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Problem Sheet 2

B4.3: Distribution Theory

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Question 1

Let $f, g \in C^1(\mathbb{R})$ and define

$$u(x) = \begin{cases} f(x) & \text{if } x < 0 \\ g(x) & \text{if } x \geqslant 0 \end{cases}$$

Explain why $u \in \mathscr{D}'(\mathbb{R})$ and calculate the distributional derivative u'. What can you say about the function

$$v(x) = \begin{cases} f(x) & \text{if } x < 0 \\ a & \text{if } x = 0 \\ g(x) & \text{if } x > 0 \end{cases}$$

where $a \in \mathbb{R}$ is a constant that is different from both $f(0)$ and $g(0)$?

Proof. Since $f, g \in C^1(\mathbb{R})$, u is continuous almost everywhere. In particular it is locally Lebesgue integrable. Hence u defines a regular distribution $T_u \in \mathscr{D}'(\mathbb{R})$.

For $\varphi \in \mathscr{D}(\mathbb{R})$, the distributional derivative

$$\langle u', \varphi \rangle := \langle u, -\varphi' \rangle = \int_{\mathbb{R}} -u(x)\varphi'(x)\,\mathrm{d}x = \int_{-\infty}^{0} -f(x)\varphi'(x)\,\mathrm{d}x + \int_{0}^{+\infty} -g(x)\varphi'(x)\,\mathrm{d}x$$

By integration by parts,

$$\langle u', \varphi \rangle = -f(0)\varphi(0) + \int_{-\infty}^{0} f'(x)\varphi(x)\,\mathrm{d}x + g(0)\varphi(0) + \int_{0}^{+\infty} g'(x)\varphi(x)\,\mathrm{d}x = \big(g(0) - f(0)\big)\varphi(0) + \int_{\mathbb{R}} u'(x)\varphi(x)\,\mathrm{d}x$$

write as $f'\chi_{(-\infty,0)} + g'\chi_{(0,\infty)}$ as u' is what you're defining.

The u' in the integrand is the almost everywhere derivative of function u.

It is clear that u and v define the same distribution in $\mathscr{D}'(\Omega)$ because $u = v$ almost everywhere. $\square$

Question 2

(a) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is piecewise continuous and $k \in \mathbb{R}$, then the function $u(x,t) = f(x - kt), (x,t) \in \mathbb{R}^2$, is locally integrable on $\mathbb{R}^2$. Conclude that it defines a distribution and show that it satisfies the one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = k^2 \frac{\partial^2 u}{\partial x^2}$$

in the sense of distributions on $\mathbb{R}^2$.

(b) Prove that $u(x,y) = \log(x^2 + y^2)$ is locally integrable on $\mathbb{R}^2$, and that we have

$$\Delta u = 4\pi\delta_0$$

in the sense of distributions on $\mathbb{R}^2$, where δ_0 is the Dirac delta function on $\mathbb{R}^2$ concentrated at the origin.

Proof. (a) I assume that piecewise continuous functions are locally bounded. Under this notion, f is locally bounded and hence locally integrable. Then $x \mapsto f(x - kt)$ and $t \mapsto f(x - kt)$ are locally integrable. By Tonelli's Theorem $u(x,t) = f(x - kt)$ is locally integrable on $\mathbb{R}^2$.

This doesn't follow from what you've said before! Instead try showing that $\forall K \subset\subset \mathbb{R}^2$ $\exists \tilde{K} \subset \mathbb{R}$ s.t $\{x - kt : (x,t) \in K\} \subset \tilde{K}$.

By Example 3.6 u defines a regular distribution T_u given by

$$\langle u, \varphi \rangle := \iint_{\mathbb{R}^2} u(x,t)\varphi(x,t)\,\mathrm{d}x\mathrm{d}t = \iint_{\mathbb{R}^2} f(x - kt)\varphi(x,t)\,\mathrm{d}x\mathrm{d}t$$

Then the distributional partial derivatives

$$\langle (\partial_t^2 - k^2\partial_x^2)\,u, \varphi \rangle = \langle u, (\partial_t^2 - k^2\partial_x^2)\,\varphi \rangle = \iint_{\mathbb{R}^2} f(x - kt)\left(\frac{\partial^2 \varphi}{\partial t^2} - k^2 \frac{\partial^2 \varphi}{\partial x^2}\right)\mathrm{d}x\mathrm{d}t$$

Change of variables $(x,t) \mapsto (v,w)$, where $v = x - kt$ and $w = x + kt$. The Jacobian $\left|\dfrac{\partial(x,t)}{\partial(v,w)}\right| = \dfrac{1}{2k}$. Let $\tilde{\varphi}(v,w) = \varphi(x,t)$.

