Peize Liu
St. Peter's College
University of Oxford

Problem Sheet 4

B4.2: Functional Analysis II

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Question 1

(a) Use Theorem 4.6.1 to prove the localisation property of Fourier series: if two (continuous) 2π-periodic functions f and g are equal in an open interval containing 0, then their Fourier series either both converge at 0 or both diverge at 0.

(b) In the lecture, we prove that there is a continuous function whose Fourier series diverges at 0. Use (a) to construct a continuous function whose Fourier series diverges at 0 and $\pi/2$.

Proof. Let $S_N(f)$ be the partial sum of the first N terms of the Fourier series of f. That is,

$$S_N f(x) = \sum_{n=-N}^{N} \mathrm{e}^{\mathrm{i}nx} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\,\mathrm{e}^{-\mathrm{i}nt}\,\mathrm{d}t = \int_{-\pi}^{\pi} f(t) D_N(x-t)\,\mathrm{d}t =: (f * D_N)(x)$$

where

$$D_N(x) := \frac{1}{2\pi} \frac{\sin\left(N+\frac{1}{2}\right)x}{\sin\frac{x}{2}}$$

is the Dirichlet kernel.

(a) Since f and g are continuous and 2π-periodic, they are bounded and hence are in $\mathrm{L}^1(-\pi,\pi)$. By assumption, $f-g=0$ in the interval (a,b) where $a<0<b$. In particular $f-g$ is α-Hölder continuous for any $\alpha \in \mathbb{R}$. By Theorem 4.6.1, we have

$$\lim_{N\to\infty} S_N(f-g)(0) = \lim_{N\to\infty} S_N f(0) - \lim_{N\to\infty} S_N g(0) = 0$$

Hence $\lim_{N\to\infty} S_N f(0)$ and $\lim_{N\to\infty} S_N g(0)$ either both converge or both diverge.

(b) First we need to construct a function whose Fourier series diverges at 0. The proof given in the lecture is not very constructive. We use the classical example due to Fejér.[1] For $x \in [0,\pi]$,

$$f(x) = \sum_{p=1}^{\infty} \frac{1}{p^2} \sin\left(\left(2^{p^3}+1\right)\frac{x}{2}\right)$$

Then we extend the domain to $\mathbb{R}$ such that f is a 2π-periodic even function.

We claim that f is continuous with Fourier series divergent at 0. Then we consider

$$g(x) := \begin{cases} f(x), & x \in \left[-\frac{\pi}{8},\frac{\pi}{8}\right] \\ \text{linear}, & x \in \left[-\frac{\pi}{4},-\frac{\pi}{8}\right] \cup \left[\frac{\pi}{8},\frac{\pi}{4}\right] \\ 0 & \text{otherwise} \end{cases}$$

and

$$h(x) := g(x) + g\left(x-\frac{\pi}{2}\right)$$

For $|x| < \frac{\pi}{8}$, $h(x) = f(x)$. By the localisation property, the Fourier series of h diverges at 0. For $\left|x-\frac{\pi}{2}\right| < \frac{\pi}{8}$, $h(x) = f\left(x-\frac{\pi}{2}\right)$. Similarly, the the Fourier series of h diverges at $\pi/2$.

Next we need to verify the properties of f. By Weierstrass M-test, the series that defines f is uniformly convergent on $\mathbb{R}$. Hence f is continuous on $\mathbb{R}$. The Fourier series of f is given by

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

The coefficients are given by

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_0^{\pi} \sum_{p=1}^{\infty} \frac{1}{p^2} \sin\left(\left(2^{p^3}+1\right)\frac{x}{2}\right)\cos nx\,\mathrm{d}x \end{aligned}$$

[1] `https://www.mathcounterexamples.net/continuous-function-with-divergent-fourier-series/`

$$= \frac{2}{\pi}\sum_{p=1}^{\infty}\frac{1}{p^2}\int_0^{\pi}\sin\left(\left(2^{p^3}+1\right)\frac{x}{2}\right)\cos nx\,\mathrm{d}x \qquad \text{(by uniform convergence)}$$
$$= \frac{2}{\pi}\sum_{p=1}^{\infty}\frac{1}{p^2}\alpha\left(2^{p^3-1},n\right)$$

where $\alpha(m,n) := \int_0^{\pi}\sin\left(\frac{2m+1}{2}x\right)\cos nx\,\mathrm{d}x = \frac{1}{2}\left(\frac{1}{m+n+\frac{1}{2}}+\frac{1}{m-n+\frac{1}{2}}\right)$.

