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Problem Sheet 1

ASO: Multivariable Calculus

May 6, 2020

Question 1

The function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} \dfrac{|xy|^\alpha}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

where $\alpha > 0$. Find the values of α for which f is

(a) continuous at $(0,0)$;

(b) differentiable at $(0,0)$.

Proof. (a) We change to polar coordinates:

$$f(r,\theta) = \begin{cases} r^{2\alpha-2}\,|\cos^\alpha\theta\sin^\alpha\theta| & \text{for } r > 0 \\ 0 & \text{for } r = 0 \end{cases}$$

If f is continuous at the origin, then $f \to 0$ as $r \to 0$ for any $\theta \in \mathbb{R}$. Hence we have $2\alpha - 2 > 0 \implies \alpha > 1$.

(b) First we compute the partial derivatives. For $|xy| > 0$, we have

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{y^\alpha}{(x^2+y^2)^2}(\alpha x^{\alpha-1}(x^2+y^2) - 2x\cdot x^\alpha) \\ &= \frac{x^{\alpha-1}y^\alpha}{(x^2+y^2)^2}((\alpha-2)x^2 + \alpha y^2) \\ &= r^{2\alpha-3}\ \sin^\alpha\theta\cos^{\alpha-1}\theta((\alpha-2)\cos^2\theta + \alpha\sin^2\theta) \end{aligned}$$

and $\dfrac{\partial f}{\partial y} = r^{2\alpha-3}\ \cos^\alpha\theta\sin^{\alpha-1}\theta((\alpha-2)\sin^2\theta + \alpha\cos^2\theta)$.

For $|xy| < 0$, we have

$$\frac{\partial f}{\partial x} = -r^{2\alpha-3}\ \sin^\alpha\theta\cos^{\alpha-1}\theta((\alpha-2)\cos^2\theta + \alpha\sin^2\theta), \quad \frac{\partial f}{\partial y} = -r^{2\alpha-3}\ \cos^\alpha\theta\sin^{\alpha-1}\theta((\alpha-2)\sin^2\theta + \alpha\cos^2\theta)$$

We observe that $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are continuous at the origin if and only if $2\alpha - 3 > 0 \implies \alpha > \dfrac{3}{2}$. By Proposition 3.1 in the notes we conclude that f is differentiable at the origin if and only if $\alpha > \dfrac{3}{2}$. □

Question 2

A function is called *homogeneous of degree* k if $f(\lambda\boldsymbol{x}) = \lambda^k f(\boldsymbol{x})$ for all $\lambda > 0$ and all $\boldsymbol{x} \in \mathbb{R}^n$.

(a) Show that if f is homogeneous of degree k, then

$$\langle \boldsymbol{\nabla} f(\boldsymbol{x}), \boldsymbol{x}\rangle = kf(\boldsymbol{x}).$$

(b) Show conversely if f satisfies the equation, then f is homogeneous of degree k.

Proof. (a) We have to assume that f is differentiable. Starting from $f(\lambda\boldsymbol{x}) = \lambda^k f(\boldsymbol{x})$, we first differentiate both sides with respect to x_i by chain rule:

$$\lambda\frac{\partial f}{\partial x_i}(\lambda\boldsymbol{x}) = \lambda^k\frac{\partial f}{\partial x_i}(\boldsymbol{x})$$

Therefore $\partial_i f(\lambda \boldsymbol{x}) = \lambda^{k-1} f(\boldsymbol{x})$ and $\boldsymbol{\nabla} f(\lambda \boldsymbol{x}) = \lambda^{k-1} \boldsymbol{\nabla} f(\boldsymbol{x})$.

Next we differentiable both sides of $f(\lambda \boldsymbol{x}) = \lambda^k f(\boldsymbol{x})$ with respect to λ by chain rule:

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\lambda \boldsymbol{x}) \cdot x_i = k\lambda^{k-1} f(\boldsymbol{x})$$

Note the LHS is $\langle \boldsymbol{\nabla} f(\lambda \boldsymbol{x}), \boldsymbol{x} \rangle$. Substituting the previous equation we have

$$\left\langle \lambda^{k-1} \boldsymbol{\nabla} f(\boldsymbol{x}), \boldsymbol{x} \right\rangle = k\lambda^{k-1} f(\boldsymbol{x}) \quad \Longrightarrow \quad \langle \boldsymbol{\nabla} f(\boldsymbol{x}), \boldsymbol{x} \rangle = kf(\boldsymbol{x})$$