Then

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \widetilde{\varphi}}{\partial v^2} + \frac{\partial^2 \widetilde{\varphi}}{\partial w^2} + 2\frac{\partial^2 \widetilde{\varphi}}{\partial v \partial w}, \qquad \frac{\partial^2 \varphi}{\partial t^2} = k^2\left(\frac{\partial^2 \widetilde{\varphi}}{\partial v^2} + \frac{\partial^2 \widetilde{\varphi}}{\partial w^2} - 2\frac{\partial^2 \widetilde{\varphi}}{\partial v \partial w}\right)$$

Hence

$$\iint_{\mathbb{R}^2} f(x-kt)\left(\frac{\partial^2 \varphi}{\partial t^2} - k^2\frac{\partial^2 \varphi}{\partial x^2}\right) \mathrm{d}x\mathrm{d}t = \iint_{\mathbb{R}^2} 2kf(v)\frac{\partial^2 \widetilde{\varphi}}{\partial v \partial w}\,\mathrm{d}v\mathrm{d}w = \int_{\mathbb{R}} 2kf(v)\left(\int_{\mathbb{R}} \frac{\partial^2 \widetilde{\varphi}}{\partial v \partial w}\,\mathrm{d}w\right)\mathrm{d}v$$

But

$$\int_{\mathbb{R}} \frac{\partial^2 \widetilde{\varphi}}{\partial v \partial w}\,\mathrm{d}w = \left.\frac{\partial \widetilde{\varphi}}{\partial v}\right|_{-\infty}^{+\infty} = 0$$

since φ is compactly supported. Hence $\left(\partial_t^2 - k^2\partial_x^2\right) u$ is the zero distribution. We deduce that

$$\frac{\partial^2 u}{\partial t^2} = k^2\frac{\partial^2 u}{\partial x^2}$$

in the sence of distributions.

(b) For any compact subset $K \subseteq \mathbb{R}^2$, we take a closed disk $\overline{B(0,R)}$ such that $K \subseteq \overline{B(0,R)}$. Then

$$\iint_K u(x,y)\,\mathrm{d}x\mathrm{d}y \leqslant \iint_{\overline{B(0,R)}} \log(x^2+y^2)\,\mathrm{d}x\mathrm{d}y = 2\pi\int_0^R r\log r^2\,\mathrm{d}r = 2\pi R^2\log R$$

is finite. We deduce that u is locally integrable on $\mathbb{R}^2$.

For $\varphi \in \mathscr{D}(\mathbb{R}^2)$,

$$\langle \nabla^2 u, \varphi\rangle := \langle u, \nabla^2\varphi\rangle = \iint_{\mathbb{R}^2} u\nabla^2\varphi\,\mathrm{d}x\mathrm{d}y$$

To prove that $\nabla^2 u = 4\pi\delta_0$, we need to prove that

$$\iint_{\mathbb{R}^2} u\nabla^2\varphi\,\mathrm{d}x\mathrm{d}y = \varphi(0)$$

Since φ is compactly supported, there exists $R > 0$ such that $\operatorname{supp}\varphi \subseteq \overline{B(0,R)}$. Let $A = \{\boldsymbol{x} \in \mathbb{R}^2 : r < \|x\| < R\}$.

$$I(r) = \iint_A u\nabla^2\varphi\,\mathrm{d}x\mathrm{d}y$$

need more reasoning here.

