Peize Liu
St. Peter's College
University of Oxford

Problem Sheet 3

Nuclear Mass and Energy
& Radioactive Decays

B4: Subatomic Physics

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Question 3.1

The radius r of a nucleus with mass number A is given by $r = r_0 A^{1/3}$ with $r_0 = 1.2$fm. What does this tell us about the nuclear force?

a) Use the Fermi gas model (assuming $N \approx Z$) to show that the energy ϵ_F of the Fermi level is given by $\epsilon_F = \frac{\hbar^2}{2mr_0^2}\left(\frac{9\pi}{8}\right)^{\frac{2}{3}}$

b) Estimate the total kinetic energy of the nucleons in an ^{16}O nucleus.

c) For a nucleus with neutron number N and proton number Z the asymmetry term in the semi-empirical mass formula is

$$\frac{a_A(N-Z)^2}{A}$$

Assuming that $(N-Z) \ll A$ use the Fermi gas model to justify this form and to estimate the value of a_A. Comment on the value obtained.

Proof. $r_0 = 1.2$ fm tells us that the nuclear force only acts in an extremely short distance. ✓ + hard spheres

a) This is a revision of statistical mechanics. The occupation number of Fermi-Dirac statistics is given by

$$N = \int_0^\infty \frac{\rho_n(\varepsilon)}{e^{\beta(\varepsilon-\mu)}+1}\,d\varepsilon, \qquad Z = \int_0^\infty \frac{\rho_p(\varepsilon)}{e^{\beta(\varepsilon-\mu)}+1}\,d\varepsilon$$

where $\rho(\varepsilon)$ is the density of states, $\beta = 1/kT$, and μ is the chemical potential.

First we compute the density of states.

$$\rho_n(k)\,dk = (2s_n+1)\frac{Vk^2}{2\pi^2}\,dk = \frac{Vk^2}{\pi^2}\,dk \implies \rho_n(\varepsilon)\,d\varepsilon = \frac{2mV\varepsilon}{\hbar^2\pi^2}\frac{\sqrt{m}}{\hbar\sqrt{2\varepsilon}}\,d\varepsilon = \frac{2mV\sqrt{2m\varepsilon}}{\pi^2\hbar^3}\,d\varepsilon = \left(\frac{2mr_0^2}{\hbar^2}\right)^{3/2}\frac{2A\sqrt{\varepsilon}}{3\pi}\,d\varepsilon$$ ✓

Similarly,

$$\rho_p(\varepsilon) = \rho_n(\varepsilon) = \left(\frac{2mr_0^2}{\hbar^2}\right)^{3/2}\frac{2A\sqrt{\varepsilon}}{3\pi}$$

For the degenerate limit of the Fermi gas ($T \to 0$), we have

$$\frac{1}{e^{\beta(\varepsilon-\mu)}+1} \to \mathbf{1}_{\{\varepsilon<\mu(T=0)\}} = \mathbf{1}_{\{\varepsilon<\varepsilon_F\}}$$

where $\varepsilon_F := \mu(T=0)$ is the Fermi energy. In the limit, we have

$$Z = \int_0^{(\varepsilon_F)_p} \rho_p(\varepsilon)\,d\varepsilon = \left(\frac{2mr_0^2}{\hbar^2}\right)^{3/2}\frac{2A}{3\pi}\int_0^{(\varepsilon_F)_p}\sqrt{\varepsilon}\,d\varepsilon = \frac{4A}{9\pi}\left(\frac{2mr_0^2(\varepsilon_F)_p}{\hbar^2}\right)^{3/2}$$

Hence

$$(\varepsilon_F)_p = \frac{\hbar^2}{2mr_0^2}\left(\frac{9\pi Z}{4A}\right)^{2/3}, \qquad (\varepsilon_F)_n = \frac{\hbar^2}{2mr_0^2}\left(\frac{9\pi N}{4A}\right)^{2/3}$$