To show that the Fourier series of f diverges at 0, it suffices to show that

$$\sum_{n=0}^{\infty}a_n = \sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\frac{1}{p^2}\alpha\left(2^{p^3-1},n\right) = \sum_{p=1}^{\infty}\frac{1}{p^2}\sum_{n=0}^{\infty}\alpha\left(2^{p^3-1},n\right) = \infty$$

For $m, N \geqslant 1$,

$$\sum_{n=0}^{N}\alpha(m,n) = \frac{1}{2}\sum_{n=0}^{N}\left(\frac{1}{m+n+\frac{1}{2}}+\frac{1}{m-n+\frac{1}{2}}\right) = \frac{1}{2}\left(\frac{1}{m+\frac{1}{2}}+\sum_{i=m-N}^{m+N}\frac{1}{i+\frac{1}{2}}\right) \geqslant 0$$

When $m = N$,

$$\sum_{n=0}^{m}\alpha(m,n) = \frac{1}{2}\left(\frac{1}{m+\frac{1}{2}}+\sum_{i=0}^{2m}\frac{1}{i+\frac{1}{2}}\right) \sim \frac{1}{2}\ln m$$

for large m. Therefore

$$\sum_{p=1}^{\infty}\frac{1}{p^2}\sum_{n=0}^{\infty}\alpha\left(2^{p^3-1},n\right) \geqslant \frac{1}{p^2}\sum_{n=0}^{2^{p^3-1}}\alpha\left(2^{p^3-1},n\right) \sim \frac{1}{2p^2}\ln\left(2^{p^3-1}\right) = \frac{p^3-1}{2p^2}\ln 2 \to \infty$$

as $p \to \infty$. This completes the argument. □

Question 2

Consider the system $\left\{e_n = \frac{1}{\sqrt{2\pi}}e^{inx}\right\}_{n\in\mathbb{Z}}$ as a subset of $X = L^1(-\pi,\pi)$.

(a) Show that $\|e_n\| = \sqrt{2\pi}$ for all n and $\|e_n - e_m\| = \frac{8}{\sqrt{2\pi}}$ for all $n \neq m$.

(b) Show that $\left\{e_n = \frac{1}{\sqrt{2\pi}}e^{inx}\right\}_{n\in\mathbb{Z}}$ is a basis of $L^1(-\pi,\pi)$, i.e. the closed linear span of $\left\{e_n = \frac{1}{\sqrt{2\pi}}e^{inx}\right\}_{n\in\mathbb{Z}}$ is $L^1(-\pi,\pi)$.

Proof. (a) For $n \in \mathbb{Z}$,

$$\|e_n\| = \int_{-\pi}^{\pi}|e_n(x)|\,\mathrm{d}x = \int_{-\pi}^{\pi}\frac{1}{\sqrt{2\pi}}\,\mathrm{d}x = \sqrt{2\pi}$$

For $n \neq m$,

$$\begin{aligned}\|e_n - e_m\| &= \int_{-\pi}^{\pi}|e_n(x)-e_m(x)|\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\sqrt{\left(\mathrm{e}^{inx}-\mathrm{e}^{imx}\right)\left(\mathrm{e}^{-inx}-\mathrm{e}^{-imx}\right)}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi}\sqrt{1-\cos(m-n)x}\,\mathrm{d}x \\ &= \sqrt{\frac{2}{\pi}}\int_{-\pi}^{\pi}\left|\sin\frac{m-n}{2}x\right|\mathrm{d}x \\ &= \frac{2\sqrt{2}}{\sqrt{\pi}|m-n|}|m-n|\int_0^{\pi}\sin u\;du \\ &= \frac{8}{\sqrt{2\pi}}\end{aligned}$$