Using Gauss-Green Formula in $\mathbb{R}^2$ and that $\boldsymbol{\nabla}\varphi = 0$ on $\|\boldsymbol{x}\| = R$,

$$I(r) = -\iint_A \boldsymbol{\nabla} u\cdot\boldsymbol{\nabla}\varphi\,\mathrm{d}x\mathrm{d}y + \oint_{\partial B(0,r)} u\boldsymbol{\nabla}\varphi\cdot\boldsymbol{n}\,\mathrm{d}s = -\oint_{\partial B(0,r)} \varphi\boldsymbol{\nabla}u\cdot\boldsymbol{n}\,\mathrm{d}s + \oint_{\partial B(0,r)} u\boldsymbol{\nabla}\varphi\cdot\boldsymbol{n}\,\mathrm{d}s$$

On $\|\boldsymbol{x}\| = r$, $u(\boldsymbol{x}) = 2\log r$. Hence

$$-\oint_{\partial B(0,r)} \varphi\boldsymbol{\nabla}u\cdot\boldsymbol{n}\,\mathrm{d}s = \oint_{\partial B(0,r)} \varphi(\boldsymbol{x})\cdot\frac{2}{r}\,\mathrm{d}s = 2\oint_{\partial S^1} \varphi(r\boldsymbol{x})\,\mathrm{d}s \to 4\pi\varphi(0)$$

as $r \to 0$. For the other integral,

$$\left|\oint_{\partial B(0,r)} u\boldsymbol{\nabla}\varphi\cdot\boldsymbol{n}\,\mathrm{d}s\right| \leqslant 2\pi r\cdot 2r\log r \sup_{\partial B(0,r)}\|\boldsymbol{\nabla}\varphi\| \to 0$$

as $r \to 0$. Hence

$$I = \lim_{r\to 0} I(r) = 4\pi\varphi(0)$$

We deduce that $\nabla^2 u = 4\pi\delta_0$. ☐

Question 3

Let $a > 0$. For each $\varphi \in \mathscr{D}(\mathbb{R})$ we let

$$\langle T_a, \varphi\rangle = \left(\int_{-\infty}^{-a} + \int_a^{\infty}\right)\frac{\varphi(x)}{|x|}\,\mathrm{d}x + \int_{-a}^{a}\frac{\varphi(x)-\varphi(0)}{|x|}\,\mathrm{d}x$$

Show that T_a hereby is well-defined and that it is a distribution on $\mathbb{R}$. Now assume that $\varphi \in \mathscr{D}(\mathbb{R})$ satisfies $\varphi(0) = 0$. Show that then

$$\langle T_a, \varphi \rangle = \int_{-\infty}^{\infty} \frac{\varphi(x)}{|x|} \mathrm{d}x$$

What distribution is $T_a - T_b$ for $0 < b < a$?

Proof. It is clear from the definition that T_a is a linear functional. First we need to verify that $\langle T_a, \varphi \rangle$ is finite for all $\varphi \in \mathscr{D}(\mathbb{R})$.

Note that both $\varphi(x) - \varphi(0)$ and x tends to 0 as $x \searrow 0$. By l'Hôptial's rule,

$$\lim_{x \searrow 0} \frac{\varphi(x) - \varphi(0)}{|x|} = \lim_{x \searrow 0} \varphi'(x) = \varphi'(0)$$

It is similar for $x \nearrow 0$. Hence the second integrand in the definition of T_a is continuous in $(-a, a)$. So the second integral is finite. Since φ is compactly supported, the first integral is also finite. We deduce that T_a is well-defined.

Next we show that T_a is a distribution. Consider $\{\varphi_n\} \subseteq \mathscr{D}(\mathbb{R})$ and $\varphi \in \mathscr{D}(\mathbb{R})$, such that $\varphi_n \to \varphi$ in $\mathscr{D}$. Let $R > a$ such that $\operatorname{supp} \varphi_n, \operatorname{supp} \varphi \subseteq [-R, R]$. Then

$$\langle T_a, \varphi_n \rangle - \langle T_a, \varphi \rangle = \left(\int_{-R}^{-a} + \int_a^R \right) \frac{\varphi_n(x) - \varphi(x)}{|x|} \mathrm{d}x + \int_{-a}^{a} \frac{\varphi_n(x) - \varphi(x) - (\varphi_n(0) - \varphi(0))}{|x|} \mathrm{d}x \to 0$$

as $n \to \infty$, by the uniform convergence $\varphi_n \to \varphi$ on $[-R, R]$. Hence T_a is a distribution. — You need a lot more reasoning - it's not okay to stick $\to 0$ on the end of the above & call that the proof!

