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Problem Sheet 4

B3.3: Algebraic Curves

8 March, 2021

Question 1

Let $\Lambda = \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$ be a lattice in $\mathbb{C}$ and let f be meromorphic and doubly periodic with respect to Λ.

Let $\Gamma(a)$ denote the solid parallelogram with vertices at $a, a+\omega_1, a+\omega_2, a+\omega_1+\omega_2$ and let $\gamma(a)$ be the boundary of $\Gamma(a)$. Choose a so that f has no zeroes or poles on $\gamma(a)$.

Let $\beta_1, \dots, \beta_s$ denote the set of poles of f inside $\gamma(a)$.

Show that

$$\sum_{i=1}^{s} \operatorname{Res}\left(f; \beta_i\right) = 0$$

Proof. Without loss of generality let $\gamma(a)$ be positively oriented. By residue theorem,

$$\oint_{\gamma(a)} f(z)\,\mathrm{d}z = 2\pi\mathrm{i} \sum_{i=1}^{s} \operatorname{Res}(f; \beta_i)$$

Since f is doubly periodic with repsect to Λ, we have $f(z) = f(z \pm \omega_1) = f(z \pm \omega_2)$ for all z in the domain of f. We have

$$\int_a^{a+\omega_1} f(z)\,\mathrm{d}z = \int_a^{a+\omega_1} f(z-\omega_2)\,\mathrm{d}z = \int_{a+\omega_2}^{a+\omega_1+\omega_2} f(z)\,\mathrm{d}z$$
$$\int_a^{a+\omega_2} f(z)\,\mathrm{d}z = \int_a^{a+\omega_2} f(z-\omega_1)\,\mathrm{d}z = \int_{a+\omega_1}^{a+\omega_1+\omega_2} f(z)\,\mathrm{d}z$$

(Abuse of notation: $\int_c^d$ means integrating along the line segment from c to d.) Hence

$$\begin{aligned}\oint_{\gamma(a)} f(z)\,\mathrm{d}z &= \left(\int_a^{a+\omega_1} + \int_{a+\omega_1}^{a+\omega_1+\omega_2} + \int_{a+\omega_1+\omega_2}^{a+\omega_2} + \int_{a+\omega_2}^{a}\right) f(z)\,\mathrm{d}z \\ &= \left(\int_a^{a+\omega_1} + \int_a^{a+\omega_2} + \int_{a+\omega_1}^{a} + \int_{a+\omega_2}^{a}\right) f(z)\,\mathrm{d}z \\ &= 0\end{aligned}$$

We deduce that

$$\sum_{i=1}^{s} \operatorname{Res}(f; \beta_i) = 0$$ □

Question 2

Consider the affine nodal cubic C_{aff} in $\mathbb{C}^2$ with equation

$$y^2 = x^3 + x^2$$

Show that the formula

$$t \mapsto \left(t^2 - 1, t - t^3\right)$$

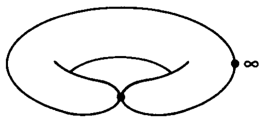
describes a map from $\mathbb{C}$ onto C_{aff}. Describe the fibres of this map (ie. the preimages of points in C_{aff}).

What can you deduce about the topology of the projective nodal cubic $y^2 z = x^3 + x^2 z$?

Proof. First note that $t \mapsto t^2 - 1$ is surjective onto $\mathbb{C}$. For any $x \in \mathbb{C}$ there exists $t \in \mathbb{C}$ such that $x = t^2 - 1$. Then $y^2 = x^2(x+1) = t^2(t^2-1)^2$. So $y = \pm t(1-t^2)$. By replacing t with $\pm t$ we can always have $y = t - t^3$. So the map $t \mapsto (t^2 - 1, t - t^3)$ is surjective onto C_{aff}.

For $(0,0) \in C_{\text{aff}}$, we note that the preimages of the point are $t = \pm 1$. For $(x, y) \in C_{\text{aff}} \setminus \{(0,0)\}$, the map $t \mapsto t^2 - 1$ is two to one, and each t corresponds to a branch $y = \pm\sqrt{x^3 + x^2}$. So $t \mapsto (t^2 - 1, t - t^3)$ is one to one for $t \neq \pm 1$.

The topology of the projective nodal cubic looks like:



The cubic resembles the torus but has a nodal singularity at the origin. (*I am not sure how to understand this directly from the equation...*) □

Question 3

Let $\wp(z)$ be the Weierstrass $\wp$-function associated to a lattice Λ. Consider the meromorphic function $\wp'(z)$ as a function from the elliptic curve $X = \mathbb{C}/\Lambda$ to the Riemann sphere.