(b) We follow the outline in the lectures:

$$\mathrm{span}\{e_n\}_{n\in\mathbb{Z}} \xrightarrow{\text{dense in}} \mathrm{C}_{\mathrm{per}}(\mathbb{R}) \xrightarrow{\text{dense in}} \mathrm{L}^1(-\pi,\pi)$$

The linear span of $\{e_n\}_{n\in\mathbb{Z}}$ clearly contains constant functions and is closed under pointwise multiplication (which is because $\{e_n\}_{n\in\mathbb{Z}}$ is closed under pointwise multiplication: $e_n(x)e_m(x) = e_{n+m}(x)$). Hence $\operatorname{span}\{e_n\}_{n\in\mathbb{Z}}$ is a subalgebra of $C_{\mathrm{per}}(\mathbb{R})$. In addition, $e_1(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}x}$ is injective on $\mathbb{R}/2\pi\mathbb{Z}$, and hence separates points. Furthermore $\mathbb{R}/2\pi\mathbb{Z}$ is compact. By the subalgebra form of the Stone-Weierstrass Theorem, $\operatorname{span}\{e_n\}_{n\in\mathbb{Z}}$ is dense in $C_{\mathrm{per}}(\mathbb{R})$.

From Functional Analysis I we know that $C_{\mathrm{per}}(\mathbb{R})$ is dense in $L^1(-\pi,\pi)$. Therefore $\operatorname{span}\{e_n\}_{n\in\mathbb{Z}}$ is dense in $L^1(-\pi,\pi)$. □

Question 3

Let X be the closed subspace of $C[-\pi,\pi]$ consisting of all continuous (on $[-\pi,\pi]$) functions f such that $f(-\pi) = f(\pi)$. For $n \in \mathbb{Z}$, define $e_n \in X$ by $e_n(t) = \frac{1}{\sqrt{2\pi}} e^{int}$ and let

$$\widehat{f}(n) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-int} dt$$

for $f \in X$. Let $\{\alpha_n\}_{n\in\mathbb{Z}}$ be a sequence in $\mathbb{C}$, and assume that for each $f \in X$ there exists a unique element $g \in X$ such that $\widehat{g}(n) = \alpha_n \widehat{f}(n)$ for all $n \in \mathbb{Z}$. Let $Tf = g$.

(a) Show that T is linear and has closed graph. Deduce that $T \in \mathscr{B}(X)$.

(b) Show that $Te_n = \alpha_n e_n$ for all $n \in \mathbb{Z}$ and that the sequence $\{\alpha_n\}_{n\in\mathbb{Z}}$ is bounded.

(c) Show that there exists a bounded linear functional φ on X such that $\varphi(e_n) = \alpha_n$ for all $n \in \mathbb{Z}$.

Proof. (a) For $f_1, f_2 \in X$, let $g_1 = T(f_1)$, $g_2 = T(f_2)$. It is clear from definition that

$$\widehat{af_1 + bf_2}(n) = a\widehat{f_1}(n) + b\widehat{f_2}(n), \qquad \widehat{ag_1 + bg_2}(n) = a\widehat{g_1}(n) + b\widehat{g_2}(n)$$

where $a, b \in \mathbb{C}$. Hence

$$\widehat{ag_1 + bg_2}(n) = \alpha_n \widehat{af_1 + bf_2}(n)$$

and $T(af_1 + bf_2) = ag_1 + bg_2 = aT(f_1) + bT(f_2)$. We deduce that T is linear.

Suppose that $\{f_k\}_{k\in\mathbb{N}} \subseteq X$ such that $f_k \to f$ and $T(f_k) \to g$ uniformly as $k \to \infty$. Then

$$\lim_{k\to\infty} \widehat{f_k}(n) = \lim_{k\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f_k(t)\,\mathrm{e}^{-\mathrm{i}nt}\,\mathrm{d}t = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t)\,\mathrm{e}^{-\mathrm{i}nt}\,\mathrm{d}t = \widehat{f}(n)$$

Hence $g = T(f)$. We deduce that $\Gamma(T) \subseteq X \times X$ is closed. By the closed graph theorem, $T \in \mathscr{B}(X)$.