For $\varphi \in \mathscr{D}(\mathbb{R})$ with $\varphi(0) = 0$,

$$\langle T_a, \varphi \rangle = \left(\int_{-\infty}^{-a} + \int_a^{\infty} \right) \frac{\varphi(x)}{|x|} \mathrm{d}x + \int_{-a}^{a} \frac{\varphi(x)}{|x|} \mathrm{d}x = \int_{-\infty}^{+\infty} \frac{\varphi(x)}{|x|} \mathrm{d}x$$

The distribution $T_a - T_b$ is given by

$$\langle T_a - T_b, \varphi \rangle = \left(\int_{-a}^{-b} + \int_b^a \right) \frac{\varphi(x) - \varphi(0)}{|x|} \mathrm{d}x - \left(\int_{-a}^{-b} + \int_b^a \right) \frac{\varphi(x)}{|x|} \mathrm{d}x = -\left(\int_{-a}^{-b} + \int_b^a \right) \frac{\varphi(0)}{|x|} \mathrm{d}x \qquad \square$$

Take out $\varphi(0)$ & integrate!

Question 4

(a) Let $\alpha \in (-n, \infty)$ and $u_\alpha(x) = |x|^\alpha$ for $x \in \mathbb{R}^n \setminus \{0\}$. Show that u_α is a regular distribution on $\mathbb{R}^n$.

(*Hint: Use polar coordinates.*)

(b) For each $r > 0$ we define the r-dilation of a test function $\varphi \in \mathscr{D}(\mathbb{R}^n)$ by the rule

$$(d_r \varphi)(x) = \varphi(rx), \quad x \in \mathbb{R}^n$$

Extend the r-dilation to distributions $u \in \mathscr{D}'(\mathbb{R}^n)$.

(c) Show that for the distribution u_α defined in (a) we have $d_r u_\alpha = r^\alpha u_\alpha$ for all $r > 0$. We express this by saying that u_α is *homogeneous of degree* α.

(d) Show that the Dirac delta function δ_0 concentrated at the origin $0 \in \mathbb{R}^n$ is homogeneous of degree $-n$.

(e) Let $u \in \mathscr{D}'(\mathbb{R}^n)$ be homogeneous of degree $\beta \in \mathbb{R}$: $d_r u = r^\beta u$ for all $r > 0$. Show that for each $j \in \{1, \dots, n\}$ the distribution $x_j u$ is homogeneous of degree $\beta + 1$ and that the distribution $D_j u$ is homogeneous of degree $\beta - 1$. Finally show that

$$\sum_{j=1}^{n} x_j D_j u = \beta u$$

This PDE is known as Euler's relation for β-homogeneous distributions.

(f) Show that a distribution $u \in \mathscr{D}'(\mathbb{R}^n)$ that satisfies (1) must be homogeneous of degree β.

Proof. (a) For any compact subset $K \subseteq \mathbb{R}^n$, we take a closed ball $\overline{B(0,R)}$ such that $K \subseteq \overline{B(0,R)}$. Then

$$\int_K \|\boldsymbol{x}\|^\alpha \, \mathrm{d}x^n \leqslant \int_{\overline{B(0,R)}} \|\boldsymbol{x}\|^\alpha \, \mathrm{d}x^n = \int_{S^{n-1}} \int_0^R r^\alpha r^{n-1} \mathrm{d}r \mathrm{d}\Omega = \int_{S^{n-1}} \mathrm{d}\Omega \frac{1}{\alpha+n} r^{\alpha+n} \Big|_0^R = \int_{S^{n-1}} \mathrm{d}\Omega \frac{1}{\alpha+n} R^{\alpha+n} < \infty$$

where we used $\alpha + n > 0$. Hence u_α is locally integrable. It defines a regular distribution in $\mathscr{D}'(\mathbb{R})$ via

$$\langle u_\alpha, \varphi \rangle = \int_{\mathbb{R}^n} \|\boldsymbol{x}\|^\alpha \varphi(\boldsymbol{x})\, \mathrm{d}x^n$$

(b) The r-dilation of a distribution $u \in \mathscr{D}'(\mathbb{R}^n)$ is $d_r u \in \mathscr{D}'(\mathbb{R}^n)$ such that for $\varphi \in \mathscr{D}(\mathbb{R})$,

$$\langle d_r u, \varphi \rangle = r^{-n} \langle u, d_{1/r}\varphi \rangle = r^{-n} \langle u, \varphi(x/r) \rangle$$

