Grade: Alpha
Great work! See my comments for the two problems you have.

Peize Liu
St. Peter's College
University of Oxford

Problem Sheet 2

B8.1: Probability, Measure & Martingales

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Remark. I attempted all Questions in Section 1.

For Question 6, I fail to show the independence because the distribution function I get is not consistent.

For Question 8, I am not able to prove that the events $\{|S_n/n| \geqslant 1\}$ are independent. You cannot. You use X_n/n instead.

Section 1

Question 1. Proof of Lemma 3.3 for any n

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $A_1, \dots, A_n$ some events in $\mathscr{F}$. Show that their generated σ-algebras are independent if and only if for any $J \subseteq \{1, \dots, n\}$

$$\mathbb{P}\left(\bigcap_{i\in J} A_i\right) = \prod_{i\in J} \mathbb{P}(A_i)$$

Proof. The direction "$\Longrightarrow$" is clear. For the "$\Longleftarrow$" direction, it suffices to prove that

$$\mathbb{P}\left(\bigcap_{i\in I} A_i \cap \bigcap_{j\in J} \Omega\setminus A_j\right) = \prod_{i\in I}\prod_{j\in J} \mathbb{P}(A_i)\mathbb{P}(\Omega\setminus A_j)$$

for any $I, J \subseteq \{1,...,n\}$ with $I\cap J = \varnothing$. We use induction on the cardinality of J. Base case $J = \varnothing$ is our assumption. For the induction case, fix $k \in J$. Write $J' = J\setminus\{k\}$. Then

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{i\in I} A_i \cap \bigcap_{j\in J} \Omega\setminus A_j\right) &= \mathbb{P}\left(\bigcap_{i\in I} A_i \cap \bigcap_{j\in J'} \Omega\setminus A_j \cap \Omega\setminus A_k\right) \\
&= \mathbb{P}\left(\bigcap_{i\in I} A_i \cap \bigcap_{j\in J'} \Omega\setminus A_j\right) - \mathbb{P}\left(\bigcap_{i\in I} A_i \cap A_k \cap \bigcap_{j\in J} \Omega\setminus A_j\right) \\
&= \prod_{i\in I}\prod_{j\in J'} \mathbb{P}(A_i)\mathbb{P}(\Omega\setminus A_j) - \prod_{i\in I}\prod_{j\in J'} \mathbb{P}(A_i)\mathbb{P}(\Omega\setminus A_j)\mathbb{P}(A_k) && \text{(induction hypothesis)} \\
&= \prod_{i\in I}\prod_{j\in J'} \mathbb{P}(A_i)\mathbb{P}(\Omega\setminus A_j)(1-\mathbb{P}(A_k)) \\
&= \prod_{i\in I}\prod_{j\in J} \mathbb{P}(A_i)\mathbb{P}(\Omega\setminus A_j)
\end{aligned}$$

which completes the induction. Hence the σ-algebra generated by $A_1, ..., A_k$ are independent. □

Question 2

Let $X_1, X_2, \dots$ be independent uniformly distributed random variables on $[0,1]$. Let A_n be the event that a record high value occurs at time n:

$$A_n = \{X_n > X_m \text{ for all } m < n\}$$

Find the probability of A_n and show that $A_1, A_2, \dots$ are independent. Deduce that, with probability one, infinitely many records occur.

Now consider *double records*, that is two records in a row. What is the probability of infinitely many double records?

Proof. Let $Y_n = \max_{1\leqslant m\leqslant n-1} X_m$. Then by independence of $X_1, ..., X_{n-1}$,

$$\mathbb{P}(Y_n \leqslant x) = \mathbb{P}\left(\bigcap_{m=1}^{n-1}\{X_m \leqslant x\}\right) = \prod_{m=1}^{n-1}\mathbb{P}(X_m \leqslant x) = x^{n-1}$$

The probability density function of Y_n is $f_{Y_n}(y) = \dfrac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}(Y_n \leqslant x) = (n-1)x^{n-2}$. Since $X_1, ..., X_n$ are independent, X_n is independent of Y_n. Hence the joint density of X_n and Y is given by $f_{X_n,Y_n}(x,y) = (n-1)y^{n-2}$. Now

$$\mathbb{P}(A_n) = \mathbb{P}(X_n > Y_n) = \iint_{\{(x,y)\in[0,1]^2 : x>y\}} f_{X_n,Y_n}(x,y)\,\mathrm{d}(x\otimes y) = \int_0^1\int_0^x (n-1)y^{n-2}\,\mathrm{d}y\mathrm{d}x = \int_0^1 x^{n-1}\,\mathrm{d}x = \frac{1}{n}$$

To prove that $(A_n)_{n=1}^\infty$ is an independent sequence of events, by Theorem 3.5 it suffices to verify that

$$\mathbb{P}\left(\bigcap_{i\in I} A_k\right) = \prod_{i\in I}\mathbb{P}(A_k)$$

for all finite subset $I\subseteq\mathbb{Z}_+$. We shall prove by induction on the cardinality of I. The base case $I=\{*\}$ is trivial.