Determine its degree and the number and ramification indices of its ramification points.

Is there a meromorphic function f on X with $f'(z) = \wp(z)$?

Proof. Let $\Lambda := \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\}$. By definition,

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left(\frac{1}{(z+\omega)^2} + \frac{1}{\omega^2}\right)$$

Since the series converges absolutely, we can differentiate termwise:

$$\wp'(z) = -2\sum_{\omega\in\Lambda} \frac{1}{(z+\omega)^3}$$

At $z = 0$, the leading term in the Laurent expansion of $\wp'$ is $-2z^{-3}$. Hence $z = 0$ is a triple pole of $\wp'$. As a map from $\mathbb{C}/\Lambda$ to $\mathbb{CP}^1$, $\wp'$ has degree 3. And $z = 0$ is a ramification point of $\wp'$ of index 3.

We know that

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$

where $e_1 = \wp(\omega_1/2)$, $e_2 = \wp(\omega_2/2)$, and $e_3 = \wp((\omega_1+\omega_2)/2)$. Moreover, we can in fact show that $e_1 + e_2 + e_3 = 0$, which gives another expression for $\wp'$:

$$(\wp')^2 = 4\wp^3 - 20c_1\wp - 28c_2$$

I have done this in Sheet 2 of Geometry of Surfaces. The proof is as follows:

Starting from

$$\wp(z) := \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$$

Note that

$$\frac{1}{(1-w)^2} = \sum_{\ell=0}^{\infty} (\ell+1) w^\ell, \qquad \text{for } |w| < 1$$

which is obtained by differentiating the geometric series. Hence for $|z| < |\omega|$:

$$\frac{1}{(z-\omega)^2} = \frac{1}{\omega^2}\sum_{\ell=0}^{\infty} (\ell+1)\left(\frac{z}{\omega}\right)^\ell$$

Therefore we obtain the Laurent expansion of $\wp$ near $z = 0$:

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left(\frac{1}{\omega^2}\sum_{\ell=0}^{\infty}(\ell+1)\left(\frac{z}{\omega}\right)^\ell - \frac{1}{\omega^2}\right) = \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}}\sum_{\ell=1}^{\infty}(\ell+1)\frac{z^\ell}{\omega^{\ell+2}} \\ &= \frac{1}{z^2} + \sum_{\ell=1}^{\infty}(\ell+1)z^\ell \sum_{\omega\in\Lambda\setminus\{0\}} \omega^{-(\ell+2)}\end{aligned}$$

By symmetry, replacing (m, n) with $(-m, -n)$, we see that the following series is zero for odd ℓ.

$$\sum_{\omega\in\Lambda\setminus\{0\}} \omega^{-(\ell+2)} = \sum_{(m,n)\neq(0,0)} (m\omega_1 + n\omega_2)^{-(\ell+2)}$$

Hence

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} (2k+1) z^{2k} \sum_{\omega\in\Lambda\setminus\{0\}} \omega^{-(2k+2)} = \frac{1}{z^2} + \sum_{k=1}^{\infty} c_k z^{2k}$$

From the Laurent expansion we find that

$$\wp = z^{-2} + c_1 z^2 + c_2 z^4 + \mathrm{O}\left(z^6\right)$$

$$\wp^3 = z^{-6} + 3c_1 z^{-2} + 3c_2 + \mathrm{O}\left(z^2\right)$$

$$(\wp')^2 = 4z^{-6} - 8c_1 z^{-2} - 16c_2 + \mathrm{O}\left(z^4\right)$$

Hence $g = (\wp')^2 - 4\wp^3 + 20c_1\wp + 28c_2 = \mathrm{O}\left(z^2\right)$ near $z = 0$. So g can be extended to a holomorphic function near $z = 0$. Since g is doubly periodic, the image $g(\mathbb{C}/\Lambda)$ is compact in $\mathbb{C}$. Hence by Liouville's Theorem g is constant. $g(z) = 0$ implies that

$$(\wp')^2 = 4\wp^3 - 20c_1\wp - 28c_2$$

By differentiating the expression we find that

$$\wp''(z) = 6\wp(z)^2 - 20c_1$$

whenever $\wp'(z) \neq 0$. If $\wp''(z) = 0$, then $\wp(z)^2 = \frac{10}{3}c_1$. There are exactly 4 points in $\mathbb{C}/\Lambda$ satisfying the equation, because $\wp$ has degree 2. Hence $\wp'$ has 4 ramification points away from $z = 0$. We claim that all of them has ramification index 2.