For $N \approx Z$, we have

$$(\varepsilon_F)_p \approx (\varepsilon_F)_n = \varepsilon_F = \frac{\hbar^2}{2mr_0^2}\left(\frac{9\pi}{8}\right)^{2/3}$$ ✓

b) The mean kinetic energies of protons and neutrons are given by

$$T_p = \int_0^{(\varepsilon_F)_p}\varepsilon\rho(\varepsilon)\,d\varepsilon = Z\frac{\int_0^{(\varepsilon_F)_p}\varepsilon^{3/2}\,d\varepsilon}{\int_0^{(\varepsilon_F)_p}\varepsilon^{1/2}\,d\varepsilon} = \frac{3}{5}Z(\varepsilon_F)_p, \qquad T_n = \frac{3}{5}N(\varepsilon_F)_n$$ ✓

For the ^{16}O nucleus, the atomic number $A = 16$ and $N = Z = 8$. Hence ✓

$$T = \frac{3}{5}Z(\varepsilon_F)_p + \frac{3}{5}N(\varepsilon_F)_n = \frac{3}{5}A\varepsilon_F = \frac{6}{5}(9\pi)^{2/3}\frac{\hbar^2}{mr_0^2} = 222.2\text{ MeV}$$ ≈

c) The mean energy

$$T = \frac{3}{5}Z(\varepsilon_F)_p + \frac{3}{5}N(\varepsilon_F)_n = \frac{3\hbar^2}{10mr_0^2}\left(\frac{9\pi}{4A}\right)^{2/3}(Z^{5/3}+N^{5/3})$$

For $(N-Z) \ll A$, we try to expand $Z^{5/3}+N^{5/3}-A^{5/3}/$ in term of the difference $x := N-Z$.

$$N^{5/3}+Z^{5/3} = \left(\frac{A+x}{2}\right)^{5/3} + \left(\frac{A-x}{2}\right)^{5/3} = \left(\frac{A}{2}\right)^{5/3}\left(\left(1+\frac{x}{A}\right)^{5/3} + \left(1-\frac{x}{A}\right)^{5/3}\right) = \left(\frac{A}{2}\right)^{5/3}\left(2+2\cdot\frac{1}{2}\frac{5}{3}\frac{2}{3}\left(\frac{x}{A}\right)^2 + O\left(\left(\frac{x}{A}\right)^4\right)\right)$$

Hence

$$T \sim \frac{3\hbar^2}{10mr_0^2}\left(\frac{9\pi}{8}\right)^{2/3}\left(A + \frac{5}{9}\frac{(N-Z)^2}{A}\right)$$

The first term is absorbed in the volume term. The second term determines the asymmetric term. The coefficient a_A is given by

$$a_A = \frac{\hbar^2}{6mr_0^2}\left(\frac{9\pi}{8}\right)^{2/3} = 11.11\,\text{MeV}$$

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Comparing to the result given in the lecture notes $a_A \sim 23$ MeV, there is a 100% difference, suggesting our model is insufficient. In our degenerate Fermi gas model, it is assumed that the protons and neutrons have no mutual interactions. This is not the case in reality. □

Question 3.2

$^9_4$Be is the only stable isotope of beryllium. Some values of $B(A,Z)$ computed from the SEMF in MeV are

	$A=6$	$A=7$	$A=8$	$A=9$	$A=10$	$A=11$
$Z=2$	24.3	14.5	11.8	-0.5	-5.7	-19.0
$Z=3$	28.1	39.3	36.5	39.2	31.7	30.2
$Z=4$	19.7	36.7	54.2	57.5	64.9	61.9
$Z=5$	-40.1	6.8	31.0	54.5	63.7	76.0
$Z=6$	-112.2	-50.4	0.7	30.1	58.5	72.5
$Z=7$	–	-135.0	-70.5	-15.6	18.7	51.5

a) Would you expect the SEMF to work well in this range of A and Z?

b) Illustrate how the binding energies predicted by the SEMF can be used to understand why $^9_4$Be is stable and $^{11}_4$Be is unstable.

c) The results of the SEMF given above predict that $^{10}_4$Be is stable. In reality, it decays slowly to $^{10}_5$B. Indicate the mechanism by which the decay occurs and calculate by how much the binding energy of $^{10}_4$Be would need to be adjusted from the SEMF value of 64.9MeV to make the decay energetically possible.

d) $^7_4$Be is found to decay only by electron capture. In what way are the binding energies predicted by the SEMF for $^7_4$Be and $^7_3$Li inconsistent with this fact?

e) Why is $^8_4$Be unstable even though in low-mass elements the $A=2Z$ isotope is often the most stable? (The measured binding energy of $^4_2$He is 28.3MeV.)

