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Problem Sheet 1

C2.2: Homological Algebra

Overall mark: α

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Section A: Introductory

Question 1

Let $A, B, C \in R$-Mod. Show that there exist canonical R-module isomorphisms

$$\mathrm{Hom}(A \oplus B, C) \cong \mathrm{Hom}(A, C) \oplus \mathrm{Hom}(B, C), \qquad \text{and} \qquad \mathrm{Hom}(A, B \oplus C) \cong \mathrm{Hom}(A, B) \oplus \mathrm{Hom}(A, C)$$

More generally, prove that

$$\mathrm{Hom}\left(\bigoplus_{i\in I} M_i, N\right) \cong \prod_{i\in I} \mathrm{Hom}(M_i, N) \qquad \text{and} \qquad \mathrm{Hom}\left(M, \prod_{i\in I} N_i\right) = \prod_{i\in I} \mathrm{Hom}(M, N_i)$$

Proof.
α

- The functor $\mathrm{Hom}(-, N) : R\text{-Mod}^{\mathrm{op}} \to R\text{-Mod}$ is a right adjoint functor to itself. Therefore it commutes with limits. Note that the direct sum is a colimit in R-Mod and hence a limit in $R\text{-Mod}^{\mathrm{op}}$. We have

$$\mathrm{Hom}\left(\bigoplus_{i\in I} M_i, N\right) \cong \prod_{i\in I} \mathrm{Hom}(M_i, N)$$

- The functor $\mathrm{Hom}(M, -) : R\text{-Mod} \to R\text{-Mod}$ is a right adjoint functor to $(- \otimes M)$. It commutes with limits and hence products. We have

$$\mathrm{Hom}\left(M, \prod_{i\in I} N_i\right) = \prod_{i\in I} \mathrm{Hom}(M, N_i)$$

□

Question 2

A monomorphism is a morphism f satisfying $[f \circ g_1 = f \circ g_2] \Longrightarrow [g_1 = g_2]$. An epimorphism is a morphism satisfying $[g_1 \circ f = g_2 \circ f] \Longrightarrow [g_1 = g_2]$.

Given $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, show (using the language of category theory) that f is a monomorphism and g is an epimorphism.

Proof. (*We assume that this is a short exact sequence.*)

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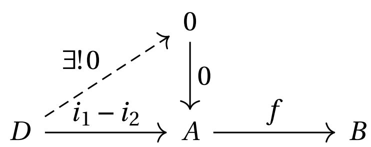
By definition, the exactness at A, B, C implies respectively that,

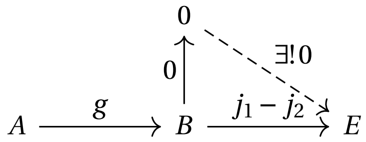
- $\ker f = (0 \to A) = 0$;
- $\ker g = f$, $\mathrm{coker}\, f = g$, $g \circ f = 0$;
- $\mathrm{coker}\, g = (B \to 0) = 0$.

Suppose that $i_1, i_2 \in \mathrm{Hom}(D, A)$ such that $f \circ i_1 = f \circ i_2$. Then $f \circ (i_1 - i_2) = 0$. By the universal property of $\ker f$, there exists a unique morphism $0 : D \to 0$ such that the diagram commutes:



Hence $i_1 - i_2 = 0$, $i_1 = i_2$, and f is a monomorphism.

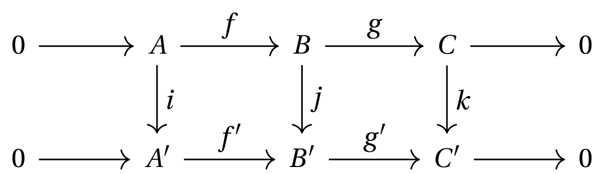
Suppose that $j_1, j_2 \in \mathrm{Hom}(B, E)$ such that $j_1 \circ g = j_2 \circ g$. Then $(j_1 - j_2) \circ g = 0$. By the universal property of $\mathrm{coker}\, g$, there exists a unique morphism $0 : 0 \to E$ such that the diagram commutes:



Hence $j_1 - j_2 = 0$, $j_1 = j_2$, and g is an epimorphism. □

Section B: Core

Question 3



Suppose i, k are isomorphisms. Show that j must then be an isomorphism.