Induction case: Let $I=\{k_1,\dots,k_n\}$, $I'=\{k_2,\dots,k_n\}$, $B_I := \bigcap_{k\in I'} A_k$.

The joint distribution of $X_1,\dots,X_{k_1}$ defines a pushforward measure on the product measure space $[0,1]^{k_1}$. Since $X_1,\dots,X_{k_1}$ are independent and identically uniformly distributed, for each $\sigma\in S_{k_1}$ there is an isomorphism of measure spaces

$$\widetilde{\sigma} : \left([0,1]^{k_1}, \mathscr{B}\left([0,1]^{k_1}\right), \mathbb{P}\circ X_1^{-1}\otimes\cdots\otimes\mathbb{P}\circ X_{k_1}^{-1}\right) \to \left([0,1]^{k_1}, \mathscr{B}\left([0,1]^{k_1}\right), \mathbb{P}\circ X_{\sigma(1)}^{-1}\otimes\cdots\otimes\mathbb{P}\circ X_{\sigma(k_1)}^{-1}\right)$$

In particular,

$$\mathbb{P}\big(A_{k_1}\big) = \mathbb{P}\left(X_{k_1} = \max_{1\leqslant m\leqslant k_1} X_m\right) = \mathbb{P}\left(X_{\sigma(k_1)} = \max_{1\leqslant m\leqslant k_1} X_m\right)$$

for any $\sigma\in S_{k_1}$. Note that $\mathbb{P}\big(X_i = X_j\big) = 0$ for any $i\neq j$, as it is the Lebesgue measure of the hyperplane $x=y$ in k_1-dimensional Euclidean space. Then we have

$$\mathbb{P}(B_I) = \sum_{i=1}^{k_1}\mathbb{P}\left(B_I\cap\left\{X_i = \max_{1\leqslant m\leqslant k_1} X_m\right\}\right) = \sum_{i=1}^{k_1}\mathbb{P}\big(B_I\cap A_{k_1}\big) = k_1\mathbb{P}\big(B_I\cap A_{k_1}\big) = \frac{\mathbb{P}(\bigcap_{k\in I}A_k)}{\mathbb{P}\big(A_{k_1}\big)}$$

By induction hypothesis, $\mathbb{P}(B_I) = \prod_{k\in I'}\mathbb{P}(A_k)$. Hence $\mathbb{P}\left(\bigcap_{k\in I} A_k\right) = \prod_{k\in I}\mathbb{P}(A_k)$. This completes the induction. We deduce that $A_1, A_2, \dots$ are independent.

Since $\sum_{n=1}^{\infty} A_n = \sum_{i=1}^{\infty}\frac{1}{n} = +\infty$, by the second Borel–Cantelli Lemma, A_n occurs infinitely often with probability 1.

Let $C_n = A_n\cap A_{n+1}$. Then C_n represents the event that a double record occurs at the $(n+1)$-st time. Then

$$\mathbb{P}(C_n) = \mathbb{P}(A_n\cap A_{n+1}) = \mathbb{P}(A_n)\mathbb{P}(A_{n+1}) = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \implies \sum_{n=1}^{\infty} C_n = 1 < \infty$$

By the first Borel–Cantelli Lemma, double records occur infinitely often with probability 0. □

Question 3. Example 3.9

On $([0,1], \mathscr{B}([0,1]), \mathrm{Leb})$ let $X_n(\omega) = \mathbf{1}_{\lfloor 2^n\omega\rfloor \text{ is odd}}$, $n\geqslant 1$, where 0 is even. Put differently, $X_n(\omega)$ is the n^{th} digit in the binary expansion of ω. Check that $(X_n)_{n\geqslant 1}$ is an i.i.d. sequence.

Proof. Let $A_n^k := \left[\frac{2k-1}{2^n}, \frac{2k}{2^n}\right)$ and $A_n := \bigcup_{k=1}^{2^{n-1}} A_n^k$. Then we have $X_n(\omega) = \mathbf{1}_{A_n}(\omega)$. Note that $m(A_n) = \frac{1}{2}$. Hence $\mathbb{P}(X_n = 0) = \mathbb{P}(X_n = 1) = \frac{1}{2}$. The random variables X_n are identically distributed. To show that they are independent, by Corollary 3.8 it suffices to show that

$$\mathbb{P}(X_1\leqslant x_1,\dots,X_n\leqslant x_n) = \mathbb{P}(X_1\leqslant x_1)\cdots\mathbb{P}(X_n\leqslant x_n)$$

for any $x_1,\dots,x_n\in\overline{\mathbb{R}}$ and $n\in\mathbb{Z}_+$. Since $\mathbb{P}(X_n\leqslant x) = m(A_n)$ for $x\in[0,1)$, it suffices to verify that

$$m\left(\bigcap_{k\in I} A_k\right) = \prod_{k\in I} m(A_k) = \frac{1}{2^{-|I|}}$$

for any finite subset $I\subseteq\mathbb{Z}_+$. We use induction on the cardinality of I. The base case $I=\{*\}$ is trivial.

