that $Q_1 \widetilde{Q}_1 = 0$ and $\widetilde{Q}_k \widetilde{Q}_1 = 0$. Similar to (2.12), $Q^{(k-1)} \circ Q^{(k-1)} = 0$ implies that

$$Q_1 Q_k(x) + Q_k \widetilde{Q}_1(x) = 0. \qquad (2.13)$$

Combining the two equations, we have $F_k \widetilde{Q}_1(x) + Q_k(x) \in Z$. Note that $\mathsf{Type}(\widetilde{Q}_1 x) = r - 1$, so that $F_k \widetilde{Q}_1(x)$ is determined up to an element in Z. Now we specify $F_k \widetilde{Q}_1(x)$ such that

$$F_k \widetilde{Q}_1(x) + Q_k(x) =: b(x) \in B.$$

Applying (2.11) to x, we obtain that $b(x) = Q_1 F_k(x)$. On the other hand, note that $b(x)$ must satisfy the constraint that $b(x) = Q_k(x)$ provided $Q_1(x) = 0$. In this case, the sub-lemma above implies that $x = \widetilde{Q}_1 y$ for some y with $\mathsf{Type}(y) = r + 1$. Hence (2.13) implies that $b(x) = -Q_1 Q_k(y)$. By fixing $b(x)$ satisfying the above constraints, $F_k(x)$ is determined up to an element in Z. □

Corollary 2.48. Minimal Model Theorem

Each L_∞-algebra $\mathfrak{g}$ is homotopy equivalent to a minimal L_∞-algebra $\mathfrak{g}_m$, which is known as the **minimal model** of $\mathfrak{g}$.

Proof. Let $\mathfrak{g} = \mathfrak{g}_m \oplus \mathfrak{g}_c$ be the decomposition of $\mathfrak{g}$ into minimal $\mathfrak{g}_m$ and linear contractible $\mathfrak{g}_c$. Note that the linear contractible part $\mathfrak{g}_c$ has trivial cohomology. The inclusion $\mathfrak{g}_m \hookrightarrow \mathfrak{g}$ and the projection $\mathfrak{g} \twoheadrightarrow \mathfrak{g}_m$ are L_∞-quasi-isomorphisms. Hence $\mathfrak{g}$ and $\mathfrak{g}_m$ are homotopy equivalent. □

> **Lemma 2.49**
>
> Suppose that $F\colon (\mathfrak{g}, Q) \to (\mathfrak{g}', Q')$ is an L$_\infty$-quasi-isomorphism of minimal L$_\infty$-algebras $\mathfrak{g}$ and $\mathfrak{g}'$. Then F is an isomorphism.

Proof. The conditions that $\mathfrak{g}$ and $\mathfrak{g}'$ are minimal imply that $\mathrm{H}^\bullet(\mathfrak{g}) = \mathfrak{g}$ and $\mathrm{H}^\bullet(\mathfrak{g}') = \mathfrak{g}'$. Since F is a quasi-isomorphism, $F_1\colon \mathfrak{g} \to \mathfrak{g}'$ is an isomorphism. Let G_1 be the inverse of F_1. By the universal property of $\mathrm{Sym}^+(\mathfrak{g}[1])$, G_1 uniquely lifts to an L$_\infty$-morphism $G\colon (\mathfrak{g}', Q') \to (\mathfrak{g}, Q)$, which is the inverse of F. □

Remark. This demonstrates that the minimal model of an L$_\infty$-algebra is unique up to isomorphism.

> **Proposition 2.50. Homotopy Equivalence of L$_\infty$-Algebras**
>
> Every L$_\infty$-quasi-isomorphism $F\colon (\mathfrak{g}, Q) \to (\mathfrak{g}', Q')$ has a quasi-inverse $G\colon (\mathfrak{g}', Q') \to (\mathfrak{g}, Q)$, that is, $\mathrm{H}^\bullet(G_1)$ is an actual inverse of $\mathrm{H}^\bullet(F_1)$. Therefore the homotopy equivalence of L$_\infty$-algebras is an equivalence relation.

Proof. We decompose $(\mathfrak{g}, Q)$ and $(\mathfrak{g}', Q')$ into the direct sum of the minimal part and the contractible part:

$$(\mathfrak{g}, Q) = (\mathfrak{g}_m, Q_m) \oplus (\mathfrak{g}_c, Q_c), \qquad (\mathfrak{g}', Q') = (\mathfrak{g}'_m, Q'_m) \oplus (\mathfrak{g}'_c, Q'_c).$$

We have a commutative diagram shown as follows, where $\widetilde{F} = \pi' \circ F \circ \iota\colon (\mathfrak{g}_m, Q_m) \to (\mathfrak{g}'_m, Q'_m)$ is an L$_\infty$-quasi-isomorphism induced by F.

$$
\begin{array}{ccc}
(\mathfrak{g}, Q) & \xrightarrow{\ \ F\ \ } & (\mathfrak{g}', Q') \\
\pi \downarrow \uparrow \iota & & \pi' \downarrow \uparrow \iota' \\
(\mathfrak{g}_m, Q_m) & \xrightarrow{\ \ \widetilde{F}\ \ } & (\mathfrak{g}'_m, Q'_m)
\end{array}
\tag{2.14}
$$

Since $\mathfrak{g}_m$ and $\mathfrak{g}'_m$ are minimal, $\widetilde{F}$ is an isomorphism. We denote by $\widetilde{G}$ the inverse of $\widetilde{F}$. Then $G = \iota \circ \widetilde{G} \circ \pi'$ is a quasi-inverse of F. □

2.4.4 Maurer–Cartan Equation, Reprise

In this part, we extend the discussions in Section 2.2.1 to incorporate L$_\infty$-morphisms into the deformation theory via differential graded Lie algebras.

Definition 2.51. Let $(\mathfrak{g}, Q)$ be an L$_\infty$-algebra. For $x \in \mathfrak{g}^1$, the **homotopy Maurer–Cartan equation** is given by

$$Q(\mathrm{e}^x - 1) = 0 \iff \sum_{n=1}^{\infty} \frac{1}{n!} Q_n(x \cdots x) = 0. \tag{2.15}$$

The set of solutions is again denoted by $\mathsf{MC}(\mathfrak{g})$. Note that when $\mathfrak{g}$ is a DGLA, (2.15) reduces to the Maurer–Cartan equation (2.2).

It is more natural to consider $\mathrm{e}^x - 1 \in \mathrm{Sym}^+(\mathfrak{g}[1])$ instead of $x \in \mathfrak{g}^1$ as the invariant object in an L$_\infty$-algebra $(\mathfrak{g}, Q)$. If $F\colon (g, Q) \to (g', Q')$ is an L$_\infty$-morphism, then $F(Q(\mathrm{e}^x - 1)) = Q'(F(\mathrm{e}^x - 1))$. It follows that if $x \in \mathsf{MC}(\mathfrak{g})$, then under the L$_\infty$-morphism F,

$$x \longmapsto \sum_{n=1}^{\infty} F_n(x \cdots x) \in \mathsf{MC}(\mathfrak{g}').$$

Remark. Note that $e^x - 1$ is not a well-defined element in $\mathrm{Sym}^+(\mathfrak{g}[1])$ without specifying the notion of convergence. We resolve this issue by only considering the L_∞-algebra $\mathfrak{g} \otimes_k \mathfrak{m}_R$ where R is either a test k-algebra (so that we have the nilpotency condition) or a formal completion (so that $e^x - 1$ is a well-defined formal power series).

As an analogue of the gauge action on a DGLA, we can define the homotopy action on an L_∞-algebra $\mathfrak{g}$ by constructing a simplicial structure on $\mathsf{MC}(\mathfrak{g})$. In the case where $\mathfrak{g}$ is a DGLA, the fundamental group of $\mathsf{MC}(\mathfrak{g})$ corresponds to $\mathsf{MC}(\mathfrak{g})$ modulo gauge action. The notion of Maurer–Cartan functor (2.5) can thus be extended to L_∞-algebras. This is beyond the scope of our exposition, as we restrict our attention to the deformation problem controlled by DGLAs. For a detailed treatment, see [SS12] and [Get09].

Lemma 2.52

Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two DGLAs. Then the direct sum of DGLAs induces the Cartesian product of Maurer–Cartan functors:

$$\overline{\mathsf{MC}}_{\mathfrak{g} \oplus \mathfrak{g}'} \simeq \overline{\mathsf{MC}}_{\mathfrak{g}} \times \overline{\mathsf{MC}}_{\mathfrak{g}'}.$$

Proof. This follows from that $\mathsf{MC}((\mathfrak{g} \oplus \mathfrak{g}') \otimes \mathfrak{m}_R) = \mathsf{MC}(\mathfrak{g} \otimes \mathfrak{m}_R) \times \mathsf{MC}(\mathfrak{g}' \otimes \mathfrak{m}_R)$ and that the gauge action is preserved in the decomposition. $\qquad\square$

Lemma 2.53

Suppose that $\mathfrak{g}$ is a linear contractible L_∞-algebra. Then $\overline{\mathsf{MC}}_{\mathfrak{g}}(\mathfrak{m}_R)$ is a singleton for any test algebra R.