Proof. *(We assume that the sequences are row-exact, and the whole diagram is commutative.)*

The result is called the **short five lemma**. We shall prove this first in the setting of R-Mod and then in general Abelian categories.

First we suppose that everything is in R-Mod, where R is a CRI (commutative ring with identity). We shall prove this by element-theoretic diagram chasing.

- j is injective.

  Let $x \in B$ such that $j(x) = 0$. The following shows that $x = 0$:

  - We have $g' \circ j(x) = 0$.
  - Since k is an isomorphism, $k^{-1} \circ g' \circ j(x) = 0$.
  - By commutativity of the right square, $g(x) = k^{-1} \circ g' \circ j(x) = 0$. Hence $x \in \ker g$.
  - Since the sequence is row-exact at B, we have $x \in \operatorname{im} f$.
  - As f is injective, there exists a unique $y \in A$ such that $x = f(y)$.
  - By commutativity of the left sqaure, $0 = j(x) = j \circ f(y) = f' \circ i(y)$.
  - Since f' is injective, $i(y) = 0$.
  - Since i is an isomorphism, $y = 0$. Hence $x = f(y) = 0$.

- j is surjective.

  Let $z \in B'$. The following shows that there exists $v \in B$ such that $j(v) = z$:

  - We have $g'(z) \in C'$.
  - Since k is an isomorphism, $k^{-1} \circ g'(z) \in C$.
  - Since g is surjective, there exists $w \in B$ such that $g(w) = k^{-1} \circ g'(z)$.
  - By commutativity of the right square, $g' \circ j(w) = k \circ g(w) = g'(z)$. Hence $j(w) - z \in \ker g'$.
  - Since the sequence is row-exact at B', we have $j(w) - z \in \operatorname{im} f'$.
  - Since f' is injective, there exists a unique $u \in A'$ such that $f'(u) = j(w) - z$.
  - Since i is an isomorphism, $i^{-1} \circ f'(u) \in A$.
  - By commutativity of the left sqaure, $j \circ f \circ i^{-1}(u) = f'(u) = j(w) - z$.
  - Hence $z = j(w - f \circ i^{-1}(u))$. We can take $v = w - f \circ i^{-1}(u)$.

We conclude that j is an isomorphism.

Then we suppose that everything is in a general Abelian category[1] A. We shall prove this by arrow-theoretic diagram chasing.

We need the following lemma:

Lemma 1

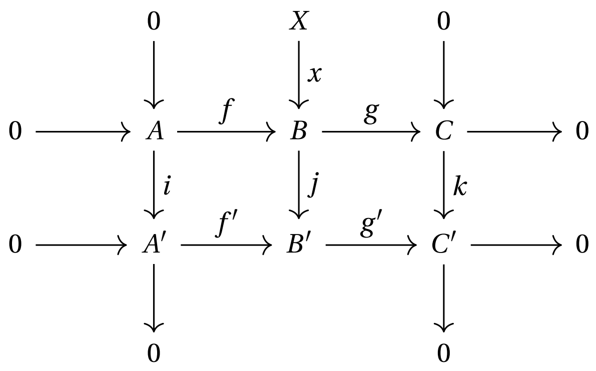
Let A be an Abelian category. $f \in \mathrm{Hom}_{\mathsf{A}}(X, Y)$.

1. f is a monomorphism if and only if $f \circ g = 0 \implies g = 0$ for any g;
2. f is an epimorphism if and only if $g \circ f = 0 \implies g = 0$ for any g;
3. f is an isomorphism if and only if f is both a monomorphism and an epimorphism.