(b) By L^2 orthonormality,

$$\widehat{e_n}(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(n-m)t}\,\mathrm{d}t = \delta_{mn}$$

Hence $\alpha_m \widehat{e_n}(m) = \alpha_n \widehat{e_n}(m)$ for all $m \in \mathbb{Z}$. We deduce that $T(e_n) = \alpha_n e_n$.

$$|\alpha_n|\|e_n\| = \|\alpha_n e_n\| = \|T(e_n)\| \leqslant \|T\|\|e_n\|$$

Hence the sequence $\{\alpha_n\}_{n\in\mathbb{Z}}$ is bounded by $\|T\|$.

(c) We define a linear map $\psi : \operatorname{span}\{e_n :\ n \in \mathbb{Z}\} \to \mathbb{C}$ by $\psi(e_n) = \alpha_n$. We claim that ψ is bounded.

(*It is hard to bound* $|\psi(f)|$ *by* $\|f\|_\infty$......)

From Question 2 we know that $\operatorname{span}\{e_n :\ n \in \mathbb{Z}\}$ is dense in X. Therefore ψ has a unique extension $\varphi \in X^*$, which satisfies $\varphi(e_n) = \alpha_n$ for all $n \in \mathbb{Z}$. □

Question 4

Consider the right shift operator on sequences $R(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ Show that as an operator on ℓ^2, R satisfies $\sigma_p(R) = \emptyset, \sigma_r(R) = \{\lambda : |\lambda| < 1\}$ and $\sigma_c(R) = \{\lambda : |\lambda| = 1\}$.

[To put thing in perspective, compare Question 7 of Sheet 4 of B4.1 from MT: If we consider T as an operator on ℓ^∞, then $\sigma_p(R) = \emptyset$, $\sigma_r(R) = \{\lambda : |\lambda| \leqslant 1\}$ and $\sigma_c(R) = \emptyset$.]

Proof. Let $L : \ell^2 \to \ell^2$ be the left shift operator: $(x_1, x_2, \ldots) \mapsto (x_2, x_3, \ldots)$. We claim that $L = R^*$. Indeed, for $x, y \in \ell^2$,

$$\langle Rx, y \rangle = \sum_{n=1}^{\infty} (Rx)_n y_n = \sum_{n=2}^{\infty} x_{n-1} y_n = \sum_{n=1}^{\infty} x_n y_{n+1} = \sum_{n=1}^{\infty} x_n (Ly)_n = \langle x, Ly \rangle$$

We claim that $\sigma_p(R) = \varnothing$ and $\sigma_p(L) = B_{\mathbb{C}}(0,1)$.

Suppose that $x \in \ker(L - \lambda\,\mathrm{id})$. Then $0 = x_{n+1} - \lambda x_n$ and hence $x_n = \lambda^{n-1} x_1$ for all $n \in \mathbb{Z}_+$. If $x \neq 0$, we have

$$\|x\|_2^2 = \sum_{n=1}^{\infty} |x_n|^2 = |x_1|^2 \sum_{n=1}^{\infty} |\lambda|^{2n-2}$$

So $x \in \ell^2$ if and only if $|\lambda| < 1$. We deduce that $\sigma_p(L) = B_{\mathbb{C}}(0,1)$.

Suppose that $x \in \ker(R - \lambda\,\mathrm{id})$. Then $x_1 = 0$ and $x_n - \lambda x_{n+1} = 0$ for all $n \in \mathbb{Z}_+$. Hence $x = 0$. We deduce that $\ker(R - \lambda\,\mathrm{id}) = \{0\}$ for all $\lambda \in \mathbb{C}$ and thus $\sigma_p(R) = \varnothing$.

We have $(R - \lambda\,\mathrm{id})^* = (L - \overline{\lambda}\,\mathrm{id})$. From Question 1.(a) of Sheet 2 we know that

$$\ker(L - \overline{\lambda}\,\mathrm{id}) = \mathrm{im}(R - \lambda\,\mathrm{id})^{\perp}$$

Hence

$$\lambda \in \sigma_r(R) \iff \overline{\mathrm{im}(R - \lambda\,\mathrm{id})} \neq \ell^2 \iff \ker(L - \overline{\lambda}\,\mathrm{id}) \neq \{0\} \iff \overline{\lambda} \in \sigma_p(L)$$

We deduce that $\sigma_r(R) = \sigma_p(L) = B_{\mathbb{C}}(0,1)$.