Proof. A linear contractible L_∞-algebra is simply a differential graded vector space. The Maurer–Cartan equation is $\mathrm{d}x = 0$ for $x \in \mathfrak{g}^1 \otimes \mathfrak{m}_R$. Write $x = \sum_i x_i \otimes a_i$ for $x_i \in \mathfrak{g}^1$ and $a_i \in \mathfrak{m}_R$. Then $\mathrm{d}x_i = 0$ for each i.

Since $\mathfrak{g}$ has trivial cohomology, $x_i = \mathrm{d}y_i$ for some $y_i \in \mathfrak{g}^0$. Then $x = \mathrm{d}(\sum_i y_i \otimes a_i) = \exp(\sum_i y_i \otimes a_i) \cdot 0$. Hence x is gauge equivalent to 0. We deduce that $\overline{\mathsf{MC}}_{\mathfrak{g}}(\mathfrak{m}_R) = \{0\}$. $\qquad\square$

Theorem 2.54. L_∞-Quasi-Isomorphism Theorem

Let $F \colon \mathfrak{g} \to \mathfrak{g}'$ be an L_∞-morphism of DGLAs. If F is an L_∞-quasi-isomorphism, then the map

$$x \longmapsto \sum_{n=1}^\infty \frac{1}{n!} F_n(x \cdots x)$$

induces an natural isomorphism of the Maurer–Cartan functors:

$$\overline{\mathsf{MC}}_{\mathfrak{g}} \simeq \overline{\mathsf{MC}}_{\mathfrak{g}'}.$$

Proof. Let $\mathfrak{g} = \mathfrak{g}_m \oplus \mathfrak{g}_c$ be a decomposition of $\mathfrak{g}$ such that $\mathfrak{g}_m$ is minimal and $\mathfrak{g}_c$ is contractible. By the previous two lemmata, we have

$$\overline{\mathsf{MC}}_{\mathfrak{g}} = \overline{\mathsf{MC}}_{\mathfrak{g}_m \oplus \mathfrak{g}_c} \simeq \overline{\mathsf{MC}}_{\mathfrak{g}_m} \times \overline{\mathsf{MC}}_{\mathfrak{g}_c} \simeq \overline{\mathsf{MC}}_{\mathfrak{g}_m}.$$

Moreover, any L_∞-morphism induces a natural transformation in the corresponding Maurer–Cartan functors. The diagram (2.14) induces the commutative diagram of Maurer–Cartan functors:

$$
\begin{array}{ccc}
\overline{\mathsf{MC}}_{\mathfrak{g}} & \xrightarrow{\ \overline{\mathsf{MC}}_F\ } & \overline{\mathsf{MC}}_{\mathfrak{g}'} \\
\Big\Updownarrow & & \Big\Updownarrow \\
\overline{\mathsf{MC}}_{\mathfrak{g}_m} & \xrightarrow{\ \overline{\mathsf{MC}}_{\overline{F}}\ } & \overline{\mathsf{MC}}_{\mathfrak{g}'_m}
\end{array}
$$

Since $F\colon \mathfrak{g} \to \mathfrak{g}'$ is an L$_\infty$-quasi-isomorphism, the induced map $\overline{F}\colon \mathfrak{g}_m \to \mathfrak{g}'_m$ is also an L$_\infty$-quasi-isomorphism. Since $\mathfrak{g}_m$ and $\mathfrak{g}'_m$ are minimal, the $\overline{F}$ is a strict isomorphism which admits a strict inverse. Therefore $\overline{\mathsf{MC}}_{\overline{F}}$ is a natural isomorphism of functors. We conclude that $\overline{\mathsf{MC}}_{\mathfrak{g}} \simeq \overline{\mathsf{MC}}_{\mathfrak{g}'}$ as claimed. $\qquad\square$

Remark. Let us digress on the result from a categorical perspective. Let $\mathsf{L}_\infty\mathsf{Alg}$ be the category of L$_\infty$-algebras and dgLie be that of differential graded Lie algebras. The reason why we need to upgrade from Theorem 2.23 to Theorem 2.54 is that there exist L$_\infty$-morphisms between DGLAs which are not morphisms of DGLAs, and quasi-isomorphisms of DGLAs may not have inverse in dgLie. The theorem tells us that, from the viewpoint of the deformation functor, we should instead work on the homotopy category $\mathsf{Ho}(\mathsf{dgLie})$, which is obtained by localisation on the quasi-isomorphisms in dgLie.

2.4.5 Formality Theorem

Definition 2.55. A differential graded Lie algebra $\mathfrak{g}$ is said to be **formal**, if it is homotopy equivalent to its cohomology $\mathrm{H}^\bullet(\mathfrak{g})$, viewed as a DGLA with induced bracket and zero differential.

The main theorem due to Kontsevich [Kon97] in this dissertation is the following:

Theorem 2.56. Formality Theorem

There exists an L$_\infty$-quasi-isomorphism $U\colon \mathscr{T}_{\mathsf{poly}}(M) \to \mathscr{D}_{\mathsf{poly}}(M)$ such that the first Taylor coefficient U_1 coincides with the HKR map $U_1^{(0)}$.

Remark. By HKR theorem, $\mathscr{T}_{\mathsf{poly}}(M)$ is quasi-isomorphic to the cohomology $\mathrm{H}^\bullet(\mathscr{D}_{\mathsf{poly}}(M))$ of $\mathscr{D}_{\mathsf{poly}}(M)$. Therefore the formality theorem implies that the DGLA $\mathscr{D}_{\mathsf{poly}}(M)$ is formal. This justifies the name of the theorem.

The formality theorem, in combination of Proposition 2.25, Proposition 2.32, and the L$_\infty$-quasi-isomorphism theorem 2.54, completely solved the classification problem of deformation quantisation on a Poisson manifold. We obtain the bijective correspondence announced in Section 1.2:

$$
\frac{\{\text{Formal Poisson structures}\}}{\text{Equivalence}} \quad \longleftrightarrow \quad \frac{\{\text{Star products}\}}{\text{Equivalence}}. \tag{1.7}
$$

The rest of this dissertation will be devoted to the proof of formality theorem. We would like to study the constraints imposed on the Taylor coefficients U_n given that U is an L$_\infty$-morphism. Let $Q_{\mathscr{T}}$ and $Q_{\mathscr{D}}$ be the co-differentials associated to the L$_\infty$-algebras $\mathscr{T}_{\mathsf{poly}}(M)$ and $\mathscr{D}_{\mathsf{poly}}(M)$. Since they are DGLAs, then $Q_j^i = 0$ for $i > j - 1$, and we have

$$
(Q_{\mathscr{D}})_1 \circ U_n + (Q_{\mathscr{D}})_2 \circ U_n^2 = U_n \circ (Q_{\mathscr{T}})_n^n + U_{n-1} \circ (Q_{\mathscr{T}})_n^{n-1}, \qquad \forall n \in \mathbb{Z}_{>0}, \tag{2.16}
$$

where we used the same notation as (2.10). To expand this expression further, we have

$$
(Q_{\mathscr{D}})_1(U_n(\xi_1 \cdots \xi_n)) + \frac{1}{2} \sum_{\sigma \in S(i,n-i)} \varepsilon_\xi(\sigma)(Q_{\mathscr{D}})_2\big(U_i(\xi_{\sigma(1)} \cdots \xi_{\sigma(i)}) \cdot U_{n-i}(\xi_{\sigma(i+1)} \cdots \xi_{\sigma(n)})\big)
$$

$$
= \sum_{i=1}^n \varepsilon_\xi^{\{i\}} U_n\Big((Q_{\mathscr{T}})_1(\xi_i) \cdot \xi_1 \cdots \widehat{\xi_i} \cdots \xi_n\Big) + \frac{1}{2} \sum_{i=1}^k \sum_{j \neq i} \varepsilon_\xi^{\{i,j\}} U_{n-1}\Big((Q_{\mathscr{T}})_2(\xi_i \cdot \xi_j) \cdot \xi_1 \cdots \widehat{\xi_i} \cdots \widehat{\xi_j} \cdots \xi_n\Big),
$$

$$
\tag{2.17}
$$

where $\varepsilon_\xi(\sigma)$ is the Koszul sign defined in Example 2.38, $\varepsilon_\xi^{\{i\}}$ and $\varepsilon_\xi^{\{i,j\}}$ are the Koszul signs associated to $\sigma = (1\,i)$ and $\sigma = (1\,\min\{i,j\})(2\,\max\{i,j\})$ respectively. From the definition of $\mathscr{D}_{\mathsf{poly}}(M)$ and $\mathscr{T}_{\mathsf{poly}}(M)$ and the décalage isomorphisms (2.9), we have

$$
(Q_{\mathscr{T}})_1(\xi) = 0, \qquad\qquad\qquad (Q_{\mathscr{T}})_2(\xi \cdot \eta) = (-1)^{|\xi| \cdot (|\eta|-1)} [\xi, \eta]_{\mathsf{SN}},
$$

$$
(Q_{\mathscr{D}})_1(f) = (-1)^{|f|} \mathrm{d}_{\mathsf{H}} f = (-1)^{|f|} [\mu, f]_{\mathsf{G}}, \qquad (Q_{\mathscr{D}})_2(f \cdot g) = (-1)^{|f| \cdot (|g|-1)} [f, g]_{\mathsf{G}}.
$$