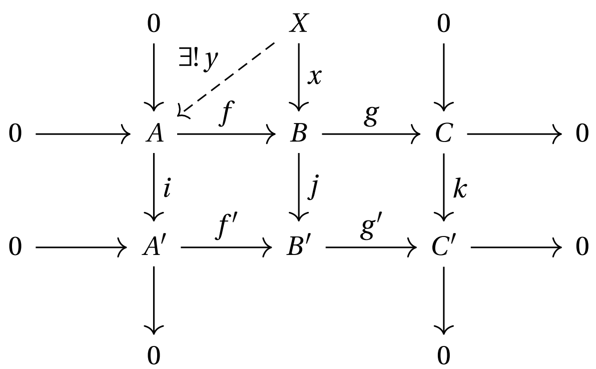
Proof. Trivial. □

- j is a monomorphism.

  Let X be a object of A and $x \in \mathrm{Hom}_{\mathsf{A}}(X, B)$ such that $j \circ x = 0$. We shall show that $x = 0$.



  We have $k^{-1} \circ g' \circ j \circ x = 0$. By commutativity of the right square, $g \circ x = 0$. Since the sequence is row-exact at B, we have $f = \ker g$. By the universal property of kernel, there exists a unique $y : X \to A$ such that the following diagram commutes:



  By the commutativity of the left square, we have $f' \circ i \circ y = j' \circ f \circ y = j \circ x = 0$. Since f' and i are monomorphisms, we must have $y = 0$. Hence $x = f \circ y = f \circ 0 = 0$.

- j is an epimorphism.

  We consider the contravariant functor $F : \mathsf{A} \to \mathsf{A}^{\mathrm{op}}$. In A^{op} we have the commutative diagram:

[1] The above method still works if we invoke the Freyd-Mitchell Embedding Theorem.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C' & \xrightarrow{F(g')} & B' & \xrightarrow{F(f')} & A' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle F(k)} & & \downarrow{\scriptstyle F(j)} & & \downarrow{\scriptstyle F(k)} & & \\
0 & \longrightarrow & C & \xrightarrow{F(g)} & B & \xrightarrow{F(f)} & A & \longrightarrow & 0
\end{array}$$

The same diagram chasing proves that $F(j)$ is a monomorphism. Hence j is an epimorphism.

We conclude that j is an isomorphism. by our lemma above. □

Question 4

Let $R := k[x,y]$ where k is a field. Let $M_1 := R^2/\langle (x,0),(y^2,-x),(0,y)\rangle$ and $M_2 := R/\langle x^2, xy, y^3\rangle$. Provide examples of non-split short exact sequences of R-modules

$$0 \longrightarrow M_1 \longrightarrow ??? \longrightarrow M_2 \longrightarrow 0$$

Proof. We wish to identify M_1 and M_2 with certain k-vector spaces with $k[x,y]$-module structure.

$M_1 = \dfrac{R^2}{\langle (x,0),(0,y),(y^2,-x)\rangle}$ is a k-vector space spanned by $\{(1,0),(0,1),(y,0),(0,x)\}$. So $M_1 \cong k^4$ as k-vector spaces. $x,y \in k[x,y]$ act on the basis vectors via:

$$\begin{aligned}
&x(1,0)=0, && x(0,1)=(0,x), && x(y,0)=y(x,0)=0, && x(0,x)=(0,x^2)=y^2(x,0)-x(y^2,-x)=0\\
&y(1,0)=(y,0), && y(0,1)=0, && y(y,0)=-(0,x), && y(0,x)=x(0,y)=0
\end{aligned}$$

Then we have a R-module isomorphism $\varphi : M_1 \to k^4$, with x,y acting on k^4 as matrices

$$T_x = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&1&0&0 \end{pmatrix}, \qquad T_y = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&0&-1&0 \end{pmatrix}$$

