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Problem Sheet 4

Gravitational Radiation

B5: General Relativity

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Question 1. The Burke-Thorne Potential.

Consider the following unusual Newtonian potential, due to Burke and Thorne:

$$\Phi = \frac{GJ_{ij}^{(5)}}{5c^7} x^i x^j$$

where $J_{ij}^{(5)}$ is the traceless (energy) moment of inertia tensor, differentiated five times with respect to time:

$$J_{ij} = c^2 \frac{d^5}{dt^5}\left[\int \rho\left(x_i x_j - \delta_{ij} r^2/3\right) dV\right] = \frac{d^5}{dt^5}\left[I_{ij} - \delta_{ij} I_{kk}/3\right]$$

$I_{ij} = c^2 \int \rho x_i x_j dV$ is the standard moment of inertia tensor. The indices i and j represent spatial Cartesian coordinates, and we use the Minkowski metric, so spatial index placement is unimportant. The radius $r^2 = x^i x^i$. Show that this potential gives rise to a force, $-\partial_i \Phi$ exactly analogous to the "radiation reaction force" in electromagnetism. In other words, show that we recover Einstein's gravitational energy loss formula,

$$\frac{dE}{dt} = \left\langle -\int \rho v_i \partial_i \Phi dV \right\rangle = -\frac{G}{5c^9}\langle \dddot{J}_{ij}\dddot{J}_{ij}\rangle$$

which states that the work done by the force, averaged over time (this is the meaning of the angle brackets $\langle\rangle$) equals the rate at which energy is lost from the system. This also works for angular momentum loss as well. Show that:

$$\frac{dL}{dt} = -\left\langle \int \epsilon^{ijk} \rho x_i \partial_j \Phi dV \right\rangle = -\frac{2G}{5c^9}\left\langle \epsilon^{imk} \dddot{J}_{mn} \ddot{J}_{in}\right\rangle$$

which states that the effect of '$\mathbf{r} \times \mathbf{F}$' torque, averaged over time, equals the angular momentum loss.

Here are some hints:

i) When in doubt, integrate by parts, either in time or in space.

ii) The equation of mass conservation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x^i} = 0$$

is used for the energy loss formula derivation. (You don't have to take the time here to prove this, though you should be familiar with it by now.)

iii) You will also need

$$J_{ij}J_{ij} = I_{ij}I_{ij} - I_{ii}I_{jj}/3$$

Show this result if you want to use it!

iv) You should find that for the angular momentum loss formula, the result holds either for the traceless moment of intertia J_{ij} or for I_{ij}

Proof. For physical reason we must have $\rho \to 0$ as $\|\boldsymbol{x}\| \to \infty$. We process the spatial integral first.

$$\begin{aligned}
-\int_{\mathbb{R}^3} \rho v_i \partial_i \Phi \, \mathrm{d}V &= \int_{\mathbb{R}^3} \left(-\partial_i(\rho v_i \Phi) + \Phi \partial_i(\rho v_i)\right) \mathrm{d}V \\
&= \int_{\mathbb{R}^3} \Phi \partial_i(\rho v_i) \, \mathrm{d}V \\
&= \int_{\mathbb{R}^3} -\Phi \partial_t \rho \, \mathrm{d}V && \text{(mass conservation)}
\end{aligned}$$

We have assumed that $\underbrace{\int_{\mathbb{R}^3} \partial_i(\rho v_i \Phi) \, \mathrm{d}V = 0}$ because we can integrate the x^i component by Fubini's Theorem and use $\rho \to 0$ as $|x_i| \to \infty$.

this vanishes because by the div. thm
it is given by an integral on the
boundary, but all quantities in the
integrand vanish at ∞

Next we process the time-average energy loss. Let τ be the period of the system.