Induction case: Let $n = \max I$, $I' = I \setminus \{n\}$, and $\ell = \max I'$. For a binary sequence

$$(a_i) = \sum_{i=1}^{\infty} 2^{-i} a_i \in [0,1], \quad a_i \in \{0,1\}$$

whether $(a_i) \in \bigcap_{k\in I'} A_k$ is uniquely determined by the values of a_i where $i \in I'$. In particular,

$$(a_i) \sim (b_i) \iff \forall i \in I' : a_i = b_i$$

defines an equivalence relation which partitions $\bigcap_{k\in I'} A_k$ into disjoint intervals, each of which has measure $2^{-\ell}$. Let B be one of the intervals. Then $B = \left[\frac{b}{2^\ell}, \frac{b+1}{2^\ell}\right) = \left[\frac{2^{n-\ell}b}{2^n}, \frac{2^{n-\ell}(b+1)}{2^n}\right)$ for some $b \in \{1,\ldots,2^\ell\}$. therefore

$$B \cap A_n = \bigcup_{k=2^{n-\ell-1}b+1}^{2^{n-\ell-1}(b+1)} A_n^k \implies m(B \cap A_n) = 2^{n-\ell-1}\frac{1}{2^n} = 2^{-\ell-1} = \frac{1}{2}m(B)$$

Hence $m\left(\bigcap_{k\in I'} A_k \cap A_n\right) = \frac{1}{2} m\left(\bigcap_{k\in I'} A_k\right)$. By induction hypothesis, $m\left(\bigcap_{k\in I'} A_k\right) = 2^{-|I'|}$. Hence $m\left(\bigcap_{k\in I} A_k\right) = 2^{-|I'|-1} = 2^{-|I|}$. This completes the induction. $\square$

Question 4

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. Let $(X_n)_{n\geqslant 1}$ be a sequence of independent identically distributed real-valued random variables such that $\mathbb{P}[X_n = 1] = p, \mathbb{P}[X_n = -1] = 1-p$ where $p \neq 1/2$. Let $S_0 := 0$ and

$$S_n := \sum_{k=1}^{n} X_k, \quad n \geqslant 1$$

Show that the probability of the event $\{S_n = 0 \text{ i.o. }\}$ is zero.

[You might, or might not, find Stirling's formula useful: $n! \sim \sqrt{2\pi n}\, n^n e^{-n}$.]

Proof. The question is discussed in Section 5.9.1 of Part A Probability.

Note that S_n is a Markov chain of period 2. So $\mathbb{P}(S_n = 0) = 0$ for odd n. For $S_{2m} = 0$, there are exactly m of $X_1, \ldots, X_{2m}$ equal to 1 and the other m equal to -1. Hence by Stirling's formula,

$$\mathbb{P}(S_{2m} = 0) = \binom{2m}{m} p^m (1-p)^m = \frac{(2m)!}{(m!)^2} p^m (1-p)^m \sim \frac{\sqrt{2\pi}(2m)^{2m+1/2} \mathrm{e}^{-2m}}{\left(\sqrt{2\pi}m^{m+1/2}\mathrm{e}^{-m}\right)^2} p^m (1-p)^m = \frac{1}{\sqrt{\pi m}}\left(4p(1-p)\right)^m$$

Since $p \neq 1/2$, $4p(1-p) < 1$. So $\mathbb{P}(S_{2m} = 0) \to 0$ exponentially as $m \to \infty$. Thus $\sum_{m=1}^{\infty} \mathbb{P}(S_{2m} = 0) < \infty$. By the first Borel-Cantelli Lemma, $\mathbb{P}\big(S_n = 0 \text{ infinitely often}\big) \leqslant \mathbb{P}\big(S_{2m} = 0 \text{ infinitely often}\big) = 0$. $\square$

Question 5

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. Let $(X_n)_{n\geqslant 1}$ be a sequence of independent identically distributed real-valued random variables such that

$$\mathbb{P}[X_n = 1] = \mathbb{P}[X_n = -1] = 1/2$$

Let $S_0 := 0$ and, for all $n \geqslant 1, S_n := \sum_{k=1}^{n} X_k$.

For $x \in \mathbb{Z}$ let

$$A_x := \{S_n = x \text{ for infinitely many } n\}, \quad B_- := \left\{\liminf_{n\to\infty} S_n = -\infty\right\}, \quad B_+ := \left\{\limsup_{n\to\infty} S_n = \infty\right\}$$

(a) Let $\mathscr{T}_n = \sigma(X_{n+1}, X_{n+2}\ldots)$ and $\mathscr{T} = \bigcap_{n=1}^{\infty}\mathscr{T}_n$. Why is $\mathscr{T}$ a σ-algebra? Show that $B_\pm$ are $\mathscr{T}$-measurable.