Solution. a) The semi-empirical mass formula works well for neclei with large atomic number A. For small neclei, the liquid drop is inaccurate and we must consider the shell model. Hence we do not expect the SEMF to work well in the range of A and Z given in the table.

b) The SEMF predicts that the stable isotope appears at which

$$\left.\frac{\partial B}{\partial Z}\right|_A = 0$$

From the table we observe that $B(3,9) < B(4,9)$ and $B(4,9) > B(5,9)$. Hence $^9_4$Be is stable. $B(3,11) < B(4,11) < B(5,11)$. Hence $^{11}_4$Be is unstable.

Look at Q : $Q < 0$ stable
$Q > 0$ unstable

c) Comparing $B(4,10)$ and $B(5,10)$ in the semi-empirical mass formula, we find that

$$B(5,10) - B(4,10) = -\frac{9}{10^{1/3}}a_C - \delta(4,10) + \frac{4}{10}a_A$$

$Q = -0.4$ MeV

$\therefore$ 0.4 MeV would be enough

Since A is small, the asymmetric term ($\propto A^{-1}$) has a larger contribution to the binding energy than the Coulomb term ($\propto A^{-1/3}$) and the pairing term ($\propto A^{-1/2}$ or $A^{-3/4}$). So it is likely that $B(4,10)$ is overestimated comparing to $B(5,10)$. The energy should be at least 1.2 MeV less than the prediction given by SEMF to make the decay energetically possible.

d) We find that $B(3,7) - B(4,7) = 2.6$ MeV is small. So the decay from ${}^7_4$Be to ${}^7_3$Li can only release a very small amount of energy. This can only be done through the electron capture:

also $3m_p + 4m_n \rightarrow 4m_p + 3m_n$

$${}^7_4\text{Be} + {}_{-1}\text{e} \rightarrow {}^7_3\text{Li} + \nu_e + 2.6\,\text{MeV}$$

$\rightarrow Q = 1.8$ MeV

in which all decay energy is carried by the electron neutrino ν_e, which has a very small mass.

e) We find that $B(4,8) < 2B(2,4)$. As the total energy of ${}^8_4$Be is greater than that of two α-particles, the decay into two ${}^4_2$He is energetically favorable. Therefore ${}^8_4$Be is highly unstable. □

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Question 3.3

Why is there such a wide range of stable nuclides in nature:

a) ^{62}Ni is the nuclide with the highest binding energy per nucleon. Why are there heavier nuclides with $A \lesssim 150$ still stable against α decay towards this configuration? Using the SEMF plot the Q value for α decay of the most stable isotope as a function of A and verify your answer (For ^{4}He use the binding energy given in problem 2.2).

b) Why does α decay for heavy nuclei dominate over the emission of a single nucleon? Why does α decay dominate over cluster decays or spontaneous fission?

c) Why do nuclei with $A < 58$ not spontaneously fuse to heavier nuclei?

d) Both, the neutron and the Λ^0 particle, decay due to the weak interaction. Why can a neutron be bound in a nucleus stable against decay while a lambda particle in a hypernucleus is not (the mass of the Λ^0 is 1116MeV/c^2, and its dominant decay is $\Lambda^0 \rightarrow N + \pi$, where N is either a proton or a neutron)?

Solution. a) The Q-value in the α-decay is given by

$$Q(Z,A) = B(Z-2, A-4) + B(2,4) - B(Z,A)$$

For the most stable isotope, we use

$$\left.\frac{\partial B}{\partial Z}\right|_A = 0 \implies \frac{A}{Z} = 2 + \frac{a_C}{2a_A}A^{2/3}$$

We use the semi-empirical mass formula:

$$B(Z,A) = a_V A - a_S A^{2/3} - a_C\frac{Z^2}{A^{1/3}} - a_A\frac{(A-2Z)^2}{A} \pm \delta(Z,A)$$

For the pairing term, $\pm\delta(Z,A)$ and $\pm\delta(Z-2,A-4)$ cancel out approximately. We can then sketch $Q(A)$ against A:

nice!