We extend the L_∞-morphism U to include the usual multiplication $U_0 \in \mathbb{R} \to \mathscr{D}_{\mathsf{poly}}(M)$ such that $U_0(1) = \mu \in \mathscr{D}_{\mathsf{poly}}^1(M)$. Let the both sides of (2.17) act on the smooth functions $f_0, ..., f_m \in \mathrm{C}^\infty(M)$. Furthermore, we make the convention that

$$
U_n(\xi_1 \cdots \xi_n)(f_1, ..., f_m) = 0
$$

if $m \neq \sum_{i=1}^n |x_i| + 1 - n$. With some careful computation, it can be shown (*cf.* Theorem VI.1 of [AMM02]) that (2.17) simplifies to

$$
\sum_{i=1}^n \sum_{j \neq i} \varepsilon_\xi^{\{i,j\}} U_{n-1}(\xi_i \diamond \xi_j \cdots \widehat{\xi_i} \cdots \widehat{\xi_j} \cdots \xi_n)(f_0, ..., f_m) = \sum_{i=0}^n \sum_{j=-1}^m \sum_{k=0}^{m-k} (-1)^{j(k+m)}
$$

$$
\sum_{\sigma \in S(i,n-i)} \varepsilon_\xi(\sigma) U_i(\xi_{\sigma(1)} \cdots \xi_{\sigma(i)}) \big(f_0, ..., f_{k-1}, U_{n-i}(\xi_{\sigma(i+1)} \cdots \xi_{\sigma(n)})(f_k, ..., f_{j+k}), f_{j+k+1}, ..., f_m\big)
$$

$$
\tag{2.18}
$$

where $\xi_i \diamond \xi_j$ is the product on $\mathscr{T}_{\mathsf{poly}}(M)$ (*cf.* (2.6) or (2.7)). (2.18) will be referred as the **formality equation**, which is our starting point of the next chapter, where we construct U_n as a sum of weighted admissible graphs.

Chapter 3

Kontsevich Quantisation

In this chapter we present the construction of the L$_\infty$-morphism from $\mathscr{T}_{\mathsf{poly}}(\mathbb{R}^d)$ to $\mathscr{D}_{\mathsf{poly}}(\mathbb{R}^d)$ and that of a star product in $\mathbb{R}^d$ following Kontsevich [Kon97]. Then we sketch the globalisation of the star product in a general smooth manifold M following [CFT02].

3.1 Construction in $\mathbb{R}^d$

This section is devoted to construct an explicit L$_\infty$-quasi-isomorphism $U \colon \mathscr{T}_{\mathsf{poly}}(\mathbb{R}^d) \to \mathscr{D}_{\mathsf{poly}}(\mathbb{R}^d)$.

3.1.1 Admissible Graphs

First we define the admissible graphs. A directed graph Γ is a pair (V_Γ, E_Γ), where V_Γ is the set of the vertices of Γ, and $E_\Gamma \subseteq V_\Gamma \times V_\Gamma$ is the set of edges of Γ. For each edge $e = (v_1, v_2) \in E_\Gamma$, $s(e) := v_1$ is called the source of e, and $t(e) := v_2$ is called the target of e.

Definition 3.1. An **admissible** graph $\Gamma \in G_{n,m}$ is a connected directed graph, satisfying the following conditions:

- $n, m \geqslant 0$ and $2n + m - 2 \geqslant 0$.
- $V_\Gamma = V_\Gamma^1 \sqcup V_\Gamma^2$, where $V_\Gamma^1 = \{1, ..., n\}$ is the set of vertices of the first type, and $V_\Gamma^2 = \{\overline{1}, ..., \overline{m}\}$ is the set of vertices of the second type.
- Γ has no edge starting from a vertex of the second type.
- Γ has neither loops nor double arrows.
- For $v \in V_\Gamma^1$, the set of edges starting from v is denoted by
$$\mathrm{Star}(v) := \{e \in E_\Gamma \colon e = (v, w) \text{ for some } w \in V_\Gamma\}$$
  $\mathrm{Star}(v)$ is a finite set. We put an order on it so that $\mathrm{Star}(v) = \left\{e_v^1, ..., e_v^{\#(\mathrm{Star}(v))}\right\}$.

We associate each $\Gamma \in G_{n,m}$ and polyvector vectors $\xi_1, ..., \xi_n$ with a polydifferential operator $B_\Gamma(\xi_1 \otimes \cdots \otimes \xi_n) \in \mathscr{D}_{\mathsf{poly}}^m(\mathbb{R}^d)$ by the following recipe.

- $B_\Gamma = 0$ unless $\xi_i \in \mathscr{T}_{\mathsf{poly}}^{k_i}(\mathbb{R}^d)$ for each $i \in V_\Gamma^1$, where $k_i = \#(\mathrm{Star}(i)) - 1$.
- Let $I \colon E_\Gamma \to \{1, ..., d\}$ be a map.
- For each $i \in V_\Gamma^1$ and $\overline{j} \in V_\Gamma^2$, put respectively the functions:

$$\left(\prod_{e \in \Gamma(-,i)} \partial_{I(e)}\right) \xi_i^{I(e_i^1), ..., I(e_i^{k_i})}, \qquad \left(\prod_{e \in \Gamma(-,\overline{j})} \partial_{I(e)}\right) f_j$$

  where $\Gamma(-, v) := \{e \in E_\Gamma \colon t(e) = v\}$.
- Multiply the functions on the vertices and sum over all possible $I \colon E_\Gamma \to \{1, ..., d\}$. This gives
$$B_\Gamma(\xi_1 \otimes \cdots \otimes \xi_n)(f_1, ..., f_m) \in \mathbb{R}.$$
  So we obtain the map $B_\Gamma \colon \mathscr{T}_{\mathsf{poly}}(\mathbb{R}^d)^{\otimes n} \to \mathscr{D}_{\mathsf{poly}}^{m-1}(\mathbb{R}^d)$.

3.1.2   Configuration Spaces

Definition 3.2. We construct the **configuration space** for the $n + m$ vertices of $\Gamma \in G_{n,m}$.

- If $m > 0$, let

$$\mathrm{Conf}_{n,m}^+ := \{(p_1, ..., p_n; q_1, ..., q_m) \in \mathbb{C}^{n+m} : p_1, ..., p_n \in \mathbb{H},\ p_i \neq p_j \text{ for } i \neq j; \tag{3.1}$$
$$q_1, ..., q_m \in \mathbb{R},\ q_1 < \cdots < q_m\}$$

$\mathrm{Conf}_{n,m}^+$ is a $(2n + m)$-dimensional smooth manifold. Let G be the 2-dimensional Lie group action on $\mathrm{Conf}_{n,m}^+$ by $z \mapsto az + b$ for $a > 0$ and $b \in \mathbb{R}$. This is a free action, and we obtain a quotient manifold

$$C_{n,m}^+ := \mathrm{Conf}_{n,m}^+ / G \tag{3.2}$$

which is a connected $(2n + m - 2)$-dimensional smooth manifold. It has the natural orientation induced by the volume form in $\mathbb{R}^{2n+m}$:

$$\Omega = \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n \wedge \mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}q_m, \qquad p_i = x_i + \mathrm{i}y_i,$$

which is inherited from $\mathrm{Conf}_{n,m}^+$ because the actions of G preserve orientation.

- If $m = 0$, let

$$\mathrm{Conf}_n := \{(p_1, ..., p_n) \in \mathbb{C}^n : p_1, ..., p_n \in \mathbb{C},\ p_i \neq p_j \text{ for } i \neq j\} \tag{3.3}$$

Let G' be the 3-dimensional Lie group action on Conf_n by $z \mapsto az + b$ for $a > 0$ and $b \in \mathbb{C}$. The quotient manifold

$$C_n := \mathrm{Conf}_n / G' \tag{3.4}$$

is a $(2n - 3)$-dimensional smooth manifold for $n \geqslant 2$.

Then we define the weight integral W_Γ associated to $\Gamma \in G_n$.

Definition 3.3. Let $\mathbb{H} := \{z \in \mathbb{C} \colon \mathrm{Im}\, z > 0\}$ be the upper half plane endowed with the hyperbolic metric $g = \dfrac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}$. It is called the **Poincaré half plane**. Let $\overline{\mathbb{H}} := \mathbb{H} \cup \mathbb{R}$.