Next, we note that R is isometric, as for $x \in \ell^2$,

$$\|Rx\|_2^2 = \sum_{n=2}^{\infty} |x_{n-1}|^2 = \sum_{n=1}^{\infty} |x_n|^2 = \|x\|_2^2$$

So we have $\sigma(R) \subseteq \overline{B_{\mathbb{C}}(0,1)}$. Since $\sigma(R)$ is compact and $B_{\mathbb{C}}(0,1) \subseteq \sigma(R)$, we have $\sigma(R) = \overline{B_{\mathbb{C}}(0,1)}$. Therefore

$$\sigma_c(R) = \sigma(R) \setminus (\sigma_p(R) \cup \sigma_r(R)) = S^1$$

In conclusion, we have $\sigma_p(R) = \varnothing$, $\sigma_c(R) = S^1$, and $\sigma_r(R) = B_{\mathbb{C}}(0,1)$. □

Question 5

Let X be a complex Hilbert space and $A \in \mathscr{B}(X)$ be normal (i.e. $A^* A = AA^*$)

(a) Show that

$$\mathrm{rad}(\sigma(A)) = \|A\|$$

Deduce that if P is a polynomial, then

$$\|P(A)\| = \sup_{\lambda \in \sigma(A)} |P(\lambda)|$$

(b) Let P be a Laurent polynomial, i.e. $P(z) = \sum_k a_k z^k$ where the summation range is finite but may contains positive as well as negative powers. Show that if A is unitary, then

$$\|P(A)\| = \sup_{\lambda\in\sigma(A)} |P(\lambda)|$$

Proof. Let S_X denotes the unit sphere of X.

(a) We first note that $\|A^{*n}A^n\| = \|A^n\|^2$, because

$$\left\|A^{*n}A^n\right\| = \sup_{x\in S_X} \langle A^{*n}A^n x, x\rangle = \sup_{x\in S_X} \langle A^n x, A^n x\rangle = \sup_{x\in S_X} \left\|A^n x\right\|^2 = \left\|A^n\right\|^2$$

Since A is normal, $A^*A = AA^*$. Furthermore A^*A is self-adjoint. Therefore

$$\left\|A^n\right\|^2 = \left\|A^{*n}A^n\right\| = \left\|(A^*A)^n\right\| = \left\|A^*A\right\|^n$$

By the Gelfand's formula,

$$\operatorname{rad}\sigma(A) = \lim_{n\to\infty} \left\|A^n\right\|^{1/n} = \lim_{n\to\infty} \left\|A^*A\right\|^{1/2} = \left\|A^*A\right\|^{1/2} = \|A\|$$

(b) Since P is a Laurent polynomial, there exists $n \in \mathbb{N}$ such that $P(z) = z^{-n}Q(z)$ where $Q \in \mathbb{C}[x]$. Hence $P(A) = A^{-n}Q(A)$. Since A is unitary, $\|A\| = \left\|A^{-1}\right\| = 1$. And we have

$$\|Q(A)\| \leqslant \left\|A^n\right\| \|P(A)\| = \|P(A)\| \leqslant \left\|A^{-n}\right\| \|Q(A)\| = \|Q(A)\|$$

So $\|P(A)\| = \|Q(A)\|$. By Theorem 8.6 of *B4.1 Functional Analysis I*, we have

$$\sigma(Q(A)) = Q(\sigma(A))$$

Therefore

$$\|P(A)\| = \|Q(A)\| = \operatorname{rad}\sigma(Q(A)) = \operatorname{rad}Q(\sigma(A)) = \sup_{\lambda\in Q(\sigma(A))} |\lambda| = \sup_{\lambda\in\sigma(A)} |Q(\lambda)|$$

Since $\sigma(A) \subseteq S^1$, we have

$$\|P(A)\| = \sup_{\lambda\in\sigma(A)} |Q(\lambda)| = \sup_{\lambda\in\sigma(A)} |\lambda|^{-n}|Q(\lambda)| = \sup_{\lambda\in\sigma(A)} |P(\lambda)|$$

□

Question 6

Let X be a complex Hilbert space and S and T be two self-adjoint bounded linear operators on X.