From the figure we observe that $Q < 0$ for $A < 150$. Hence the α-decay of a stable necleus of atomic number $A < 150$ is not energetically favourable.

b) In the decay of a heavy atom, the ejected particle must tunnel through the electromagnetic potential barrier. By one-dimensional WKB approximation, the probability of tunnelling is proportional to e^{-2G}, where G is the Gamow factor given by

$$G = \frac{\sqrt{2m}}{\hbar}\int \sqrt{V(r) - Q}\,\mathrm{d}r = \frac{\sqrt{2mQ}}{\hbar}\int \sqrt{\frac{Z'(Z-Z')e^2}{4\pi\varepsilon_0 Q r} - 1}\,\mathrm{d}r \approx \sqrt{\frac{2m}{Q}}\frac{Z'(Z-Z')e^2}{8\hbar\varepsilon_0}$$

The approximation is for small Q.

We observe that $G \propto Z'(Z-Z')$. So G is large for small ejected Z'. This explains why the α-decay dominates over cluster decays and spontaneous fission.

In addition, the α-particle has a huge binding energy per necleuon. This explains why α-decay dominates over the emission of single proton or neutron.

c) Nuclear fusion also needs to overcome the electromagnetic potential barrier. The fusion probability is a combination of a tunnelling probability and a Maxwell-Boltzmann distribution.

$$P_{\text{fusion}} \propto \exp\left(-\frac{\alpha}{\sqrt{E}} - \frac{E}{k_B T}\right)$$

So for the fusion to happen spontaneously ($A < 58$), we need a very high temperature.

d) Note that

$$m_n - m_p - m_e = 939.565 - 938.277 - 0.511 = 0.777 \text{ MeV}$$
$$m_\Lambda - m_n - m_{\pi^0} = 1116 - 938 - 140 = 38 \text{ MeV}$$

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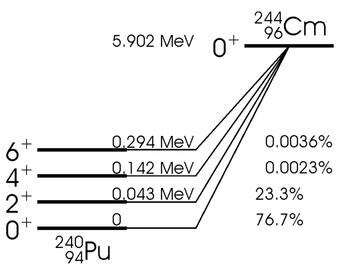
Both the decay of free neutron and free Λ^0 are energetically favourable. But the Q-value of the decay of neutron is extremely small comparing to the Q-value of Λ^0. So a neutron can be easily bound in a nucleus stable against decay. But for Λ^0 to be bound stably in a hypernucleus, the hypernucleus must have a very high binding energy, which is unlikely to happen in the real world. □

Question 3.4

The figure below shows the α-decay scheme of $^{244}_{96}$Cm and $^{240}_{94}$Pu.



Justify that we would expect the rates to satisfy an equation of the form

$$\ln R = A - \frac{B(Z-2)}{\sqrt{Q}}$$

with the parameters A and B. The Q value for the ground state to ground state transition is 5.902MeV and for this transition $A = 132.8$ and $B = 3.97(\text{MeV})^{1/2}$ when R is in s^{-1}. The branching ratio for this transition is given in the figure. Calculate the mean life of ^{244}Cm.

Estimate the transition rate from the ground state of ^{244}Cm to the 6^+ level of ^{240}Pu using the same values for A and B and compare to the branching ratio given in the figure. Suggest a reason for any discrepancy.

[*Hint: what form does the Schrödinger equation take for angular momentum quantum number $l \neq 0$?*]

Solution. The decay rate per nucleus is given by

$$R = a\mathrm{e}^{-2G}$$

where G is the Gamow factor in Question 3.(b). From which we obtain the Geiger-Nuttall law:

$$\ln R = \ln a - G \ln 2 = A - \frac{B(Z-2)}{\sqrt{Q}}$$

since $G \propto (Z-2)/\sqrt{Q}$. With $A = 132.8$, $B = 3.97\ (\text{MeV})^{1/2}$ and $Z = 96$, the transition rate from the ground state of ${}^{244}_{96}\text{Cm}$ to the ground state of ${}^{240}_{94}\text{Pu}$ is $R_0 = 9.172 \times 10^{-10}\ \text{s}^{-1}$. The total transition rate is given by