- For $p \in \mathbb{H}$, let $L(p, \infty)$ be the vertical half line from p to infinity;
- For $p \neq q \in \mathbb{H}$, let $L(p, q)$ be the geodesic from p to q, which is an arc of a circle centred on the real line;
- Let $\varphi(p, q)$ be the angle from $L(p, \infty)$ to $L(p, q)$. By simple planimetry (*cf.* [Kel03]) we find that

$$\varphi(p, q) = \arg\left(\frac{q - p}{q - \bar{p}}\right) = \frac{1}{2\mathrm{i}} \log\left(\frac{q - p}{q - \bar{p}} \cdot \frac{\bar{q} - p}{\bar{q} - \bar{p}}\right). \tag{3.5}$$

We extend $\varphi(p, q)$ to $p \neq q \in \overline{\mathbb{H}}$.

Definition 3.4. For $e \in E_\Gamma$, we define the **angle map** $\varphi_e \colon C_{n,m} \to S^1$ as

$$\varphi_e(p_1, ..., p_n; q_1, ..., q_m) = \varphi(p_{s(e)}, p_{t(e)}).$$

We define a differential form

$$\omega_\Gamma := \bigwedge_{e \in E_\Gamma} \mathrm{d}\varphi_e \tag{3.6}$$

where the ordering of the 1-forms in the product is the one induced on the set of all edges by the ordering on the source vertices v and the ordering on the set $\mathrm{Star}(v)$.

> **Proposition 3.5. Compactification of Configuration Spaces**
>
> There exists a compact smooth manifold with corners $\overline{C}_{n,m}^{+}$ (*resp.* $\overline{C}_n$) whose interior is the open configuration space $C_{n,m}^{+}$ (*resp.* C_n) such that the angle map φ and the differential form ω_Γ extends smoothly to the corresponding compactification.

Remark. A manifold with corners X is a second-countable Hausdorff topological space such that every point has a neighbourhood homeomorphic to $\mathbb{R}^{n-k} \times \mathbb{R}_{\geqslant 0}^{k}$ for some k, which is called the depth of that point. The boundary ∂X is the set of those points with depth $k > 0$. The subset of points with depth k is an $(n-k)$-submanifold $S^k(X)$ of X, the submanifolds of which are called the **strata** of codimension k. This notion is a natural generalisation of a manifold with boundary. The definition can be extended straightforward to smooth manifolds with corners, where the transition maps between charts are required to preserve the corners.

Proof. We construct a map

$$\psi\colon \mathrm{Conf}_{n,m}^{+} \to \mathbb{T}^{(2n+m)(2n+m-1)} \times (\mathbb{RP}^2)^{(2n+m)(2n+m-1)(2n+m-2)}$$

where $\mathbb{T}^{(2n+m)(2n+m-1)} = \underbrace{S^1 \times \cdots \times S^1}_{(2n+m)(2n+m-1)}$ is a torus. For each pair of distinct points $a, b \in \{p_1, \bar{p}_1, ..., p_n, \bar{p}_n, q_1, ..., q_m$,

we associate an angle $\arg(a-b) \in S^1$; for each triple of distinct points $a, b, c \in \{p_1, \bar{p}_1, ..., p_n, \bar{p}_n, q_1, ..., q_m\}$, we associative a point $[a-b : b-c : c-a] \in \mathbb{RP}^2$. These data uniquely determine the map ψ. Since these data are preserved by the action of G, ψ descends to the quotient:

$$\widetilde{\psi}\colon C_{n,m}^{+} \to X := \mathbb{T}^{(2n+m)(2n+m-1)} \times (\mathbb{RP}^2)^{(2n+m)(2n+m-1)(2n+m-2)},$$

which is injective. Therefore $C_{n,m}^{+}$ is embedded into X. Let $\overline{C}_{n,m}^{+}$ be the closure of $\widetilde{\psi}(C_{n,m}^{+})$ in this space. This is a manifold with corners. $\qquad\square$

Definition 3.6. The **weight integral** for $\Gamma \in G_{n,m}$ is defined to be

$$W_\Gamma := \begin{cases} \displaystyle\prod_{k=1}^{n} \frac{1}{\#(\mathrm{Star}(k))!} \frac{1}{(2\pi)^{\#(E_\Gamma)}} \int_{\overline{C}_{n,m}^{+}} \omega_\Gamma, & \text{if } \#(E_\Gamma) = 2n + m - 2, \\ 0, & \text{otherwise.} \end{cases} \tag{3.7}$$

Remark. The requirement $\#(E_\Gamma) = 2n + m - 2$ ensures that the degree of the form ω_Γ matches the dimension of the configuration space. It also ensures that U_n has the correct degree (see the beginning of Section 3.2).

Now we define the n-th Taylor coefficient U_n by

$$U_n = \sum_{m=0}^{\infty} \sum_{\Gamma \in G_{n,m}} W_\Gamma B_\Gamma. \tag{3.8}$$

Now we are just one step away from the formality theorem in $\mathbb{R}^d$. The proof will be completed in Section 3.2.

3.1.3 Explicit Formula of Kontsevich Star Product

From Theorem 2.54 and Equation (3.8), a Poisson bivector field π induces the Kontsevich star product in the following way:

Theorem 3.7. Kontsevich Star Product

Let $\pi \in \mathcal{T}^1_{\mathsf{poly}}(\mathbb{R}^d)$ be a Poisson bivector field on $\mathbb{R}^d$. Then

$$\star = \mu + \sum_{n=1}^{\infty} \frac{\hbar^n}{n!} U_n(\pi \cdots \pi) \in \mathscr{D}^1_{\mathsf{poly}}(\mathbb{R}^d)[\![\hbar]\!]$$

defines a star product on $\mathbb{R}^d$ such that $f \star g - g \star f = \hbar\pi(f,g)$ modulo $\hbar^2$.

From the previous discussion, we can develop a set of rules for the computation of the star product $f \star g$. Some constructions can be simplified. For example, we only use the admissible graphs in $G_{n,2}$ where $\#(\mathrm{Star}(k)) = 2$ for $k = 1, ..., n$. The Lie group action of G on configuration space $C^+_{n,2}$ fix the two vertices of the second type to $0, 1 \in \mathbb{R}$ respectively, which gives an isomorphism:

$$C^+_{n,2} \simeq H_n := \{(p_1, ..., p_n) \in \mathbb{C}^n : p_1, ..., p_n \in \mathbb{H}, \ p_i \neq p_j \text{ for } i \neq j\}.$$

The explicit formula for the Kontsevich star product is given by

$$f \star g = fg + \sum_{n=1}^{\infty} \frac{\hbar^n}{2^n(2\pi)^{2n}n!} \sum_{\Gamma \in G_{n,2}} B_{\Gamma,\pi}(f,g) \int_{\overline{H}_n} \omega_\Gamma, \tag{3.9}$$

where $B_{\Gamma,\pi}(f,g) := B_\Gamma(\pi \cdots \pi)(f,g)$ according to the definition in Section 3.1.1 and ω_Γ is the differential form (3.6).

Example 3.8. As an easy example, we compute the star product when $\pi = \pi^{ij}\partial_i \wedge \partial_j$ has constant coefficients. Since any derivative on π^{ij} vanishes, we only need to sum over the admissible graphs with no edges targeting at the vertices of first type V^1_Γ. The admissible graphs with non-vanishing weight W_Γ has $2n$ edges. Therefore the only contributing admissible graph in $G_{n,2}$ is of the following form, which is denoted by Γ_n.
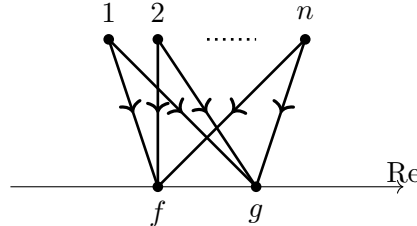
The differential operator associated to Γ_n is given by

$$B_{\Gamma_n,\pi}(f,g)(x) = \prod_{k=1}^{n} \left(\sum_{i_k,j_k=1}^{2n} \pi^{i_k,j_k} \frac{\partial}{\partial x^{i_k}} \frac{\partial}{\partial y^{j_k}} \right) f(x)g(y) \Bigg|_{x=y}.$$

The differential form ω_{Γ_n} is given by

$$\omega_{\Gamma_n} = \mathrm{d}\varphi_{(1,\overline{1})} \wedge \mathrm{d}\varphi_{(1,\overline{2})} \wedge \cdots \wedge \mathrm{d}\varphi_{(n,\overline{1})} \wedge \mathrm{d}\varphi_{(n,\overline{2})}$$

where, following (3.5), the differential 1-forms are given by $\mathrm{d}\varphi_{(i,\overline{j})}(p_1, ..., p_n, q_1, q_2) = 2\mathrm{d}\arg(q_j - p_i)$, where $q_1 = 0$ and $q_2 = 1$ are fixed. The compactification $\overline{H}_n$ is parametrised as

$$\overline{H}_n \simeq \overline{\{(\arg(0 - p_1), \arg(1 - p_1), ..., \arg(0 - p_n), \arg(1 - p_n)) : p_1, ..., p_n \in \mathbb{H} \text{ distinct}\}}$$
$$\simeq [-\pi, 0]^{2n}.$$

Hence the star product is given by

$$(f \star g)(x) = (fg)(x) + \sum_{n=1}^{\infty} \frac{\hbar^n}{2^n (2\pi)^{2n} n!} B_{\Gamma_n, \pi}(f, g)(x) \cdot \int_{\overline{H}_n} \omega_\Gamma$$

$$= (fg)(x) + \sum_{n=1}^{\infty} \frac{\hbar^n}{2^n n!} \prod_{k=1}^{n} \left(\sum_{i_k, j_k = 1}^{2n} \pi^{i_k, j_k} \frac{\partial}{\partial x^{i_k}} \frac{\partial}{\partial y^{j_k}} \right) f(x) g(y) \Bigg|_{x=y}.$$

We have thus shown that in this case the Kontsevich star product coincides with the Weyl–Moyal product (1.8).