We should check that this definition is consistent on the regular distributions. For $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$\langle d_r u, \varphi \rangle = \int_{\mathbb{R}^n} u(r\boldsymbol{x})\varphi(\boldsymbol{x})\, \mathrm{d}x^n = r^{-n} \int_{\mathbb{R}^n} u(\boldsymbol{t})\varphi(\boldsymbol{t}/r)\, \mathrm{d}t^n = r^{-n} \langle u, d_{1/r}\varphi \rangle$$

(c) $d_r u_\alpha = u_\alpha(r\boldsymbol{x}) = \|r\boldsymbol{x}\|^\alpha = r^\alpha \|\boldsymbol{x}\|^\alpha = r^\alpha u_\alpha$. Then u_α is homogeneous of degree α.

Is this linear & cts ??

No! You need to prove using the defn from (b)

(d) For $\varphi \in \mathscr{D}(\mathbb{R})$,

$$\langle d_r \delta_0, \varphi \rangle = r^{-n} \langle \delta_0, d_{1/r}\varphi \rangle = r^{-n}\varphi(0) = r^{-n} \langle \delta_0, \varphi \rangle$$

Hence $d_r \delta_0 = r^{-n}\delta_0$. We say that δ_0 is homogeneous of degree $-n$.

(e) For $\varphi \in \mathscr{D}(\mathbb{R})$,

$$\begin{aligned}
\langle d_r x_j u, \varphi \rangle &= r^{-n} \langle x_j u, d_{1/r}\varphi \rangle \\
&= r^{-n} \langle u, x_j d_{1/r}\varphi \rangle \\
&= r^{-n+1} \langle u, d_{1/r}(x_j \varphi(\boldsymbol{x})) \rangle \\
&= r \langle d_r u, x_j \varphi \rangle \\
&= r^{\beta+1} \langle u, x_j \varphi \rangle \\
&= r^{\beta+1} \langle x_j u, \varphi \rangle
\end{aligned}$$

Hence $d_r x_j u = r^{\beta+1} x_j u$ and $x_j u$ is homogeneous of degree $\beta + 1$.

$$\begin{aligned}
\langle d_r D_j u, \varphi \rangle &= r^{-n} \langle D_j u, d_{1/r}\varphi \rangle \\
&= r^{-n} \langle u, -D_j \varphi(\boldsymbol{x}/r) \rangle \\
&= r^{-n-1} \langle u, -d_{1/r} D_j \varphi \rangle \\
&= r^{-1} \langle d_r u, -D_j \varphi \rangle \\
&= r^{\beta-1} \langle u, -D_j \varphi \rangle \\
&= r^{\beta-1} \langle D_j u, \varphi \rangle
\end{aligned}$$

Hence $d_r D_j u = r^{\beta-1} D_j u$ and $D_j u$ is homogeneous of degree $\beta - 1$.

The proof of the Euler's relation for distributions is essentially the same as for functions. Starting from $d_r u = r^\beta u$, we can differentiate both sides by r. Since the derivatives are transmitted to the test function φ, we still have the chain rule for distributions:

$$\frac{\partial}{\partial r} d_r u = \frac{\partial}{\partial r}(r^\beta u) \implies \sum_{j=1}^n x_j d_r D_j u = \beta r^{\beta-1} u$$

Since $D_j u$ is homogeneous of degree $\beta - 1$, we have

$$\sum_{j=1}^n x_j r^{\beta-1} D_j u = \beta r^{\beta-1} u \implies \sum_{j=1}^n x_j D_j u = \beta u$$

show more details here please!!!