$$R = \frac{R_0}{76.7\%} = 1.195 \times 10^{-9}$$

The mean lifetime is given by

$$\tau = \frac{1}{R} = 8.36 \times 10^{8}\ \text{s}$$

If we use the same value for A and B, then the transition rate from the ground state of ${}^{244}_{96}\text{Cm}$ to the 6^+ level of ${}^{240}_{94}\text{Pu}$ is $R_6 = 1.722 \times 10^{-11}\ \text{s}^{-1}$. The branching ratio is given by

$$\frac{R_6}{R} = 1.44\% > 0.0036\%$$

The calculated branching ratio is significantly larger than the observed ratio.

We look at the Schrödinger's equation for hydrogen-like atom:

$$-\frac{\hbar^2}{2m}\nabla^2 |\psi\rangle + \frac{Z(Z-2)e^2}{4\pi\varepsilon_0 r}|\psi\rangle = E|\psi\rangle$$

For a state with angular momentum number ℓ,

$$\mathrm{L}^2|\psi\rangle = \ell(\ell+1)\hbar^2|\psi\rangle$$

If we use the separation of variables $\psi(r,\theta,\varphi) = R(r,\theta)Y(\theta,\varphi)$, then the radial function $R(r,\theta)$ satisfies

$$\left(-\frac{\hbar^2}{2m}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right) + \frac{\hbar^2\ell(\ell+1)}{2mr^2} + \frac{Z(Z-2)e^2}{4\pi\varepsilon_0 r}\right)R(r,\theta) = ER(r,\theta)$$

The effective potential is given by

$$V_{\text{eff}}(r) = \frac{\hbar^2\ell(\ell+1)}{2mr^2} + \frac{Z(Z-2)e^2}{4\pi\varepsilon_0 r} > \frac{Z(Z-2)e^2}{4\pi\varepsilon_0 r}$$

exactly!

7/7 So the effective potential is higher than the Coulomb potential, which suggests a smaller transition rate. □

Question 3.5

Which terms in the SEMF are responsible for the existence of a viable chain reaction of thermal-neutron-induced uranium fission? What distinguishes the isotopes of uranium that support such a reaction?

The fission of ${}^{235}\text{U}$ by thermal neutrons is asymmetric, the most probable mass numbers of fission fragments being 93 and 140. Where are the daughter nuclei in relation to the valley of stability and what happens to them subsequently? Use the semi-empirical mass formula to find the most probable value for Z for these mass numbers and estimate the energy released in fission of ${}^{235}_{92}\text{U}$ and hence the mass of ${}^{235}_{92}\text{U}$ consumed each second in typical commercial nuclear reactor with a power of 1 GW.

In the construction of a nuclear fission reactor an important role is often played by water, heavy water or graphite. Describe this role and explain why these materials are suitable.

Why is the fissile material not completely mixed up with the moderator?

Solution. The pairing term is responsible for the chain reaction of thermal-neutron-induced uranium fission. More specifically, when a ${}^{235}_{92}\text{U}$ captures a thermal neutron and forms ${}^{236}_{92}\text{U}^*$, the pairing energy is released, which is enough for overcoming the Coulomb potential barrier and inducing the subsequential fission. ${}^{238}_{92}\text{U}$, on the other hand, forms ${}^{239}_{92}\text{U}^*$ when capturing a thermal neutron. There is no release of pairing energy and the extra energy required to overcome the potential barrier is much higher. This explains why chain reaction of induced fission is not possible for ${}^{238}_{92}\text{U}$. + asymmetry, Coulomb terms

Using the relation obtained by the semi-empirical mass formula

$$\frac{dB}{dZ} = 0 \longleftrightarrow \quad \frac{N}{Z} = 1 + \frac{a_C}{2a_A} A^{2/3} \quad \text{maximise } \frac{dE}{dZ} = 0$$

We find that when $A = 140$, the most probable $Z = 58$; when $A = 93$, the most probable $Z = 40$. 56 39

After the fission of ${}^{235}_{92}\text{U}$, the two daughter particles have 92 protons combined. The result isotopes are not stable because they have too many neutrons. Then they undergo β^--decay to get rid of the excessive neutrons.