$M_2 = \dfrac{R}{\langle x^2, xy, y^3\rangle}$ is a k-vector space spanned by $\{1,x,y,y^2\}$. $x,y \in k[x,y]$ act on the basis vectors via:

$$\begin{aligned}
&x\cdot 1 = x, && x\cdot x = 0, && x\cdot y = 0, && x\cdot y^2 = y\cdot xy = 0\\
&y\cdot 1 = y, && y\cdot x = 0, && y\cdot y = y^2, && y\cdot y^2 = 0
\end{aligned}$$

Then we have a R-module isomorphism $\psi : M_2 \to k^4$, with x,y acting on k^4 as matrices

$$S_x = \begin{pmatrix} 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}, \qquad S_y = \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&0&1&0 \end{pmatrix}$$

By our construction we automatically have $[T_x, T_y] = 0$ and $[S_x, S_y] = 0$.

Now let $M = k^8$ be a R-module such that x,y acting on M as matrices

$$M_x = \begin{pmatrix} T_x & O\\ O & S_x \end{pmatrix}, \qquad M_y = \begin{pmatrix} T_y & A\\ O & S_y \end{pmatrix}$$

Since $xy = yx$, we must have $[M_x, M_y] = 0$. A must satisfies $T_x A - A S_x = 0$. By observation this is satisfied by

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

As k-vector spaces, we have $M = k^8 \cong k^4 \oplus k^4 \cong M_1 \oplus M_2$, with the inclusion map $f : M_1 \to M$ and the projection map $g : M \to M_2$. Now we have the short exact sequence of k-vector spaces:

$$0 \longrightarrow M_1 \xrightarrow{\ f\ } M \xrightarrow{\ g\ } M_2 \longrightarrow 0$$

f is an R-module homomorphism, because $x \cdot f(\boldsymbol{v}) = M_x f(\boldsymbol{v}) = M_x(\boldsymbol{v}, 0)^\top = (T_x\boldsymbol{v}, 0) = f(T_x\boldsymbol{v}) = f(x \cdot \boldsymbol{v})$ and similiarly for y. The same argument shows that g is an R-module homomorphism. So the short exact sequence is in fact of R-modules.

We claim that the short exact sequence does not split. If it splits, $M \cong M_1 \oplus M_2$ as R-modules. We have

$$y \cdot (\boldsymbol{v}_1, \boldsymbol{v}_2)^\top = M_y(\boldsymbol{v}_1, \boldsymbol{v}_2)^\top = (T_y\boldsymbol{v}_1 + A\boldsymbol{v}_2, S_y\boldsymbol{v}_2) \neq (T_y\boldsymbol{v}_1, S_y\boldsymbol{v}_2) = (y \cdot \boldsymbol{v}_1, y \cdot \boldsymbol{v}_2)$$

which is a contradiction. □

Question 5

Prove that every short exact sequence of $\mathbb{Z}$-modules of the form $0 \to A \to B \to \mathbb{Z} \to 0$ splits.

Prove that every short exact sequence of $\mathbb{Z}$-modules of the form $0 \to \mathbb{Q} \to B \to C \to 0$ splits.

Proof. 1. $\mathbb{Z}$ is a **projective** $\mathbb{Z}$-module. We define $\iota : \mathbb{Z} \to B$ by $\iota(1) = 1$. For any surjective $g : B \to \mathbb{Z}$, we have $g \circ \iota = \mathrm{id}_{\mathbb{Z}}$.

Consider the short exact sequence

$$0 \longrightarrow A \xrightarrow{\ f\ } B \underset{\iota}{\overset{g}{\rightleftarrows}} \mathbb{Z} \longrightarrow 0$$

Let $(f + \iota) : A \oplus \mathbb{Z} \to B$ be the $\mathbb{Z}$-module homomorphism such that $(f + \iota)(a, n) = f(a) + \iota(n)$. Consider the following diagram:

$$\begin{CD} 0 @>>> A @>f>> B @>g>> \mathbb{Z} @>>> 0 \\ @. @| @A(f+\iota)AA @| @. \\ 0 @>>> A @>i_A>> A \oplus \mathbb{Z} @>\pi_{\mathbb{Z}}>> \mathbb{Z} @>>> 0 \end{CD}$$

We check the commutativity: For $a \in A$, $(f + \iota) \circ i_A(a) = (f + \iota)(a, 0) = f(a)$. For $(a, n) \in A \oplus \mathbb{Z}$, $g \circ (f + \iota)(a, n) = g \circ f(a) + g \circ \iota(n) = n = \pi_{\mathbb{Z}}(a, n)$. Hence the diagram is commutative.