(f) Let $v = r^{-\beta} d_r u - u \in \mathscr{D}'(\mathbb{R}^{n+1})$. Then

$$\frac{\partial v}{\partial r} = -\beta r^{-\beta-1} d_r u + r^{-\beta} \sum_{j=1}^n x_j d_r D_j u = r^{-\beta-1} d_r \left(\sum_{j=1}^n x_j D_j u - \beta u \right) = 0$$

Hence $\left\langle v, -\dfrac{\partial\psi}{\partial r}\right\rangle = 0$ for any $\psi(\boldsymbol{x},r) \in \mathscr{D}(\mathbb{R}^{n+1})$. But for every $\psi(\boldsymbol{x},r) \in \mathscr{D}'(\mathbb{R}^{n+1})$,

$$\psi(\boldsymbol{x},r) = \frac{\partial}{\partial r}\int_{-\infty}^{r} \psi(\boldsymbol{x},r')\,\mathrm{d}r'$$

Hence $\langle v,\psi\rangle = 0$ for all $\psi(\boldsymbol{x},r) \in \mathscr{D}'(\mathbb{R}^{n+1})$. $v = 0$. We deduce that $d_r u = r^{\beta} u$. So u is homogeneous of degree β. □

Question 5

Show that δ_a, the Dirac delta function concentrated at $a \in \mathbb{R}$, satisfies the equation

$$(x-a)u = 0$$

Find the general solution $u \in \mathscr{D}'(\mathbb{R})$ to (2).

(Hint: See Corollary 1.10 in the Lecture Notes.)

Proof. Let $u = \delta_a$. Then for any $\varphi \in \mathscr{D}(\mathbb{R})$,

$$\langle (x-a)\delta_a, \varphi\rangle = \langle \delta_a, (x-a)\varphi(x)\rangle = (a-a)\varphi(a) = 0$$

Hence $(x-a)\delta_a = 0$. ✓

no cpt support → not well defined

Let $u \in \mathscr{D}'(\mathbb{R})$ such that $(x-a)u = 0$. Note that for any $\varphi \in \mathscr{D}(\mathbb{R})$, there exists $\psi \in \mathscr{D}(\mathbb{R})$ such that $\varphi(x) = \varphi(a) + (x-a)\psi(x)$ and $\varphi'(a) = \psi(a)$. Then

$$\langle u,\varphi\rangle = \langle u,\varphi(a)\rangle + \langle (x-a)u,\psi\rangle = \langle u,\varphi(a)\rangle = \varphi(a)\langle u,1\rangle = c\langle \delta_a,\varphi\rangle$$

for some constant $c \in \mathbb{R}$. Hence the general solution is given by $u = c\delta_a$. □

right idea though!

Question 6. Distribution defined by principal value integral

Define for each $\varphi \in \mathscr{D}(\mathbb{R})$

$$\left\langle \mathrm{pv}\left(\frac{1}{x}\right),\varphi\right\rangle = \lim_{a\to 0^+}\left(\int_{-\infty}^{-a} + \int_a^{\infty}\right)\frac{\varphi(x)}{x}\,\mathrm{d}x$$

(a) Show that hereby $\mathrm{pv}\left(\frac{1}{x}\right) \in \mathscr{D}'(\mathbb{R})$ and that it is homogeneous of order -1 (see Problem 4). Check that

$$\frac{\mathrm{d}}{\mathrm{d}x}\log|x| = \mathrm{pv}\left(\frac{1}{x}\right)$$

(b) Show that $u = \mathrm{pv}\left(\frac{1}{x}\right)$ solves the equation

$$xu = 1$$

in the sense of $\mathscr{D}'(\mathbb{R})$. What is the general solution $u \in \mathscr{D}'(\mathbb{R})$ to (3)?

not good notation - leave as limit until proven existence.

Proof. (a) $\mathrm{pv}(1/x)$ is given by

$$\left\langle \mathrm{pv}\left(\frac{1}{x}\right),\varphi\right\rangle = \lim_{a\to 0^+}\left(\int_{-\infty}^{-a} + \int_a^{\infty}\right)\frac{\varphi(x)}{x}\,\mathrm{d}x = \lim_{a\to 0^+}\int_a^{\infty}\frac{\varphi(x)-\varphi(-x)}{x}\,\mathrm{d}x = \int_0^{\infty}\frac{\varphi(x)-\varphi(-x)}{x}\,\mathrm{d}x$$

By l'Hôpital's rule,

$$\lim_{x\to 0^+}\frac{\varphi(x)-\varphi(-x)}{x} = \lim_{x\to 0^+}\left(\varphi'(x)+\varphi'(-x)\right) = 2\varphi'(0)$$

Hence the integrand is bounded near $x = 0$. In particular $\mathrm{pv}(1/x)$ is well-defined. It is clear that $\mathrm{pv}(1/x)$ is a linear functional. Following the same argument in Question 3 we deduce that $\mathrm{pv}(1/x) \in \mathscr{D}'(\mathbb{R})$.