(b) We define $g : \mathbb{R}^n \times \mathbb{R} \backslash \{0\} \to \mathbb{R}$ by

$$g(\boldsymbol{x}, \lambda) = \lambda^{-k} f(\lambda \boldsymbol{x}) - f(\boldsymbol{x})$$

Fix $\boldsymbol{x} \in \mathbb{R}^n$. Consider the derivative with respect to λ:

$$\begin{aligned} \frac{\partial g}{\partial \lambda}(\boldsymbol{x}, \lambda) &= \lambda \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(\lambda \boldsymbol{x}) - k\lambda^{-k-1} f(\lambda \boldsymbol{x}) \\ &= \lambda^{-k} \left(\langle \boldsymbol{\nabla} f(\lambda \boldsymbol{x}), \boldsymbol{x} \rangle - k\lambda^{-1} f(\lambda \boldsymbol{x}) \right) \\ &= \lambda^{-k-1} \left(\langle \boldsymbol{\nabla} f(\lambda \boldsymbol{x}), \lambda \boldsymbol{x} \rangle - kf(\lambda \boldsymbol{x}) \right) \\ &= 0 \end{aligned}$$

by the equation. Moreover, we know that $g(\boldsymbol{x}, 1) = 0$. By identity theorem, $g(\boldsymbol{x}, \lambda) = 0$ for all $\lambda \in \mathbb{R} \backslash \{0\}$ and $\boldsymbol{x} \in \mathbb{R}^n$. Hence we conclude that f is homogeneous of degree k. □

Question 3

The function $f : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases}$$

Show that all the directional derivatives of f exist at the origin, but f is not differentiable at the origin.

Proof. We express f in the polar coordinates:

$$f(r, \theta) = \begin{cases} \dfrac{r^3 \cos\theta \sin^2\theta}{r^2 \cos^2\theta + r^4 \sin^4\theta} & \text{for } r > 0 \\ 0 & \text{for } r = 0 \end{cases}$$

We consider the derivative of f at 0 in the direction of $(\cos\theta, \sin\theta)$. For $\cos\theta = 0$, f is identically zero. Hence $\partial_\theta f(0) = 0$. For $\cos\theta \neq 0$:

$$\partial_\theta f(0) = \lim_{r \to 0} \frac{f(r, \theta) - f(0)}{r} = \lim_{r \to 0} \frac{r^2 \cos\theta \sin^2\theta}{r^2 \cos^2\theta + r^4 \sin^4\theta} = \frac{\sin^2\theta}{\cos\theta}$$

In summary, all directional derivatives exist:

$$\partial_\theta f(0) = \begin{cases} \dfrac{\sin^2\theta}{\cos\theta} & \text{for } \theta \neq \dfrac{\pi}{2}, \dfrac{3\pi}{2} \\ 0 & \text{for } \theta = \dfrac{\pi}{2} \text{ or } \dfrac{3\pi}{2} \end{cases}$$

In particular, $\dfrac{\partial f}{\partial x}(0) = \dfrac{\partial f}{\partial y}(0) = 0$. Therefore $\boldsymbol{\nabla} f(0) = 0$. If f is differentiable at the origin, then we have $\partial_\theta f(0) = \langle \boldsymbol{\nabla} f(0), (\cos\theta, \sin\theta) \rangle = 0$ for all $\theta \in \mathbb{R}$. But $\partial_{\pi/4} f = \dfrac{\sqrt{2}}{2} \neq 0$, which is a contradiction. Hence f is not differentiable at the origin. □

Question 4

In this question we use the Hilbert-Schmidt matrix norm

$$\|A\| = \left(\sum_{i,j} A_{i,j}^2\right)^{\frac{1}{2}}$$

Show that if H has Hilbert-Schmidt norm less than 1, then $I - H$ is invertible (you may assume that $\|AB\| \leqslant \|A\| \, \|B\|$).

Proof. The inverse of $I - H$ is $A := \sum_{n=0}^{\infty} H^n$. We shall prove that the series on the RHS converges. Since $M_{n\times n}(\mathbb{R})$ is finite-dimensional over $\mathbb{R}$, it is trivially a Banach space (Bolzano-Weierstrass Theorem). Hence it suffices to prove that the series converges absolutely.

For $k \in \mathbb{N}$, $\sum_{n=0}^{k} \|H^n\| \leqslant \sum_{n=0}^{k} \|H\|^n$, which follows from the hint. Hence we have $\sum_{n=0}^{\infty} \|H^n\| \leqslant \sum_{n=0}^{\infty} \|H\|^n < \infty$ since $\|H\| < 1$. We infer that $\sum_{n=0}^{\infty} H^n$ converges.