3.2 Proof of Formality Theorem

To show that the coefficients U_n given in (3.8) defines an L_∞-quasi-morphism $U \colon \mathscr{T}_{\mathsf{poly}}(\mathbb{R}^d) \to \mathscr{D}_{\mathsf{poly}}(\mathbb{R}^d)$, it remains to show that

(a) U_n respects the grading on $\mathscr{T}_{\mathsf{poly}}(\mathbb{R}^d)$ and $\mathscr{D}_{\mathsf{poly}}(\mathbb{R}^d)$;

(b) $U_1 = U_1^{(0)}$, which is the HKR map (2.8) and is already proven to be a quasi-isomorphism;

(c) U_n satisfies the constraint equation (2.16).

For (a), we show that for $\Gamma \in G_{n,m}$, $W_\Gamma B_\Gamma$ is a morphism of graded vector spaces of degree $1 - n$ from $\bigwedge^n (\mathscr{T}_{\mathsf{poly}}(\mathbb{R}^d)) \to \mathscr{D}_{\mathsf{poly}}^{m-1}(\mathbb{R}^d)$. Indeed, W_Γ is non-zero only if

$$2n + m - 2 = \#(E_\Gamma) = \sum_{k=1}^{n} \#(\mathrm{Star}(k)) = \sum_{k=1}^{n} (|\xi_i| + 1) = |\xi_1 \otimes \cdots \otimes \xi_n| + n.$$

So $\deg U_n = (m-1) - (2n + m - 2 - n) = 1 - n$ as claimed.

3.2.1 Checking Quasi-Isomorphism

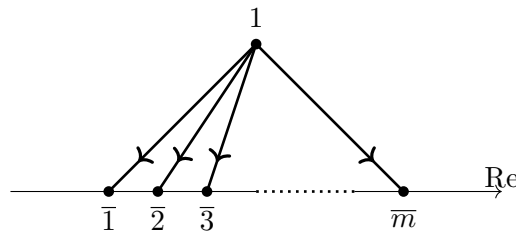For (b), the proof is simply a checking for the constructions.

Lemma 3.9. U is a Quasi-Isomorphism

The first Taylor coefficient $U_1 \colon \mathscr{T}_{\mathsf{poly}}(\mathbb{R}^d) \to \mathscr{D}_{\mathsf{poly}}(\mathbb{R}^d)$ defined by (3.8) satisfies $U_1 = U_1^{(0)}$, where $U_1^{(0)}$ is the HKR map defined in (2.8).

Proof. By (3.8), we have

$$U_1 = \sum_{m=0}^{\infty} \sum_{\Gamma \in G_{1,m}} W_\Gamma B_\Gamma = \sum_{m=0}^{\infty} W_{\Gamma_m} B_{\Gamma_m}$$

where Γ_m is the unique graph in $G_{1,m}$ as shown below:

The differential operator associated to Γ_m is given by

$$B_{\Gamma_m}(\xi)(f_1, ..., f_m) = \sum_{i_1,...,i_m=1}^{d} \xi^{i_1,...,i_m} \partial_{i_1} f_1 \cdots \partial_{i_m} f_m.$$

The computatuion of the weight integral is similar to that in Example 3.8:

$$W_{\Gamma_m} = \frac{1}{m!(2\pi)^m} \int_{\overline{C}_{1,m}^+} \mathrm{d}\varphi_{(1,\overline{1})} \wedge \cdots \wedge \mathrm{d}\varphi_{(1,\overline{m})} = \frac{1}{m!(2\pi)^m}(2\pi)^m = \frac{1}{m!}.$$

The map U_1 thus takes the form

$$U_1(\xi)(f_1, ..., f_m) = \frac{1}{m!} \sum_{i_1,...,i_m=1}^{d} \xi^{i_1,...,i_m} \partial_{i_1} f_1 \cdots \partial_{i_m} f_m.$$

In particular, if $\xi = X_1 \wedge \cdots \wedge X_n$ for $X_1, ..., X_n \in \Gamma(\mathbb{R}^d, \mathrm{T}\mathbb{R}^d)$, then

$$\xi^{i_1,...,i_m} = \sum_{\sigma \in S_m} \mathrm{sgn}(\sigma) X_1^{\sigma(i_1)} \cdots X_m^{\sigma(i_m)} \implies U_1(\xi)(f_1, ..., f_m) = \frac{1}{m!} \det(X_i(f_j))_{i,j}.$$

We conclude that U_1 agrees with the HKR map $U_1^{(0)}$. $\square$

3.2.2 Formality Equation from Boundary Strata

For (c), the construction in Section 3.1 allows us to put the formality equation (2.18) into the form

$$\sum_{\substack{\Gamma \in G_{n,m} \\ \#(E_\Gamma)=2n+m-3}} c_\Gamma B_\Gamma(\xi_1 \cdots \xi_n)(f_1, ..., f_m) = 0. \tag{3.10}$$

where we only sum over $\Gamma \in G_{n,m}$ for $\#(E_\Gamma) = 2n + m - 3$, since only these graphs can have non-vanishing weights W_Γ by definition. The equation (3.10) is satisfied if we can prove that $c_\Gamma = 0$ for all such graphs Γ. In the following subsections, we shall prove that

---

Lemma 3.10

Let $c_\Gamma \in \mathbb{R}$ be the coefficient associated to $\Gamma \in G_{n,m}$ from (2.18) and (3.10). Then

$$c_\Gamma = \prod_{k=1}^{n} \frac{1}{\#(\mathrm{Star}(k))!} \frac{1}{(2\pi)^{\#(E_\Gamma)}} \int_{\partial \overline{C}_{n,m}^+} \omega_\Gamma.$$

Having proven this, we invoke the Stokes' theorem for manifolds with corners:

Lemma 3.11. Stokes' Theorem with Corners

Let X be an n-dimensional oriented smooth manifold with corners. Let X_1 be the union of strata of X of codimension 0 and 1. Then ∂X_1 is a $(n-1)$-dimensional manifold with boundary with orientation induced from X_1. Let ω be a compactly supported differential $(n-1)$-form on M. We have

$$\int_X \mathrm{d}\omega = \int_{\partial X_1} \omega.$$

Proof. See [Con]. $\square$

Then $c_\Gamma = 0$ follows from that ω_Γ is a closed form:

$$c_\Gamma = \int_{\partial \overline{C}_{n,m}^+} \omega_\Gamma = \int_{\overline{C}_{n,m}^+} \mathrm{d}\omega_\Gamma = \int_{\overline{C}_{n,m}^+} \mathrm{d}\left(\bigwedge_{e \in E_\Gamma} \mathrm{d}\varphi_e\right) = 0.$$

To prove Lemma 3.10, we need to analyse the integral $\int_{\partial \overline{C}_{n,m}^+} \omega_\Gamma$, which has only contributions from boundary strata of codimension 1. Informally, these strata represent the degenerate configurations where some points collapse together. They can be classified into the following two types:

S1: The vertices $S := \{k_1, ..., k_i\}$ $(i \geqslant 2)$ of the first type collapse together to a point in the upper half plane $\mathbb{H}$. Such boundary strata can locally be expressed as a product $\partial_S \overline{C}_{n,m}^+ \simeq C_i \times C_{n-i+1,m}^+$.

S2: The vertices $S := \{k_1, ..., k_i\}$ $(i \geqslant 0)$ of the first type and $S' := \{\overline{\ell+1}, ..., \overline{\ell+j}\}$ $(j \geqslant 0)$ of the second type collapse together to a point on the real line $\mathbb{R}$, where $2i + j \geqslant 2$ and $i + j \leqslant n + m - 1$. Such boundary strata can locally be expressed as a product $\partial_{S,S'} \overline{C}_{n,m}^+ \simeq C_{i,j}^+ \times C_{n-i,m-j+1}^+$.