By short five lemma, $(f + \iota)$ is an isomorphism. Hence the short exact sequence splits.

2. We claim that $\mathbb{Q}$ is an **injective** $\mathbb{Z}$-module.

By Baer's criterion, it suffices to prove that for every ideal I of $\mathbb{Z}$, every $\mathbb{Z}$-module homomorphism $I \to \mathbb{Q}$ lifts to a $\mathbb{Z}$-module homomorphism $\mathbb{Z} \to \mathbb{Q}$. The ideals of $\mathbb{Z}$ are $n\mathbb{Z}$ and 0. Trivially, $0 : 0 \to \mathbb{Q}$ lifts to $0 : \mathbb{Z} \to \mathbb{Q}$. For $\varphi : n\mathbb{Z} \to \mathbb{Q}$, φ is uniquely determined by $\varphi(n)$. It lifts to $\psi : \mathbb{Z} \to \mathbb{Q}$, where $\psi(x) = \frac{x}{n}\varphi(n)$. This proves the claim.

Consider the short exact sequence

$$0 \longrightarrow \mathbb{Q} \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \longrightarrow 0$$

By universal property of injective module $\mathbb{Q}$, there exists $r : B \to \mathbb{Q}$ such that $r \circ f = \mathrm{id}_{\mathbb{Q}}$:

$$\begin{array}{ccc} & & \mathbb{Q} \\ & \mathrel{\diagup\!\!\diagup} & \uparrow{\scriptstyle \exists r} \\ \mathbb{Q} & \xrightarrow{\ f\ } & B \end{array}$$

Let $(r,g) : B \to \mathbb{Q} \oplus C$ be the $\mathbb{Z}$-module homomorphism with $b \mapsto (r(b), g(b))$. Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Q} & \xrightarrow{\ f\ } & B & \xrightarrow{\ g\ } & C & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle (r,g)} & & \| & & \\ 0 & \longrightarrow & \mathbb{Q} & \xrightarrow{\ i_{\mathbb{Q}}\ } & \mathbb{Q} \oplus C & \xrightarrow{\ \pi_C\ } & C & \longrightarrow & 0 \end{array}$$

The commutativity is obvious. By short five lemma, (r,g) is an isomorphism. Hence the sequence splits. □

Question 6

Prove that $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_{p\text{: prime}} \mathbb{Z}\left[\frac{1}{p}\right]/\mathbb{Z}$.

Proof. $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ is the colimit $\varinjlim_n \mathbb{Z}/p^n\mathbb{Z}$, and also is the ring generated by $\mathbb{Z}$ and $1/p$ in $\mathbb{Q}$. For each p, the embedding $\mathbb{Z}[\frac{1}{p}] \hookrightarrow \mathbb{Q}$ descends to the embedding $\varphi_p : \mathbb{Z}[\frac{1}{p}]/\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z}$. We define the $\mathbb{Z}$-module homomorphism:

$$\varphi = \sum_{p \in P} \varphi_p : \bigoplus_{p \in P} \mathbb{Z}\left[\frac{1}{p}\right]/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$$

where P is the set of primes.