(b) Deduce, using Kolmogorov's 0-1 law, that $\mathbb{P}[B_-] \in \{0,1\}$ and $\mathbb{P}[B_+] \in \{0,1\}$. Prove that $\mathbb{P}[B_-] = \mathbb{P}[B_+]$.

(c) Using a Borel-Cantelli lemma show that, for all $k \geqslant 1$,

$$\limsup_{n\to\infty}(S_{n+k} - S_n) = k \text{ a.s.}$$

(d) Deduce from (c) that $\mathbb{P}\left[B_-^{\mathrm{c}} \cap B_+^{\mathrm{c}}\right] = 0$, and therefore that $\mathbb{P}[B_-] = \mathbb{P}[B_+] = 1$.

Conclude that, for all $x \in \mathbb{Z}, \mathbb{P}[A_x] = 1$.

Proof. (a) $\mathscr{T}$ is a σ-algebra because it is an intersection of the σ-algebras $\mathscr{T}_n$. Now we show that $B_+ \in \mathscr{T}$. The case of B_- is analogous. Fix $n \in \mathbb{Z}_+$, we show that $B_+ \in \mathscr{T}_n$.

$$\begin{aligned}B_+ &= \left\{\limsup_{n\to\infty} S_n = \infty\right\} = \bigcap_{m=1}^{\infty}\left\{\limsup_{n\to\infty} S_n \geqslant m\right\} = \bigcap_{m=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{\ell\geqslant k}\{S_\ell \geqslant m\} = \bigcap_{m=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{\ell\geqslant k}\{S_\ell - S_n \geqslant m\}\\ &= \bigcap_{m=1}^{\infty}\bigcap_{k=1}^{\infty}\bigcup_{\ell\geqslant k}\left\{\sum_{i=n+1}^{\ell} X_i \geqslant m\right\}\end{aligned}$$

By definition $X_{n+1}, X_{n+2}, \ldots$ are $\mathscr{T}_n$-measurable functions. Hence $\sum_{i=n+1}^{\ell} X_i$ is $\mathscr{T}_n$-measurable for any ℓ. Hence $\left\{\sum_{i=n+1}^{\ell} X_i \geqslant m\right\} \in \mathscr{T}_n$. Hence $B_+ \in \mathscr{T}_n$. Hence $B_+ \in \mathscr{T}$.

(b) The same reasoning in (a) shows that $\limsup_{n\to\infty} S_n$ and $\liminf_{n\to\infty} S_n$ are $\mathscr{T}$-measurable random variables. By Kolmogorov's 0-1 law, they are constant almost surely. Hence $\mathbb{P}(B_+), \mathbb{P}(B_-) \in \{0,1\}$. Let $T_n = -S_n = \sum_{k=1}^{n} -X_k$. Then T_n and S_n has the same distribution. Hence

$$\mathbb{P}(B_-) = \mathbb{P}\Big(\liminf_{n\to\infty} S_n = -\infty\Big) = \mathbb{P}\Big(\liminf_{n\to\infty} -T_n = -\infty\Big) = \mathbb{P}\Big(\limsup_{n\to\infty} T_n = \infty\Big) = \mathbb{P}\Big(\limsup_{n\to\infty} S_n = \infty\Big) = \mathbb{P}(B_+)$$

(c) We have

$$\begin{aligned}\left\{\limsup_{n\to\infty}(S_{n+k} - S_n) = k\right\} &= \bigcap_{m=1}^{\infty}\left\{S_{n+k} - S_n \geqslant k - \frac{1}{m} \text{ for infinitely many } n\right\}\\ &= \{S_{n+k} - S_n = k \text{ for infinitely many } n\}\\ &= \{X_{n+1} = \cdots = X_{n+k} = 1 \text{ for infinitely many } n\}\end{aligned}$$

Let $A_n = \{X_{n+1} = \cdots = X_{n+k} = 1\}$. Then $\mathbb{P}(A_n) = 2^{-k}$ and $A_0, A_k, A_{2k}, \ldots$ are independent. We have $\sum_{n=0}^{\infty}\mathbb{P}(A_{nk}) = \infty$. By the second Borel–Cantelli Lemma, $\mathbb{P}\big(X_{n+1} = \cdots = X_{n+k} = 1 \text{ for infinitely many } n\big) = 1$. Hence $\limsup_{n\to\infty}(S_{n+k} - S_n) = k$ almost surely.