You still should write out the argument.

For $r > 0$,

$$\left\langle d_r\,\mathrm{pv}\left(\frac{1}{x}\right),\varphi\right\rangle = r^{-1}\left\langle \mathrm{pv}\left(\frac{1}{x}\right), d_{1/r}\varphi\right\rangle = r^{-1}\int_0^{\infty}\frac{\varphi(rx)-\varphi(-rx)}{x}\,\mathrm{d}x = r^{-1}\int_0^{\infty}\frac{\varphi(t)-\varphi(-t)}{t}\,\mathrm{d}t = \left\langle r^{-1}\,\mathrm{pv}\left(\frac{1}{x}\right),\varphi\right\rangle$$

↑ NO. You need to keep in the form of limits & work from there.

This isn't justified as written – you need to write it w. the limit from the prev. part of the qu.

Hence pv$(1/x)$ is homogeneous of degree -1.

For $\varphi \in \mathscr{D}(\mathbb{R})$,

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}x}\log|x|, \varphi \right\rangle = \langle \log|x|, -\varphi' \rangle = -\int_{\mathbb{R}} \log|x|\varphi'(x)\,\mathrm{d}x$$
$$= -\int_{-\infty}^{0} \log(-x)\varphi'(x)\,\mathrm{d}x - \int_{0}^{\infty} \log(x)\varphi'(x)\,\mathrm{d}x$$
$$= -\int_{0}^{\infty} \log x\left(\varphi'(x)+\varphi'(-x)\right)\mathrm{d}x$$
$$= -\log x\left(\varphi(x)-\varphi(-x)\right)_0^{\infty} + \int_0^{\infty} \frac{1}{x}\left(\varphi(x)-\varphi(-x)\right)\mathrm{d}x$$

For $x \to \infty$, $\log x\left(\varphi(x)-\varphi(-x)\right) = 0$ because φ is compactly supported. For $x \to 0^+$, by l'Hôptial's rule,

$$\lim_{x\to 0} \log x\left(\varphi(x)-\varphi(-x)\right) = \lim_{x\to 0} -x\log^2 x\left(\varphi'(x)+\varphi'(-x)\right) = 0$$

Hence

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}x}\log|x|, \varphi \right\rangle = \int_0^{\infty} \frac{1}{x}\left(\varphi(x)-\varphi(-x)\right)\mathrm{d}x = \left\langle \mathrm{pv}\left(\frac{1}{x}\right), \varphi \right\rangle$$

We deduce that $\frac{\mathrm{d}}{\mathrm{d}x}\log|x| = \mathrm{pv}\left(\frac{1}{x}\right)$.

(b) For $\varphi \in \mathscr{D}(\mathbb{R})$,

again – need limits.

$$\left\langle x\,\mathrm{pv}\left(\frac{1}{x}\right), \varphi \right\rangle = \int_0^{\infty} (\varphi(x)-\varphi(-x))\,\mathrm{d}x = \int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}x = \langle 1, \varphi \rangle$$

Hence pv$(1/x)$ is a solution to $xu = 1$.

Let $u \in \mathscr{D}'(\mathbb{R})$ such that $xu = 1$. Note that for any $\varphi \in \mathscr{D}(\mathbb{R})$, there exists $\psi \in \mathscr{D}(\mathbb{R})$ such that $\varphi(x) = \varphi(0) + x\psi(x)$ and $\varphi'(0) = \psi(0)$. Then

not well defined

$$\langle u, \varphi \rangle = \langle u, \varphi(0) \rangle + \langle xu, \psi \rangle = \varphi(0)\langle u, 1 \rangle + \left\langle 1, \frac{\varphi(x)-\varphi(0)}{x} \right\rangle = \varphi(0)\langle u, 1 \rangle + \int_{\mathbb{R}} \frac{\varphi(x)-\varphi(0)}{x}\,\mathrm{d}x$$
$$= \varphi(0)\langle u, 1 \rangle + \int_0^{\infty} \frac{\varphi(x)-\varphi(-x)}{x}\,\mathrm{d}x = \left\langle \langle u, 1 \rangle \delta_0 + \mathrm{pv}\left(\frac{1}{x}\right), \varphi \right\rangle$$

Hence the general solution is given by $u = c\delta_0 + \mathrm{pv}\left(\frac{1}{x}\right)$ for some constant $c \in \mathbb{R}$. ☐

Better proof: let v solve $xv = 1$.