Finally, $(I - H)\sum_{n=0}^{\infty} H^n = \sum_{n=0}^{\infty} H^n - \sum_{n=1}^{\infty} H^n = I$. Hence $I - H$ is invertible with inverse $\sum_{n=0}^{\infty} H^n$. □

Remark. As suggested in the proof, the result holds for any Banach space, with Hilbert-Schmidt norm replaced by the operator norm.

Question 5

Let $M_{n\times n}(\mathbb{R})$ denote the vector space of $n \times n$ real matrices. Show that the derivative at the identity of the determinant function

$$\det : \; M_{n\times n}(\mathbb{R}) \to \mathbb{R}$$

is

$$\mathrm{d}(\det)_I : \; h \mapsto \operatorname{tr} h$$

Deduce that the derivative at an arbitrary invertible matrix A is

$$\mathrm{d}(\det)_A : \; h \mapsto \det \mathrm{A} \cdot \operatorname{tr}(A^{-1}h)$$

Proof. For $\varepsilon > 0$ and $h \in M_{n\times n}(\mathbb{R})$, we shall find an expansion of $\det(I + \varepsilon h)$. We have:

$$\det(I + \varepsilon h) = \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n} a_{i,\sigma(i)}$$

where $a_{i,j} = \varepsilon h_{i,j}$ if $i \neq j$ and $a_{i,j} = 1 + \varepsilon h_{i,j}$ if $i = j$. Then we have

$$\det(I + \varepsilon h) = \prod_{i=1}^{n}(1 + \varepsilon h_{i,i}) + o(\varepsilon) = 1 + \varepsilon \sum_{i=1}^{n} h_{i,i} + o(\varepsilon) = \det I + \operatorname{tr} h + o(\varepsilon)$$

From the expression we infer that $\det$ is differentiable at I with differential $\mathrm{d}(\det)_I : \; h \mapsto \operatorname{tr} h$.

For $A \in \mathrm{GL}(n, \mathbb{R})$, we have:

$$\det(A + \varepsilon h) = \det A \cdot \det(I + \varepsilon A^{-1}h) = \det A + \det A \cdot \operatorname{tr}(A^{-1}h) + o(\varepsilon)$$

Therefore $\det$ is differentiable at A with differential $\mathrm{d}(\det)_A:\ h \mapsto \det A \cdot \varepsilon\, \mathrm{tr}(A^{-1}h)$. □

Question 6

(a) Show that the set $\mathrm{GL}(n,\mathbb{R})$ of invertible matrices is an open set in $M_{n\times n}(\mathbb{R})$.

(b) Show that the derivative of the inversion map $\mathrm{Inv} : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ is

$$\mathrm{d}(\mathrm{Inv})_A:\ h \mapsto -A^{-1}hA^{-1}$$

Hint: look at the case where A is the identity first.

Proof. (a) For $A \in \mathrm{GL}(n,\mathbb{R})$, let $\delta > 0$ such that $\delta < \left\|A^{-1}\right\|^{-1}$. Then for $B \in B(A,\delta)$, $\|A - B\| < \delta$. We have $B = A - (A - B) = A(I - A^{-1}(A - B))$. Since $\left\|A^{-1}(A-B)\right\| \leqslant \left\|A^{-1}\right\| \|A - B\| < \left\|A^{-1}\right\| \delta < 1$, by Question 4 we know that $I - A^{-1}(A - B)$ is invertible. Hence B is invertible. $B \in \mathrm{GL}(n,\mathbb{R})$. We conclude that $\mathrm{GL}(n,\mathbb{R})$ is open in $M_{n\times n}(\mathbb{R})$.

(b) For $A \in \mathrm{GL}(n,\mathbb{R})$, there exists $\delta > 0$ such that $B(A,\delta) \subseteq \mathrm{GL}(n,\mathbb{R})$. For $h \in \mathrm{GL}(n,\mathbb{R})$ with $\|h\| < \delta$:

$$\begin{aligned}
(A+h)^{-1} &= (I + A^{-1}h)^{-1}A^{-1} \\
&= \left(\sum_{n=0}^{\infty}(-A^{-1}h)^n\right)\cdot A^{-1} && \text{by Question 4} \\
&= \big(I - A^{-1}h + o(\|h\|)\big)\cdot A^{-1} \\
&= A^{-1} - A^{-1}hA^{-1} + o(\|h\|)
\end{aligned}$$

Hence $\mathrm{Inv}(A+h) - \mathrm{Inv}\,A = \mathrm{d}(\mathrm{Inv})_A(h) + o(\|h\|)$ where $\mathrm{d}(\mathrm{Inv}_A) : h \mapsto -A^{-1}hA^{-1}$. □