- φ is surjective:

  Let $\alpha = \dfrac{m}{n} \in \mathbb{Q}$. We use induction on n to show that $\alpha + \mathbb{Z} \in \bigoplus_{p \in P} \mathbb{Z}\left[\frac{1}{p}\right]/\mathbb{Z}$. If n is prime, then $\alpha \in \mathbb{Z}[\frac{1}{p}]$. If n is not prime, then $n = pq$ for some $p, q < n$ such that $\gcd(p,q) = 1$, and by Bezóut's Lemma $\alpha = \dfrac{a}{p} + \dfrac{b}{q}$ for some $a, b \in \mathbb{Z}$. By induction hypothesis, $\dfrac{a}{p} + \mathbb{Z}, \dfrac{b}{q} + \mathbb{Z} \in \bigoplus_{p \in P} \mathbb{Z}\left[\frac{1}{p}\right]/\mathbb{Z}$. Therefore $\alpha + \mathbb{Z} \in \bigoplus_{p \in P} \mathbb{Z}\left[\frac{1}{p}\right]/\mathbb{Z}$.

- φ is injective:

  Suppose that $\varphi(\alpha_1, \ldots, \alpha_n, 0, \ldots) = \sum_{k=1}^{n} \varphi_{p_k}(\alpha_k) = 0$. Each $\varphi_{p_k}(\alpha_k)$ is of the form $\dfrac{x_k}{p_k^{\ell_k}}$ where either $x_k = 0$ or $\gcd(x_k, p_k) = 1$. Then we have

  $$\sum_{k=1}^{n} x_k p_1^{\ell_1} \cdots \widehat{p_k^{\ell_k}} \cdots p_n^{\ell_n} = 0$$

  By modulo $p_k^{\ell_k}$ we have $p_k^{\ell_k} \mid x_k$. Hence $x_k = 0$. We conclude that $(\alpha_1, \ldots, \alpha_n, 0, \ldots) = 0$. Henec φ is injective.

In conclusion, φ defines a $\mathbb{Z}$-module isomorphism from $\bigoplus_{p \in P} \mathbb{Z}\left[\frac{1}{p}\right]/\mathbb{Z}$ to $\mathbb{Q}/\mathbb{Z}$. □

Question 7

Prove that in general, $\mathrm{Hom}\left(M, \bigoplus_{i \in I} N_i\right) \ncong \bigoplus_{i \in I} \mathrm{Hom}(M, N_i)$.

Proof. Consider the $\mathbb{Z}$-modules

$$N_i = \mathbb{Z}/p^i\mathbb{Z}, \qquad M = \bigoplus_{i>0} N_i = \bigoplus_{i>0} \mathbb{Z}/p^i\mathbb{Z}$$

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For each $\varphi \in \bigoplus_{i\in I} \operatorname{Hom}(M, N_i)$, we have $\varphi = (\varphi_1, \ldots, \varphi_n, 0, \ldots)$ for some n, where $\varphi_i : M \to \mathbb{Z}/p^i\mathbb{Z}$ is a $\mathbb{Z}$-module homomorphism. Hence $p^n\varphi = (p^n\varphi_1, \ldots, p^n\varphi_n, 0, \ldots) = 0$. So every element in $\bigoplus_{i\in I} \operatorname{Hom}(M, N_i)$ has finite order.

On the other hand, the identity homomorphism

$$\mathrm{id}_M \in \operatorname{Hom}(M, M) = \operatorname{Hom}\left(M, \bigoplus_{i\in I} N_i\right)$$

has infinite order, because for any $n \in \mathbb{Z}$, there exists $k \in \mathbb{N}$ such that $p^k > |n|$, and $n\,\mathrm{id}$ is nonzero on the k-th component of M.

In conclusion, $\operatorname{Hom}\left(M, \bigoplus_{i\in I} N_i\right)$ and $\bigoplus_{i\in I} \operatorname{Hom}(M, N_i)$ are not isomorphic as $\mathbb{Z}$-modules. □

Section C: Optional

Question 8

Prove that the natural inclusion $\bigoplus_{i\in\mathbb{N}} \mathbb{Z} \to \operatorname{Hom}\left(\prod_{i\in\mathbb{N}} \mathbb{Z}, \mathbb{Z}\right)$ is an isomorphism.

Proof. `https://www-users.mat.umk.pl//~gregbob/seminars/2008.11.07b.pdf` presents a good proof. □