(a) Let $\lambda \notin \sigma(T)$. Use the fact that $\sigma\left((T-\lambda I)^{-1}\right) = (\sigma(T)-\lambda)^{-1}$ (a form of spectral mapping theorem) and Gelfand's formula to show that

$$\left\|(T-\lambda I)^{-1}\right\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(T))}$$

Deduce that $I + (T-\lambda I)^{-1}(S-T)$ is invertible if

$$\|S-T\| < \operatorname{dist}(\lambda, \sigma(T))$$

Hence, show under this latter assumption that $\lambda \notin \sigma(S)$.

(b) Use (a) to show that

$$\|S-T\| \geqslant \operatorname{dist}_H(\sigma(S), \sigma(T))$$

where the Hausdorff distance $\operatorname{dist}_H(A,B)$ between two closed subsets A and B of $\mathbb{C}$ is defined by

$$\operatorname{dist}_H(A,B) = \max\left(\sup_{a\in A}\min_{b\in B}|a-b|, \sup_{b\in B}\min_{a\in A}|a-b|\right)$$

Proof. (a) Since T is self-adjoint and $T-\lambda\,\mathrm{id}$ is invertible, then $(T-\lambda\,\mathrm{id})^{-1}$ is also self-adjoint. In particular,

$$\operatorname{rad}\sigma\big((T-\lambda\,\mathrm{id})^{-1}\big)=\big\|(T-\lambda\,\mathrm{id})^{-1}\big\|$$

On the other hand, we have

$$\operatorname{rad}\sigma\big((T-\lambda\,\mathrm{id})^{-1}\big)=\sup_{\eta\in\sigma\left((T-\lambda\,\mathrm{id})^{-1}\right)}|\eta|=\sup_{\eta\in(\sigma(T)-\lambda)^{-1}}|\eta|=\sup_{\eta\in\sigma(T)}\frac{1}{|\eta-\lambda|}=\frac{1}{\inf_{\eta\in\sigma(T)}|\eta-\lambda|}=\frac{1}{\operatorname{dist}(\lambda,\sigma(T))}$$

Hence

$$\big\|(T-\lambda\,\mathrm{id})^{-1}\big\|=\frac{1}{\operatorname{dist}(\lambda,\sigma(T))}$$

By convergence of Neumann series, $I+(T-\lambda\,\mathrm{id})^{-1}(S-T)$ is invertible if $\big\|(T-\lambda\,\mathrm{id})^{-1}(S-T)\big\|\leqslant 1$. From the result above, it suffices to have $\|S-T\|\leqslant\operatorname{dist}(\lambda,\sigma(T))$.

Finally,

$$S-\lambda\,\mathrm{id}=(T-\lambda\,\mathrm{id})+(S-T)=(T-\lambda\,\mathrm{id})\big(\mathrm{id}+(T-\lambda\,\mathrm{id})^{-1}(S-T)\big)$$

By the assumption, $S-\lambda\,\mathrm{id}$ is invertible. Hence $\lambda\notin\sigma(S)$.

(b) The contrapositive of the result of (a) is that $\lambda\in\sigma(S)$ implies that $\|S-T\|\geqslant\operatorname{dist}(\lambda,\sigma(T))$. Hence

$$\|S-T\|\geqslant\sup_{\lambda\in\sigma(S)}\operatorname{dist}(\lambda,\sigma(T))=\sup_{\lambda\in\sigma(S)}\inf_{\eta\in\sigma(T)}|\lambda-\eta|=\sup_{\lambda\in\sigma(S)}\min_{\eta\in\sigma(T)}|\lambda-\eta|$$

The last equality follows from that $\sigma(T)$ is compact. Note that S and T are symmetric in the above inequality. Therefore we have

$$\|S-T\|\geqslant\sup_{\lambda\in\sigma(T)}\min_{\eta\in\sigma(S)}|\lambda-\eta|$$

and hence $\|S-T\|\geqslant\operatorname{dist}_H(\sigma(S),\sigma(T))$. □