(d) We have

$$B_+^{\mathrm{c}} \cap B_-^{\mathrm{c}} = \left\{\limsup_{n\to\infty} S_n - \liminf_{n\to\infty} S_n < \infty\right\} = \bigcup_{m=1}^{\infty}\{\forall\, n \in \mathbb{Z}_+ : |S_n| \leqslant m\}$$

But

$$\{\forall\, n \in \mathbb{Z}_+ : |S_n| \leqslant m\} \subseteq \{\forall\, n \in \mathbb{Z}_+ : |S_{n+3m} - S_n| \leqslant 2m\} \subseteq \left\{\limsup_{n\to\infty}|S_{n+3m} - S_n| \leqslant 2m\right\}$$

which has probability 0 by (c). Hence $\mathbb{P}\big(B_+^{\mathrm{c}} \cap B_-^{\mathrm{c}}\big) = 0$. Then $\mathbb{P}(B_+ \cup B_-) = 1$. Since $\mathbb{P}(B_+) = \mathbb{P}(B_-) \in \{0,1\}$, we deduce that $\mathbb{P}(B_+) = \mathbb{P}(B_-) = 1$.

Finally, for $x \in \mathbb{R}$ we have

$$\begin{aligned}A_x^{\mathrm{c}} &= \{S_n = x \text{ for finitely many } n\} = \{\exists N \in \mathbb{N}\ \forall\, n > N\ S_n > x\}\bigcup\{\exists N \in \mathbb{N}\ \forall\, n > N\ S_n < x\}\\ &= \left\{\liminf_{n\to\infty} S_n > x\right\}\bigcup\left\{\limsup_{n\to\infty} S_n < x\right\} \subseteq B_+^{\mathrm{c}} \cup B_-^{\mathrm{c}}\end{aligned}$$

Then $\mathbb{P}(A_x^{\mathrm{c}}) \leqslant \mathbb{P}(B_+^{\mathrm{c}} \cup B_-^{\mathrm{c}}) \leqslant \mathbb{P}(B_+^{\mathrm{c}}) + \mathbb{P}(B_-^{\mathrm{c}}) = 0$. We conclude that $\mathbb{P}(A_x) = 1$. □

Question 6

Let X_1, X_2 be independent exponentially distributed random variables with parameter one. Let $Y_1 = \min\{X_1, X_2\}$ and $Y_2 = \max\{X_1, X_2\} - Y_1$. Show that Y_1 and Y_2 are independent. Generalize this to the case of three independent exponential random variables with parameter one.

Proof. $X_1, X_2 \sim \mathrm{Exp}(1)$ i.i.d. Then the distribution functions are $F_{X_i}(x) = \mathbb{P}(X_i \leqslant x) = 1 - \mathrm{e}^{-x}$. The distribution function of Y_1:

$$F_{Y_1}(x) = 1 - \mathbb{P}(Y_1 > x) = 1 - \mathbb{P}(X_1 > x)\mathbb{P}(X_2 > x) = 1 - \mathrm{e}^{-2x}$$

The distribution function of $Z = \max\{X_1, X_2\}$:

$$F_Z(x) = \mathbb{P}(Z \leqslant x) = \mathbb{P}(X_1 \leqslant x)\mathbb{P}(X_2 \leqslant x) = (1 - \mathrm{e}^{-x})^2$$

Then the distribution function of Y_2:

$$F_{Y_2}(x) = \mathbb{P}(Z - Y_1 \leqslant x) = \int_0^\infty \mathbb{P}\big(Z - Y_1 \leqslant x \mid Y_1 = y\big) f_{Y_1}(y)\,\mathrm{d}y$$

Seems to me you assumed independence here? Try to evaluate joint integral.

$$= \int_0^\infty \mathbb{P}\big(Z \leqslant x + y\big) f_{Y_1}(y)\,\mathrm{d}y = \int_0^\infty \big(1 - \mathrm{e}^{-(x+y)}\big)^2 \cdot 2\,\mathrm{e}^{-2y}\,\mathrm{d}y$$
$$= 1 - \frac{4}{3}\mathrm{e}^{-x} + \frac{1}{2}\mathrm{e}^{-2x}$$

The joint distribution function of Y_1 and Y_2:

$$\begin{aligned} F_{Y_1,Y_2}(y_1, y_2) &= \mathbb{P}\big(Y_1 \leqslant y_1, Y_2 \leqslant y_2\big) = \mathbb{P}\big(\min\{X_1, X_2\} \leqslant y_1, \max\{X_1, X_2\} \leqslant y_2 - y_1\big) \\ &= \mathbb{P}\big(Z \leqslant y_2 - y_1\big) - \mathbb{P}\big(\min\{X_1, X_2\} > y_1, \max\{X_1, X_2\} \leqslant y_2 - y_1\big) \\ &= \mathbb{P}\big(Z \leqslant y_2 - y_1\big) - \mathbb{P}\big(y_1 < X_1 \leqslant y_2 - y_1\big)\mathbb{P}\big(y_1 < X_2 \leqslant y_2 - y_1\big) \\ &= \big(1 - \mathrm{e}^{-(y_2 - y_1)}\big)^2 - \big(\mathrm{e}^{-y_1} - \mathrm{e}^{-(y_2 - y_1)}\big)^2 \end{aligned}$$