A type of fission equation of ${}^{235}_{92}\text{U}$ is given by

$${}^{1}_{0}\text{n} + {}^{235}_{92}\text{U} \rightarrow {}^{140}_{56}\text{Ba} + {}^{93}_{36}\text{Kr} + 3{}^{1}_{0}\text{n}$$

From the semi-empirical mass formula, we can know that binding energies:

$$B(92, 235) = 1760,6 \text{ MeV}, \qquad B(56, 140) = 1153.4 \text{ MeV}, \qquad B(36, 93) = 781.6 \text{ MeV}$$

Hence the Q-value of the reaction is given by

$$Q = B(56, 140) + B(36, 93) - B(92, 235) = 174.4 \text{ MeV} = 2.79 \times 10^{-11} \text{ J}$$

For a reactor with power 1 GW, the number of atoms used per second is

$$\frac{1 \text{ GW}}{2.79 \times 10^{-11} \text{ J}} = 3.58 \times 10^{19} \text{ s}^{-1}$$

The mass used per second is

$$\frac{3.58 \times 10^{19} \times 235}{6.022 \times 10^{23}} = 1.397 \times 10^{-5} \text{ kg} \quad ✓$$

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The moderators are materials that used to slow down the ejected neutrons during fission to thermal energies before they are captured by ${}^{238}_{92}\text{U}$. This allows the chain reaction to happen. ✓ □

Question 3.6

The Fermi theory of neutron β-decay predicts that the rate of electrons emitted with momentum between p and $p + dp$ is given by

$$\Gamma(p_e)\, dp_e = \frac{G_F^2}{2\pi^3 \hbar^7 c^3} p_e^2 (Q - T_e)^2\, dp_e$$

where T_e is the kinetic energy of the electron, and Q is the energy released in the reaction.

a) Sketch this spectrum and justify the form of this result. What are the assumptions used to derive this result?

b) Show that for $Q \gg m_e c^2$ the total rate is proportional to Q^5.

c) What spin states are allowed for the combined system of the electron and the neutrino?

d) What modification to the above equation will be required for nuclear β-decay? Why are transitions between initial and final nuclei with angular momenta differing by more than $\hbar$ suppressed?

Solution. a) Sketch of Γ against p_e:

The formula is based on Fermi golden rule. In the derivation, the wave functions of the electron and the anti-neutrino are approximated by the normalization constants:

$m_\nu \approx 0$

$$\Psi_e \sim \Psi_{\bar{\nu}_e} \sim V^{-1/2}$$

$m_p, m_n \gg Q$

which is called the allowed approximation. In addition, the Coulomb interaction between the electron and the proton is neglected.

b) In the ultra-relativistic limit, $E_e \sim T_e \sim p_e c$. Then the rate

$$\Gamma = \int_0^{Q/c} \frac{G_F^2}{2\pi^3 \hbar^7 c^3} p_e^2 (Q - T_e)^2 \, \mathrm{d}p_e \propto \int_0^Q E_e^2 (Q - E_e)^2 \, \mathrm{d}E_e \propto Q^5$$

c) The allowed spin states are (Page 65 in Lecture notes):

- Fermi decays: the electron and the neutrino have anti-parallel spin. For $L = 0$ (orbital angular momentum of the electron/neutrino system) this means no change in the nuclear spin and these decays will be superallowed.
- Gamow-Teller decays: the electron and the neutrino have parallel spin. For $L = 0$, the change in the proton spin ΔI can be 0 or ± 1 and these decays will be allowed.

d) The modified equation for the transition rate is given by

$$\Gamma(p_e)\, \mathrm{d}p_e = \frac{G_F^2 |M|^2}{2\pi^3 \hbar^7 c^3} F(Z_D, p_e) p_e^2 (Q - T_e)^2 \, \mathrm{d}p_e$$

The modifications are the matrix element $|M|^2$, which depends on the wave function before and after the decay, and the Fermi factor $F(Z_D, p_e)$, which describes a correction due to the electromagnetic interction of the escaping electron with the whole daughter nucleus. □

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