By Riemann-Hurwitz formula,

$$\chi(\mathbb{C}/\Lambda) = \deg\wp' \cdot \chi(\mathbb{CP}^1) - \sum_{z\in\mathbb{C}/\Lambda} \left(\nu_{\wp'}(z) - 1\right)$$

Hence $\sum_{z\in\mathbb{C}/\Lambda} \left(\nu_{\wp'}(z) - 1\right) = 6$. We already know that $\nu_{\wp'}(0) = 3$, and that there are four more points such that $\nu_{\wp'}(z) \geqslant 2$. We must have $\nu_{\wp'}(z) = 2$ at these points.

Suppose that there exists a meromorphic function f on $\mathbb{C}/\Lambda$ such that $f' = \wp$. At $z = 0$, the leading term of the Laurent expansion of f is $-1/z$. Hence f has degree 1. This implies that f has no ramification points. This is impossible by Riemann-Hurwitz formula:

$$0 = \chi(\mathbb{C}/\Lambda) \neq \deg f \cdot \chi(\mathbb{CP}^1) = 2$$

Such meromorphic function f does not exist. □

Question 4

Let E be an elliptic curve, that is, a Riemann surface of genus 1, and let p be a point on E.

Calculate $\ell(mp)$ for $m = 1, 2, 3, \ldots$

Deduce that there exist meromorphic functions f and g on E with, respectively, a double pole at a and a triple pole at a, and no other poles. Describe $\mathscr{L}(mp)$ for $m = 1, 2, 3, 4, 5$ in terms of the functions f and g.

By considering $\mathscr{L}(6p)$, deduce that we have a polynomial relation between f and g, and interpret your results in terms of the Weierstrass $\wp$-function.

Proof. For $m \in \mathbb{Z}_+$, by Riemann-Roch Theorem:

$$\ell(mp) - \ell(\kappa - mp) = \deg(mp) + 1 - g = m$$

Since E has genus 1, the canonical divisor κ has degree $2g - 2 = 0$. Hence

$$\deg(\kappa - mp) = \deg(\kappa) - m = -m < 0$$

Then $\ell(\kappa - mp) = 0$. We deduce that $\ell(mp) = m$.

Next we study the structure of $\mathscr{L}(mp)$. We know that a meromorphic function on E is uniquely determined up to a constant by the number of zeros and poles. For $\varphi \in \mathscr{L}(mp)$, by definition $(\varphi) + mp \geqslant 0$. In other words, φ is either entire or has a unique pole at p of multiplicity not greater than p. By Liouville's Theorem, entire functions on E are constant.

Since $\dim \mathscr{L}(2p) - \ell(2p) = 2$, there exists non-constant functions in $\mathscr{L}(2p)$. Let f be such a function. As discussed in Question 3, f cannot have a simple pole at p. So f must have a double pole at p.

Similarly, there exists a meromorphic function g with a unique triple pole at p, which means $g \in \mathscr{L}(3p) \setminus \mathscr{L}(2p)$.

In general, $\mathscr{L}(mp) = \langle 1, \varphi_2, ..., \varphi_m \rangle$, where φ_k is a meromorphic function with a unique pole of multiplicity k at p.

Note that $f^i g^j$ is a meromorphic function with a unique pole of multiplicity $2i + 3j$ at p. We have:

$$\mathscr{L}(p) = \langle 1 \rangle, \quad \mathscr{L}(2p) = \langle 1, f \rangle, \quad \mathscr{L}(3p) = \langle 1, f, g \rangle, \quad \mathscr{L}(4p) = \langle 1, f, g, f^2 \rangle, \quad \mathscr{L}(5p) = \langle 1, f, g, f^2, fg \rangle$$

We note that f^3 and g^2 are meromorphic functions with a hextuple pole at p. That is, $f^2, g^3 \in \mathscr{L}(6p) \setminus \mathscr{L}(5p)$. Since $\ell(6p) = 6$, the functions $\{1, f, g, f^2, fg, f^3, g^2\}$ are linearly dependent. There exists $c_0, ..., c_6 \in \mathbb{C}$ such that

$$c_0 + c_1 f + c_2 g + c_3 f^2 + c_4 fg + c_5 f^3 + c_6 g^2 = 0$$

which is a non-trivial polynomial relation between f and g.

The Weierstrass $\wp$-function is a meromorphic function on the torus $\mathbb{C}/\Lambda$ (which is homeomorphic to the elliptic curve C_Λ) with a unique double pole at $z = 0$. The derivative $\wp'$ is a meromorphic function with a unique triple pole at $z = 0$. We already know the polynomial relation between $\wp$ and $\wp'$:

$$\wp'^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3) \qquad \square$$