□

Question 7

For $X \in \mathscr{L}^2(\Omega, \mathscr{F}, \mathbb{P})$ show that $c = \mathbb{E}[X]$ attains the infimum in

$$\inf_{c \in \mathbb{R}} \mathbb{E}\big[(X - c)^2\big]$$

Proof. By linearity of the expectation,

$$\begin{aligned} \mathbb{E}\big[(X - c)^2\big] &= \mathbb{E}\big[(X - \mathbb{E}[X] + \mathbb{E}[x] - c)^2\big] = \mathbb{E}\big[(X - \mathbb{E}[X])^2 + (\mathbb{E}[X] - c)^2 + 2(X - \mathbb{E}[X])(\mathbb{E}[X] - c)\big] \\ &= \mathbb{E}\big[(X - \mathbb{E}[X])^2\big] + (\mathbb{E}[X] - c)^2 + 2(\mathbb{E}[X] - \mathbb{E}[X])(\mathbb{E}[X] - c) = \mathbb{E}\big[(X - \mathbb{E}[X])^2\big] + (\mathbb{E}[X] - c)^2 \\ &\geqslant \mathbb{E}\big[(X - \mathbb{E}[X])^2\big] \end{aligned}$$

with equality holds if and only if $c = \mathbb{E}[X]$. □

Question 8

Let $(X_n)_{n \geqslant 2}$ be a sequence of independent random variables such that

$$\mathbb{P}[X_n = n] = \mathbb{P}[X_n = -n] = \frac{1}{2n\log n}; \quad \mathbb{P}[X_n = 0] = 1 - \frac{1}{n\log n}$$

Let $S_n = X_2 + \ldots + X_n$. Prove that $\frac{S_n}{n} \to 0$ in probability, but not almost surely.

[Hint: show that the variance of S_n is bounded from above by $n^2/\log(n)$ and deduce the convergence in probability; use a Borel-Cantelli lemma to consider the almost sure convergence].

Proof. From the given distribution we can compute that $\mathbb{E}[X_n] = 0$ and $\operatorname{Var}(X_n) = \dfrac{n}{\log n}$. By independence of $X_1, \dots, X_n$ we have

$$\operatorname{Var}\left(\frac{S_n}{n}\right) = \frac{1}{n^2}\operatorname{Var}(S_n) = \frac{1}{n^2}\sum_{k=2}^{n}\frac{k}{\log k} < \frac{1}{n^2}\cdot n\frac{n}{\log n} = \frac{1}{\log n}$$

By Chebyshev's inequality, for any $\varepsilon > 0$,

$$\mathbb{P}\left(\left|\frac{S_n}{n} - \mathbb{E}\left[\frac{S_n}{n}\right]\right| > \varepsilon\right) \leqslant \frac{\operatorname{Var}(S_n/n)}{\varepsilon^2} \implies \mathbb{P}\left(\left|\frac{S_n}{n}\right| > \varepsilon\right) \leqslant \frac{1}{\varepsilon^2 \log n} \to 0$$

as $n \to \infty$. Hence $S_n/n \to 0$ in probability.

Consider the events $A_n := \{|S_n| \geqslant n\}$. Then

$$\begin{aligned}
\mathbb{P}(A_n) &= \mathbb{P}(|S_n| \geqslant n) \\
&\geqslant \mathbb{P}(S_{n-1} > 0 \wedge X_n = n) + \mathbb{P}(S_{n-1} \leqslant 0 \wedge X_n = -n) \\
&= \mathbb{P}(S_{n-1} > 0)\mathbb{P}(X_n = n) + \mathbb{P}(S_{n-1} \leqslant 0)\mathbb{P}(X_n = -n) \\
&= \mathbb{P}(S_{n-1} > 0)\frac{1}{2n\log n} + \mathbb{P}(S_{n-1} \leqslant 0)\frac{1}{2n\log n} \\
&= \frac{1}{2n\log n}
\end{aligned}$$

Hence $\sum_{n=2}^{\infty}\mathbb{P}(A_n) \geqslant \sum_{n=2}^{\infty}\frac{1}{2n\log n} = +\infty$. *If we can prove that the sequence of events $\{A_n\}$ are independent*, then by the Second Borel–Cantelli Lemma, we have $\mathbb{P}(A_n \text{ infinitely often}) = 1$. But

Consider the event |X_n|>=n.

$$\{A_n \text{ infinitely often}\} = \left\{\left|\frac{S_n}{n}\right| \geqslant 1 \text{ for infinitely many } n\right\} \subseteq \left\{\lim_{n\to\infty}\frac{S_n}{n} \to 0\right\}^{c}$$

Hence

$$\mathbb{P}\left(\lim_{n\to\infty}\frac{S_n}{n} \to 0\right) = 0$$

The convergence is not almost sure. $\square$

Question 9. Lemma 5.15

Let X, Y be two positive random variables such that

$$x\mathbb{P}(X \geqslant x) \leqslant \mathbb{E}\left[Y\mathbf{1}_{\{X\geqslant x\}}\right], \quad \forall x > 0$$

Show that for, $p > 1$ and $q = p/(p-1)$, we have $\|X\|_p \leqslant q\|Y\|_p$.