The products represent what Kontsevich [Kon97] called "looking through a magnifying glass", as shown in the following diagram:
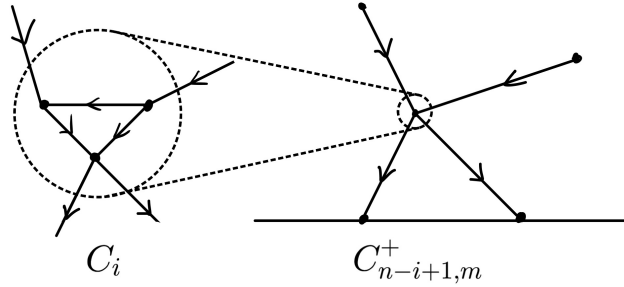

Figure 3.1: Type S1 boundary stratum through a magnifying glass.

Suppose that we start with an admissible graph Γ. Let Γ_0 be the subgraph of Γ spanned by the collapsing vertices, and Γ_1 be the quotient of Γ by Γ_0. There are cases where Γ_1 fails to be admissible:

- Γ_1 may have multiple arrows.
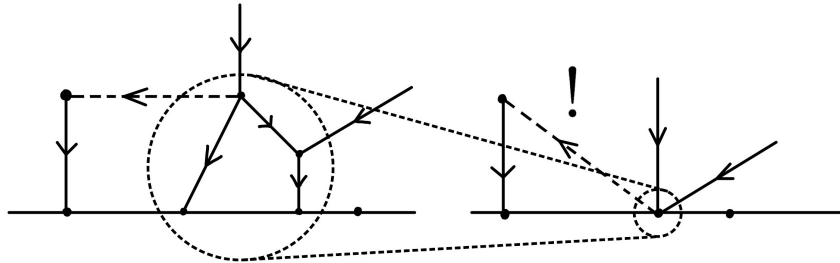- Γ_1 may have a bad edge, as shown in the following diagram:

Figure 3.2: Type S2 boundary stratum with a bad edge.

In both cases, we note that the corresponding weight integrals vanish. Therefore we can safely neglect these cases. The decomposition of a stratum $F \simeq F_0 \times F_1$ induces the decomposition of the weight integral:

$$\int_F \omega_\Gamma = \pm \int_{F_0} \omega_{\Gamma_0} \int_{F_1} \omega_{F_1}.$$

It requires great caution when dealing with the orientations and signs in the integrals. For details refer to [AMM02].

Type S1: $i = 2$

Now we consider type S1 strata with $i = 2$. In this case, Γ_0 contains a single edge e. Suppose that $s(e) = p_i$ and $t(e) = p_j$. Then Γ_1 is obtained by identifying p_i with p_j and contracting the edge (if exists) from p_i to p_j. The polydifferential operator associated to this stratum is given by

$$B_{\Gamma,R}(\xi_1 \cdots \xi_n)(f_0, ..., f_m) = B_{\Gamma_1}(\xi_i \diamond \xi_j \cdots \widehat{\xi_i} \cdots \widehat{\xi_j} \cdots \xi_n)(f_0, ..., f_m).$$

Type S1: $i \geqslant 3$

We prove that the integral vanishes for $i \geqslant 3$.

Lemma 3.12. Vanishing of Kontsevich Integrals

For $n \geqslant 3$ and $\Gamma \in G_{n,0}$, the weight integral over the configuration space C_n is zero:

$$\int_{\overline{C}_n} \omega_\Gamma = 0.$$

This result implies that type S1 strata with $i \geqslant 3$ does not contribute to the coefficient c_Γ. For the proof we need a result from distribution theory, which is stated below:

Lemma 3.13. Distributional Forms

Let U be a complex manifold with compactification $\overline{U}$. For a differential form ω on U such that the coefficients of ω and $d\omega$ are locally integrable over $\overline{U}$, we denote by $\mathscr{I}(\omega)$ the corresponding distributional form, that is, differential form with coefficients in the space of distributions. Then $\mathscr{I}$ commutes with the exterior differential: $d(\mathscr{I}(\omega)) = \mathscr{I}(d\omega)$. In addition, the integral $\int_U \omega$ is absolutely convergent and is equal to $\int_{\overline{U}} \mathscr{I}(\omega)$.

Proof. See Lemma 6.6.1 of [Kon97].   □

Proof of Lemma 3.12. We pick the edge e_1. Using the action of the Lie group G', we may identify the configuration space as a subset of $\overline{\mathbb{H}}^n$ where $s(e_1)$ is fixed to the origin and $t(e_1)$ lies on the unit circle. This provides a decomposition $\overline{C}_n \simeq S^1 \times U$ where U is a complex manifold. The weight integral factors accordingly:

$$\int_{\overline{C}_n} \omega_\Gamma = \int_{S^1 \times U} d\varphi_{e_1} \wedge \bigwedge_{i=2}^{2n-3} d(\varphi_{e_i} - \varphi_{e_1}) = 2\pi \int_U \bigwedge_{i=2}^{2n-3} d\varphi_{e_i} = 2\pi \int_U \bigwedge_{i=1}^{2n-4} d\arg z_i,$$

where z_i is the difference in the complex coordinates of $s(e_{i+1})$ and $t(e_{i+1})$. Now we employ a trick using logarithm. We claim that

$$\int_U \bigwedge_{i=1}^{2n-4} d\arg z_i = \int_U \bigwedge_{i=1}^{2n-4} d\log|z_i|. \tag{3.11}$$

Indeed, both $\mathrm{d}\arg z_i$ and $\mathrm{d}\log|z_i|$ can be expressed as a sum of a holomorphic 1-form and an anti-holomorphic 1-form:

$$\mathrm{d}\arg z_i = \frac{1}{2\mathrm{i}}\left(\mathrm{d}\log z_i - \mathrm{d}\log\overline{z}_i\right); \qquad \mathrm{d}\log|z_i| = \frac{1}{2}\left(\mathrm{d}\log z_i + \mathrm{d}\log\overline{z}_i\right). \tag{3.12}$$

We know that the integration of a $(k, 2N-k)$-form on a $2N$-dimensional complex manifold vanishes unless $k = N$. The non-vanishing contributions of (3.12) are the same in the integration, which justifies (3.11). Finally, using the fact from distribution theory which is stated above, we have

$$\int_U \bigwedge_{i=1}^{2n-4} \mathrm{d}\log|z_i| = \int_U \mathrm{d}\left(\log|z_1|\bigwedge_{i=2}^{2n-4}\mathrm{d}\log|z_i|\right) = \int_{\overline{U}} \mathscr{I}\left(\mathrm{d}\left(\log|z_1|\bigwedge_{i=2}^{2n-4}\mathrm{d}\log|z_i|\right)\right)$$

$$= \int_{\overline{U}} \mathrm{d}\left(\mathscr{I}\left(\log|z_1|\bigwedge_{i=2}^{2n-4}\mathrm{d}\log|z_i|\right)\right) = 0 \qquad \qquad \square$$

Type S2

Now we consider type S2 strata. Suppose that Γ_0 contains the vertices $k_1, ..., k_i, \overline{\ell+1}, ..., \overline{\ell+j}$. The polydifferential operator associated to this stratum is given by

$$B_{\Gamma,F}(\xi_J)(f_0, ..., f_m) = B_{\Gamma_1}(\xi_{\sigma(1)}\cdots\xi_{\sigma(i)})\left(f_0, ..., f_\ell, B_{\Gamma_0}(\xi_I)(f_{\ell+1}, ..., f_{\ell+j}), f_{\ell+j+1}, ..., f_m\right)$$

where the multi-indices $I = (k_1, ..., k_i)$ and $J = \{1, ..., n\}\setminus I$ (with increasing order).

These polydifferential operators match those appeared in the formality equation 2.16. The matching of the coeffcients $c_{\Gamma,F}$ with $\int_F \omega_\Gamma$ is more subtle and we refer to [AMM02]. This concludes the proof of Lemma 3.10 and thus of the formality theorem 2.56.

3.3 Globalisation of Star Product

Kontsevich proved the formality theorem on general manifolds in [Kon97] and [Kon01]. In this section we summarise a slightly different approach of the globalisation of Kontsevich star product following Cattaneo, Felder and Tomassini in [CFT02] and [CF01]. Note that this approach proves a weaker result, as the general formality theorem does not follow immediately from the existence of star products. Another approach due to Dolgushev ([Dol05a], [Dol05b]) directly globalises the formality theorem is stronger, but we will not cover it due to length limit.

3.3.1 Formal Geometry

The first step of globalisation is to construct two vector bundles over the Poisson manifold M, which fibre-wise represents the Poisson algebras and their quantisations. The idea is that two smooth functions with the same Taylor coeffcients are indistinguishable from the star product.