$\Rightarrow x\left(v - \mathrm{pv}\left(\frac{1}{x}\right)\right) = 0$

$\Rightarrow v - \mathrm{pv}\left(\frac{1}{x}\right) = c\delta_0$ by qu. 5

$\Rightarrow$ general sol^n is $\mathrm{pv}\,\frac{1}{x} + c\delta_0$.

Hence pv$(1/x)$ is homogeneous of degree -1.

For $\varphi \in \mathscr{D}(\mathbb{R})$,

$$\begin{aligned}
\left\langle \frac{\mathrm{d}}{\mathrm{d}x}\log|x|, \varphi \right\rangle &= \langle \log|x|, -\varphi' \rangle = -\int_{\mathbb{R}} \log|x|\varphi'(x)\,\mathrm{d}x \\
&= -\int_{-\infty}^{0} \log(-x)\varphi'(x)\,\mathrm{d}x - \int_{0}^{\infty} \log(x)\varphi'(x)\,\mathrm{d}x \\
&= -\int_{0}^{\infty} \log x \left(\varphi'(x) + \varphi'(-x)\right) \mathrm{d}x \\
&= -\log x \left(\varphi(x) - \varphi(-x)\right)_0^\infty + \int_0^\infty \frac{1}{x}\left(\varphi(x) - \varphi(-x)\right) \mathrm{d}x
\end{aligned}$$

For $x \to \infty$, $\log x \left(\varphi(x) - \varphi(-x)\right) = 0$ because φ is compactly supported. For $x \to 0^+$, by l'Hôptial's rule,

$$\lim_{x\to 0} \log x \left(\varphi(x) - \varphi(-x)\right) = \lim_{x \to 0} -x\log^2 x \left(\varphi'(x) + \varphi'(-x)\right) = 0$$

Hence

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}x}\log|x|, \varphi \right\rangle = \int_0^\infty \frac{1}{x}\left(\varphi(x) - \varphi(-x)\right) \mathrm{d}x = \left\langle \mathrm{pv}\left(\frac{1}{x}\right), \varphi \right\rangle$$

We deduce that $\frac{\mathrm{d}}{\mathrm{d}x}\log|x| = \mathrm{pv}\left(\frac{1}{x}\right)$.

(b) For $\varphi \in \mathscr{D}(\mathbb{R})$,

$$\left\langle x\,\mathrm{pv}\left(\frac{1}{x}\right), \varphi \right\rangle = \int_0^\infty (\varphi(x) - \varphi(-x))\,\mathrm{d}x = \int_{-\infty}^\infty \varphi(x)\,\mathrm{d}x = \langle 1, \varphi \rangle$$

Hence pv$(1/x)$ is a solution to $xu = 1$.

Let $u \in \mathscr{D}'(\mathbb{R})$ such that $xu = 1$. Note that for any $\varphi \in \mathscr{D}(\mathbb{R})$, there exists $\psi \in \mathscr{D}(\mathbb{R})$ such that $\varphi(x) = \varphi(0) + x\psi(x)$ and $\varphi'(0) = \psi(0)$. Then

$$\begin{aligned}
\langle u, \varphi \rangle = \langle u, \varphi(0) \rangle + \langle xu, \psi \rangle = \varphi(0)\langle u, 1\rangle + \left\langle 1, \frac{\varphi(x) - \varphi(0)}{x} \right\rangle = \varphi(0)\langle u, 1\rangle + \int_{\mathbb{R}} \frac{\varphi(x) - \varphi(0)}{x}\,\mathrm{d}x \\
= \varphi(0)\langle u, 1\rangle + \int_0^\infty \frac{\varphi(x) - \varphi(-x)}{x}\,\mathrm{d}x = \left\langle \langle u, 1\rangle \delta_0 + \mathrm{pv}\left(\frac{1}{x}\right), \varphi \right\rangle
\end{aligned}$$

Hence the general solution is given by $u = c\delta_0 + \mathrm{pv}\left(\frac{1}{x}\right)$ for some constant $c \in \mathbb{R}$. □