Proof. First we prove the inequality in the case $X \in \mathscr{L}^p$. The case $Y \notin \mathscr{L}^p$ is trivial so we assume that $Y in \mathscr{L}^p$.

Let $f(x,\omega) = \mathbf{1}_{\{(x,\omega):0\leqslant x\leqslant X(\omega)\}}$ be a map from $[0,\infty) \times \Omega$ to $\mathbb{R}$. Then we have

$$\int_0^\infty f(x,\omega)\,\mathrm{d}x = X(\omega), \qquad \int_\Omega f(x,\omega)\,\mathrm{d}\mathbb{P}(\omega) = \mathbb{P}(X \geqslant x)$$

The assumption can be expressed as

$$x\int_\Omega f\,\mathrm{d}\mathbb{P} \leqslant \int_\Omega Yf\,\mathrm{d}\mathbb{P} \implies x^p\int_\Omega f\,\mathrm{d}\mathbb{P} \leqslant x^{p-1}\int_\Omega Yf\,\mathrm{d}\mathbb{P}$$

Integrating on $[0,\infty)$ and using Fubini's Theorem:

$$\int_0^\infty x^p\int_\Omega f\,\mathrm{d}\mathbb{P}\mathrm{d}x \leqslant \int_0^\infty\int_\Omega x^{p-1}Yf\,\mathrm{d}\mathbb{P}\mathrm{d}x \implies \int_\Omega\left(\int_0^\infty x^pf\,\mathrm{d}x\right)\mathrm{d}\mathbb{P} \leqslant \int_\Omega\left(\int_0^\infty x^{p-1}f\,\mathrm{d}x\right)Y\,\mathrm{d}\mathbb{P}$$

$$\Longrightarrow \frac{1}{p}\int_\Omega X^p \,\mathrm{d}\mathbb{P} \leqslant \frac{1}{p-1}\int_\Omega X^{p-1} Y \,\mathrm{d}\mathbb{P}$$
$$\Longrightarrow \mathbb{E}[X^p] \leqslant q\mathbb{E}[X^{p-1}Y]$$

Since $q = p/(1-p)$, $1 = p^{-1} + q^{-1}$. By Hölder's inequality:

$$\mathbb{E}[X^{p-1}Y] \leqslant \|X^{p-1}\|_q \|Y\|_p = \|X^{p/q}\|_q \|Y\|_p = \|Y\|_p \left(\int_\Omega X^p \,\mathrm{d}\mathbb{P}\right)^{1/q} = \|X\|_p^{p/q} \|Y\|_p$$

Hence

$$\|X\|_p^p = \mathbb{E}[X^p] \leqslant q\mathbb{E}[X^{p-1}Y] \leqslant q\|X\|_p^{p/q}\|Y\|_p \Longrightarrow \|X\|_p \leqslant q\|Y\|_p$$

Next we assume that $X \notin \mathscr{L}^p$. Consider $X_n := \max\{X, n\}$ for $n \in \mathbb{N}$. Then X_n is bounded on Ω. Since Ω has finite measure, we have $X_n \in \mathscr{L}^p$. Moreover, X_n satifies the inequality:

$$x\mathbb{P}(X_n \geqslant x) \leqslant x\mathbb{P}(X \geqslant x) \leqslant \mathbb{E}\left[Y\mathbf{1}_{\{X\geqslant x\}}\right]$$

By the discussion above, we deduce that $\|X_n\|_p \leqslant q\|Y\|_p$. Since $X_n \uparrow X$, by Monotone Convergence Theorem we have

$$\|X\|_p = \lim_{n\to\infty} \|X_n\|_p \leqslant q\|Y\|_p$$ □

Question 10

Let $(\Omega, \mathscr{F}, \mu)$ be a probability space and $Y \in \mathscr{L}^1$. Show that $\{X \in \mathscr{L}^0 : |X| \leqslant |Y|\}$ is a uniformly integrable family of random variables. Suppose now that $X_1, X_2, \ldots \in \mathscr{L}^1$ and $\mathbb{E}[|X_n - Y|] \to 0$ as $n \to \infty$. Show that $\{X_n : n \geqslant 1\}$ is uniformly integrable.