Definition 3.14. Let M be a d-dimensional smooth manifold. The k-th jet bundle $\mathrm{J}^k(M)$ over M is defined fibre-wise as follows: for $x \in M$, $\mathrm{J}^k_x(M)$ is the set of functions $f \in \mathrm{C}^\infty(M)$ quotient by the equivalence relation that $f \sim g$ if and only if the partial derivatives of f and g at p agree up to order k. The **infinite jet bundle** $\mathrm{J}^\infty(M)$ is obtained by taking the projective limit[1] of the forgetful maps $\mathrm{J}^{k+1}(M) \to \mathrm{J}^k(M)$. That is, for

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[1]Technically we have to specify the topology with respect to which we are taking the limit. Heuristically we consider $\mathrm{J}^\infty(M)$ as a formal projective limit in the category of smooth manifolds.

$f, g \in C^\infty(M)$, $[f] = [g] \in J^\infty(M)$ if and only if f and g have the same Taylor expansion at some $x \in M$.

We give a local characterisation of $J^\infty(M)$. Let $M^{\mathrm{coor}} := J_0^\infty(\mathbb{R}^d, M)$ be the fibre at $0 \in \mathbb{R}^d$ of the infinite jet bundle $J^\infty(\mathbb{R}^d, M)$. Note that $\mathrm{GL}(d, \mathbb{R})$ acts on M^{coor} by linear diffeomorphisms, which induces the quotient manifold $M^{\mathrm{aff}} := M^{\mathrm{coor}}/\mathrm{GL}(d, \mathbb{R})$. In particular, M^{aff} is a vector bundle over M whose fibres are contractible, and hence admits a section $\varphi^{\mathrm{aff}} \colon M \to M^{\mathrm{aff}}$. We define the **associated bundle** $\widetilde{E}_0$ over M^{aff} by taking the fibre product of M^{coor} with a formal power series:

$$\widetilde{E}_0 := M^{\mathrm{coor}} \times_{\mathrm{GL}(d,\mathbb{R})} \mathbb{R}\big[\big[y^1, ..., y^d\big]\big].$$

Then the pull-back via φ^{aff} provides a vector bundle $E_0 := (\varphi^{\mathrm{aff}})^* \widetilde{E}_0$ over M, with fibres $\mathbb{R}\big[\big[y^1, ..., y^d\big]\big]$. Note that E_0 is isomorphic to $J^\infty(M)$, given by identifying the jets of f at p with the Taylor expansion of $f \circ \varphi^{-1}$ at 0, where $\varphi_x \colon M \to \mathbb{R}^d$ is a chosen coordinate chart around $x \in M$ and $\varphi_x(x) = 0$.

Now consider M as a Poisson manifold. Let $E_0 = J^\infty(M)$ be its infinite jet bundle. The canonical map $C^\infty(M) \to E_0$ transports the Poisson structure of $C^\infty(M)$ to E_0.

Proposition 3.15. Grothendieck Connection

There exists a canonical connection D_0 on E_0 such that

1. D_0 is a derivation: $D_0(fg) = f D_0 g + g D_0 f$ for $f, g \in \Gamma(M, E_0)$.
2. D_0 is flat: $D_0^2 = 0$. This implies that the E_0-valued differential forms
$$\Omega^\bullet(E_0) := \Omega^\bullet(M) \otimes_{C^\infty(M)} \Gamma(M, E_0)$$
   is a cochain complex with respect to the connection D_0. It provides the cohomology groups $H^\bullet(E_0, D_0)$.
3. The cohomology is concentrated at zeroth degree: $H^n(E_0, D_0)$ for $n > 0$.
4. The canonical map $C^\infty(M) \to E_0$ induces a Poisson isomorphism between $C^\infty(M)$ and the algebra of D_0-horizontal sections $H^0(E_0, D_0)$.

We call D_0 the **Grothendieck connection** of E_0.

Proof. See §2 of [CF01].   □

On the other hand, let $\widetilde{E}$ be the associated bundle of $\mathbb{R}[\![\hbar]\!]$-modules

$$\widetilde{E} := M^{\mathrm{coor}} \times_{\mathrm{GL}(d,\mathbb{R})} \mathbb{R}\big[\big[y^1, ..., y^d, \hbar\big]\big]$$

over M^{aff}. And let $E := (\varphi^{\mathrm{aff}})^* \widetilde{E} = E_0[\![\hbar]\!]$ be a vector bundle over M. We shall regard E as the fibre-wise quantisation of E_0. More explicitly we would like to prove that

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Theorem 3.16. Deformation Quantisation on a Poisson Manifold

Let (M, π) be a Poisson manifold and E be the vector bundle over M constructed above. There exists a flat connection $\overline{D} = D_0 + \sum_{n=1}^\infty \hbar^n D_n$ of E such that the $\mathbb{R}[\![\hbar]\!]$-algebra $H^0(E, \overline{D})$ is a quantisation of $C^\infty(M)$.

Following the spirit of Fedosov's construction ([Fed94]), the next step is to construct a connection of E and transport the Kontsevich star product to the sections $\Gamma(M, E)$ of E. The formality theorem in $\mathbb{R}^d$ provides such constructions.

3.3.2 Star Product and Connection on Deformed Bundle

Recall that Kontsevich's construction (3.8) provides an L_∞-quasi-isomorphism $U \colon \mathscr{T}_{\mathsf{poly}}(\mathbb{R}^d) \to \mathscr{D}_{\mathsf{poly}}(\mathbb{R}^d)$. We make the following definition:

Definition 3.17. Let π be a Poisson bivector field and $\xi, \eta \in \Gamma(\mathbb{R}^d, \mathrm{T}\mathbb{R}^d)$. Define:

$$P(\pi) := \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} U_k(\pi \cdots \pi),$$

$$A(\xi, \pi) := \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} U_{k+1}(\xi \cdot \pi \cdots \pi),$$

$$F(\xi, \eta, \pi) := \sum_{k=0}^{\infty} \frac{\hbar^k}{k!} U_{k+2}(\xi \cdot \eta \cdot \pi \cdots \pi).$$

Note that $P(\pi) \in \mathscr{D}^1_{\mathsf{poly}}(\mathbb{R}^d)[\![\hbar]\!]$ is exactly the star product. We expect that $A(-, \pi)$ and $F(-, -, \pi)$ behave like a connection 1-form and its curvature 2-form.

Definition 3.18. The Lie algebra $\Gamma(\mathrm{T}\mathbb{R}^d)$ of vector fields acts on the space of *local polynomial maps* $\mathscr{U}$ (*cf.* §3 of [CFT02]). We consider the **Chevalley–Eilenberg complex**: [2]

$$C^\bullet_{\mathsf{CE}}(\mathrm{T}\mathbb{R}^d, \mathscr{U}) := \mathrm{Hom}_{\mathbb{R}}(\textstyle\bigwedge^\bullet \Gamma(\mathrm{T}\mathbb{R}^d), \mathscr{U}),$$

where $S \in C^p_{\mathsf{CE}}(\mathrm{T}\mathbb{R}^d, \mathscr{U})$ sends $X_1 \wedge \cdots \wedge X_p$ to $S(X_1 \wedge \cdots \wedge X_p, \pi) \in \mathscr{D}_{\mathsf{poly}}(\mathbb{R}^d)$ which depends polynomially on π. Therefore $P \in C^0_{\mathsf{CE}}(\mathrm{T}\mathbb{R}^d, \mathscr{U})[\![\hbar]\!]$, $A \in C^1_{\mathsf{CE}}(\mathrm{T}\mathbb{R}^d, \mathscr{U})[\![\hbar]\!]$, and $F \in C^2_{\mathsf{CE}}(\mathrm{T}\mathbb{R}^d, \mathscr{U})[\![\hbar]\!]$.

The Chevalley–Eilenberg differential $\delta \colon C^p_{\mathsf{CE}}(\mathrm{T}\mathbb{R}^d, \mathscr{U}) \to C^{p+1}_{\mathsf{CE}}(\mathrm{T}\mathbb{R}^d, \mathscr{U})$ is defined as

$$
\begin{aligned}
(\delta S)(X_1, \ldots, X_k + 1, \pi) :={} & \sum_{i=1}^{k+1} (-1)^i \frac{\mathrm{d}}{\mathrm{d}t} S\Big(X_1, \ldots, \widehat{X}_i, \ldots, X_{k+1}, \big(\Phi_X^t\big)_* \pi\Big)\Big|_{t=0} \\
& + \sum_{i<j} (-1)^{i+j} S\Big([X_i, X_j], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{k+1}, \pi\Big),
\end{aligned}
$$

where Φ_X^t is the flow along the vector field X. From this differential and the formality equation (2.18) follows the lemma:

Lemma 3.19

  (i) $P \circ (P \otimes \mathrm{id} - \mathrm{id} \otimes P) = 0$;

  (ii) $(\delta P)(\xi, \pi) = P(\pi) \circ (A(\xi, \pi) \otimes \mathrm{id} + \mathrm{id} \otimes A(\xi, \pi)) - A(\xi, \pi) \circ P(\pi)$;

  (iii) $(\delta A)(\xi, \eta, \pi) = P(\pi) \circ (F(\xi, \eta, \pi) \otimes \mathrm{id} - \mathrm{id} \otimes F(\xi, \eta, \pi)) - A(\xi, \pi) \circ A(\eta, \pi) + A(\eta, \pi) \circ A(\xi, \pi)$;

  (iv) $(\delta F)(\xi, \eta, \zeta, \pi) = -A(\xi, \pi) \circ F(\eta, \zeta, \pi) - A(\eta, \pi) \circ F(\zeta, \xi, \pi) - A(\zeta, \pi) \circ F(\xi, \eta, \pi)$.

In the lemma, (i) is the associativity equation of the star product, and (ii),(iii),(iv) describe the changes of P, A, F under a coordinate transformation induced by the vector field ξ. These identifies are useful in the construction of the Fedosov connection D in Proposition 3.23. In addition, the following lowest order properties of P, A, F are required.

---

[2] In general, let $\mathfrak{g}$ be a Lie algebra and V be a $\mathfrak{g}$-module. We can similar construct the Chevalley–Eilenberg complex $C^\bullet_{\mathsf{CE}}(\mathfrak{g}, V)$. The cohomology of this complex is called the *Lie algebra cohomology of $\mathfrak{g}$ with value in V*. See Chapter 7 of [Weib95] for details.

> **Lemma 3.20**
>
> (i) $P(\pi)(f \otimes g) = fg + \pi(\mathrm{d}f, \mathrm{d}g)\hbar + \mathcal{O}(\hbar^2)$.
>
> (ii) $A(\xi, \pi) = \xi + \mathcal{O}(\hbar)$, and $A(\xi, \pi) = \xi$ if ξ is a linear vector field.
>
> (iii) $F(\xi, \eta, \pi) = \mathcal{O}(\hbar)$.
>
> (iv) $P(\pi)(1 \otimes f) = P(\pi)(f \otimes 1) = f$.
>
> (v) $A(\xi, \pi)1 = 0$.

With these tools in hand, we can now lift the star product to the deformed bundle E. A key observation in [CFT02] is that, since the Kontsevich star product is $\mathrm{GL}(d, \mathbb{R})$-equivariant, after taking an open cover of M by contractible coordinate charts and fixing the representatives of $\mathrm{GL}(d, \mathbb{R})$-equivalence classes on each chart, we may assume that E is a *trivial bundle* over M, with fibres isomorphic to $\mathbb{R}[\![y^1, ..., y^d, \hbar]\!]$. A section $f \in \Gamma(M, E)$ is a map $x \mapsto f_x$, where $f_x = f_x(y) \in \mathbb{R}[\![y^1, ..., y^d, \hbar]\!]$.

Definition 3.21. Let $\star = P(\pi)$ be the Kontsevich star product on $\mathbb{R}^d$. It induces an associative product on the space $\Gamma(M, E)$ of sections of E by:

$$(f \star g)_x := P(\pi_x)(f_x \otimes g_x) = P((\varphi_x^{-1})_*\pi)(f_x \otimes g_x),$$

where $\varphi_x \colon U \to \mathbb{R}^d$ is the coordinate chart around x.

Definition 3.22. We define a connection $D \colon \Gamma(M, E) \to \Omega(M) \otimes_{\mathrm{C}^\infty(M)} \Gamma(M, E)$ of E by

$$(Df)_x := \mathrm{d}_x f + A_x^M f,$$

where d_x is the de Rham differential of $x \mapsto f_x$ with value in $\mathbb{R}[\![y^1, ..., y^d, \hbar]\!]$, and A_x^M is a connection 1-form satisfying $A_x^M(\xi) := A(\widehat{\xi}_x, \pi_x)$ for $\xi \in \Gamma(M, \mathrm{T}M)$. Here $\widehat{\xi}_x$ is defined by

$$\widehat{\xi}_x(y) = -\sum_{i,j,k=1}^{\dim M} \xi^i \left(\left(\frac{\partial \varphi_x}{\partial y} \right)^{-1} \right)_j^k \frac{\partial \varphi_x^j}{\partial x^i} \frac{\partial}{\partial y^k},$$

which is a formal vector field in y and depends on the coordinate chart φ_x. Furthermore, it can be shown from Lemma 3.20 that D is independent of the choice of φ_x and hence is a well-defined global connection of E.

3.3.3   Fedosov's Construction

Similar to Fedosov's construction ([Fed94]) for a star product on the symplectic manifold, we would like to construct a flat Fedosov connection $\overline{D}$ of E so that the $\overline{D}$-horizontal sections are isomorphic to the $\mathbb{R}[\![\hbar]\!]$-algebra $\mathrm{C}^\infty(M)[\![\hbar]\!]$.

> **Proposition 3.23**
>
> Let $F^M \in \Omega^2(E)$ be the E-valued 2-form $x \mapsto F_x^M$, where $F_x^M(\xi, \eta) := F(\widehat{\xi}_x, \widehat{\eta}_x, \pi_x)$ for $\xi, \eta \in \Gamma(M, \mathrm{T}M)$. Then D is a **Fedosov connection** with **Weyl curvature** F^M. More specifically, for $f, g \in \Gamma(M, E)$,
>
> 1. $D(f \star g) = Df \star g + f \star Dg$;
> 2. $D^2 f = \left[F^M, f\right]_\star := F^M \star f - f \star F^M$;
> 3. Bianchi identity: $DF^M = 0$.

To prove Theorem 3.16, we need to deform D into a *flat connection* $\overline{D}$ which is still Fedosov. We need the following lemma:

> **Lemma 3.24**
>
> Suppose that D is a Fedosov connection on E with Weyl curvature F. Then for $\gamma \in \Omega^1(E)$, $\overline{D} := D + [\gamma, -]_\star$ is a Fedosov connection with Weyl curvature $F + D\gamma + \gamma \star \gamma$.

The construction of a flat Fedosov connection $\overline{D}$ is now reduced to finding a solution to the differential equation $F^M + D\gamma + \gamma \star \gamma = 0$. It can be proven that γ can be constructed order by order as a result of the vanishing of higher cohomology of E_0:

> **Lemma 3.25**
>
> Suppose that D_0 is a flat connection of E_0 with $\mathrm{H}^2(E_0, D_0) = 0$. Then there exists $\gamma \in \hbar\Omega^1(E)$ such that $\overline{D} := D + [\gamma, -]_\star$ has zero Weyl curvature.

Finally, we would like to construct a quantisation from $\mathrm{H}^0(E_0, D_0)$ to $\mathrm{H}^0(E, \overline{D})$. Let $\mathrm{End}(E_0)$ be vector bundle over M of the fibre endomorphisms of E_0. Then $\Omega^\bullet(\mathrm{End}(E_0))$ is a differential graded algebra, with the differential given by the super-commutator:

$$D_0(\Phi) := D_0 \circ \Phi - (-1)^p \Phi \circ D_0, \qquad \Phi \in \Omega^p(E_0).$$

Note that if $D = D_0 + \sum_{n=1}^\infty \hbar^n D_n$ is a connection on $E = E_0[\![\hbar]\!]$, then $D_n \in \Omega^1(\mathrm{End}(E_0))$ for $n \in \mathbb{Z}_{>0}$.

> **Lemma 3.26**
>
> Suppose that $D = D_0 + \sum_{n=1}^\infty \hbar^n D_n$ is a flat Fedosov connection on E, and $\mathrm{H}^1(\mathrm{End}(E_0), D_0) = 0$. Then there exists a formal power series $\rho = \mathrm{id} + \sum_{n=1}^\infty \hbar^n \rho_n \in \Omega^0(\mathrm{End}(E_0))[\![\hbar]\!]$ which induces an $\mathbb{R}[\![\hbar]\!]$-isomorphism $\mathrm{H}^0(E_0, D_0)[\![\hbar]\!] \to \mathrm{H}^0(E, \overline{D})$.

The proof of the lemma is similar to the previous one, where ρ is constructed recursively as a result of the vanishing of cohomology. Combining Proposition 3.15 and Lemma 3.26, we obtain an isomorphism $\tau \colon \mathrm{C}^\infty(M)[\![\hbar]\!] \to \mathrm{H}^0(E, \overline{D})$. An explicit star product on $\mathrm{C}^\infty(M)$ is given by

$$(f \star g)(x) = \left[\tau^{-1}(\tau(f)_x(y) \star \tau(g)_x(y)) \right]_{y=0}.$$

It remains to check that $f \star g$ satisfies the conditions for a star product on $\mathrm{C}^\infty(M)$ and that it recovers the Poisson bivector π in the first order of $\hbar$. For the details refer to [CFT02]. This concludes the proof of Theorem 3.16.

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