Proof. Let $\{X_n\}$ be a sequence in $A = \{X \in \mathscr{L}^0 : |X| \leqslant |Y|\}$. By Reverse Fatou's Lemma,

$$\limsup_{n\to\infty} \int_\Omega X_n \,\mathrm{d}\mathbb{P} \leqslant \int_\Omega \limsup_{n\to\infty} X_n \,\mathrm{d}\mathbb{P} \leqslant \int_\Omega \sup_{X\in A} X \,\mathrm{d}\mathbb{P} \Longrightarrow \sup_{X\in A}\int_\Omega X \,\mathrm{d}\mathbb{P} \leqslant \int_\Omega \sup_{X\in A} X \,\mathrm{d}\mathbb{P}$$

Let $f_n = \sup_{X\in A} X\mathbf{1}_{\{|X|>n\}}$. Then $f_n \to 0$ pointwise (if we allow random variables to take infinite values on a null set, then this is a almost everywhere convergence). Since $f_n \leqslant |Y| \in \mathscr{L}^1$, by Dominated Convergence Theorem,

$$\lim_{n\to\infty}\int_\Omega f_n \,\mathrm{d}\mathbb{P} = \int_\Omega \lim_{n\to\infty} f_n \,\mathrm{d}\mathbb{P} = 0$$

Hence

$$\lim_{n\to\infty} \sup_{X\in A}\int_\Omega X\mathbf{1}_{\{|X|>n\}} \,\mathrm{d}\mathbb{P} \leqslant \lim_{n\to\infty}\int_\Omega f_n \,\mathrm{d}\mathbb{P} = 0$$

We deduce that A is uniformly integrable.

Next, suppose that $X_n \to Y$ in $\mathscr{L}^1$. We have

$$\int_{\{|X_n|>K\}} |X_n| \,\mathrm{d}\mathbb{P} \leqslant \int_{\{|X_n|>K\}} |Y| \,\mathrm{d}\mathbb{P} + \int_{\{|X_n|>K\}} |X_n - Y| \,\mathrm{d}\mathbb{P}$$

for each $n \in \mathbb{Z}_+$. Fix $\varepsilon > 0$. Since $|Y| \in \mathscr{L}^1$, by Dominated Convergence Theorem,

$$\int_E |Y| \,\mathrm{d}\mathbb{P} \to 0 \text{ as } \mathbb{P}(E) \to 0$$

There exists $\delta > 0$ such that

$$\mathbb{P}(E) < \delta \Longrightarrow \int_E |Y| \,\mathrm{d}\mathbb{P} < \frac{\varepsilon}{2}$$

Since $\mathbb{E}[|X_n - Y|] \to 0$, by Lemma 4.14 $\mathbb{E}[|X_n|] \to \mathbb{E}[|Y|]$. In particular $\sup_{n\in\mathbb{Z}_+} \mathbb{E}[|X_n|] < M$ for some $M > 0$. By Markow's inequality,

$$\mathbb{P}(|X_n| > K) \leqslant \frac{\mathbb{E}[X_n]}{K} \leqslant \frac{M}{K}$$

Let $K_0 = M/\delta$. So whenever $K > K_0$,

$$\mathbb{P}(|X_n| > K) \leqslant \frac{M}{K} < \delta \implies \sup_{n\in\mathbb{Z}_+} \int_{\{|X_n|>K\}} |Y| \, d\mathbb{P} < \frac{\varepsilon}{2}$$

$\mathbb{E}[|X_n - Y|] \to 0$ implies that there exists $N \in \mathbb{N}$ such that $\mathbb{E}[|X_n - Y|] < \varepsilon/2$ whenever $n > N$. For $n \in \{1, ..., N\}$, there exists $K_n > 0$ such that

$$K > K_n \implies \int_{\{|X_n|>K\}} |X_n - Y| \, d\mathbb{P} < \frac{\varepsilon}{2}$$

Hence for $K > \max\{K_0, K_1, ..., K_n\}$, for $n > N$,

$$\int_{\{|X_n|>K\}} |X_n - Y| \, d\mathbb{P} \leqslant \mathbb{E}[|X_n - Y|] < \frac{\varepsilon}{2}$$

and for $n \leqslant N$,

$$\int_{\{|X_n|>K\}} |X_n - Y| \, d\mathbb{P} < \frac{\varepsilon}{2}$$

Hence

$$\sup_{n\in\mathbb{Z}_+} \int_{\{|X_n|>K\}} |X_n - Y| \, d\mathbb{P} < \frac{\varepsilon}{2}$$

Hence

$$\sup_{n\in\mathbb{Z}_+} \int_{\{|X_n|>K\}} |X_n| \, d\mathbb{P} \leqslant \sup_{n\in\mathbb{Z}_+} \int_{\{|X_n|>K\}} |Y| \, d\mathbb{P} + \sup_{n\in\mathbb{Z}_+} \int_{\{|X_n|>K\}} |X_n - Y| \, d\mathbb{P} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

We conclude that $\{X_n\}$ is uniformly integrable. $\square$