$$
\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{1}{\tau}\int_0^\tau \left(-\int_{\mathbb{R}^3} \rho v_i \partial_i \Phi \,\mathrm{d}V\right)\mathrm{d}t \\
&= \frac{1}{\tau}\int_0^\tau \left(\int_{\mathbb{R}^3} -\Phi \partial_t \rho \,\mathrm{d}V\right)\mathrm{d}t \\
&= \frac{1}{\tau}\int_{\mathbb{R}^3} \left(\int_0^\tau -\Phi \partial_t \rho \,\mathrm{d}t\right)\mathrm{d}V \\
&= \frac{1}{\tau}\int_{\mathbb{R}^3} \left(\int_0^\tau \left(-\partial_t(\rho\Phi) + \rho\partial_t\Phi\right)\mathrm{d}t\right)\mathrm{d}V \\
&= \frac{1}{\tau}\int_{\mathbb{R}^3} \left(\int_0^\tau \rho\partial_t\Phi \,\mathrm{d}t\right)\mathrm{d}V \\
&= \frac{1}{\tau}\int_{\mathbb{R}^3} \left(\int_0^\tau \rho \frac{G}{5c^7} J^{(6)}_{jk} x^j x^k \,\mathrm{d}t\right)\mathrm{d}V \\
&= \frac{1}{\tau}\int_0^\tau \left(\int_{\mathbb{R}^3} \rho x^j x^k \,\mathrm{d}V\right)\frac{G}{5c^7} J^{(6)}_{jk}\,\mathrm{d}t \\
&= \frac{1}{\tau}\int_0^\tau \frac{G}{5c^9} J^{(6)}_{jk} I_{jk}\,\mathrm{d}t \\
&= -\frac{1}{\tau}\int_0^\tau \frac{G}{5c^9} \dddot{J}_{jk} \dddot{I}_{jk}\,\mathrm{d}t && \text{(partial integration 3 times)}
\end{aligned}
$$

We claim that $\dddot{J}_{jk}\dddot{I}_{jk} = \dddot{J}_{jk}\dddot{J}_{jk}$.

$$
\begin{aligned}
\dddot{J}_{jk}\dddot{J}_{jk} &= \left(\dddot{I}_{jk} - \delta_{jk} I_{mm}/3\right)\left(\dddot{I}_{jk} - \delta_{jk} I_{nn}/3\right) \\
&= \dddot{I}_{jk}\dddot{I}_{jk} - 2\dddot{I}_{jj}\dddot{I}_{mm}/3 + 3\cdot\dddot{I}_{mm}\dddot{I}_{nn}/9 \\
&= \dddot{I}_{jk}\dddot{I}_{jk} - \dddot{I}_{jj}\dddot{I}_{\ell\ell}/3 \\
&= \dddot{I}_{jk}\left(\dddot{I}_{jk} - \dddot{I}_{jk}\delta_{jk}\dddot{I}_{\ell\ell}/3\right) \\
&= \dddot{I}_{jk}\dddot{J}_{jk}
\end{aligned}
$$

Hence we deduce that

$$
\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{G}{5c^9}\langle \dddot{I}_{jk}\ddddot{J}_{jk}\rangle = -\frac{G}{5c^9}\langle \dddot{J}_{jk}\ddddot{J}_{jk}\rangle
$$

Now we turn to angular momentum. We process the spatial integral:

$$
-\int_{\mathbb{R}^3} \varepsilon^{ijk}\rho x_i \partial_j \Phi \,\mathrm{d}V = -\int_{\mathbb{R}^3} \varepsilon^{ijk}\rho x_i \frac{GJ^{(5)}_{mn}}{5c^7}\partial_j(x^m x^n)\,\mathrm{d}V
$$

We claim that $J^{(5)}_{mn}\partial_j(x^m x^n) = 2J^{(5)}_{jm}x^m$. For $m,n \neq j$, $\partial_j(x^m x^n) = 0$. For $m = j, n \neq j$ or $m \neq j, n = j$, $J^{(5)}_{mn}\partial_j(x^m x^n) = J^{(5)}_{mn}x^m$ because $J^{(5)}_{mn}$ is symmetric. For $m = n = j$, $\partial_j(x^m x^n) = 2x^m$. This justifies the claim.

Hence

$$
\begin{aligned}
-\int_{\mathbb{R}^3} \varepsilon^{ijk}\rho x_i \partial_j \Phi \,\mathrm{d}V &= -\int_{\mathbb{R}^3} \varepsilon^{ijk}\rho x_i \frac{2GJ^{(5)}_{jm}}{5c^7} x^m \,\mathrm{d}V \\
&= -\varepsilon^{ijk}\frac{2GJ^{(5)}_{jm}}{5c^7}\int_{\mathbb{R}^3} \rho x^i x^m \,\mathrm{d}V \\
&= -\varepsilon^{ijk}\frac{2G}{5c^9} J^{(5)}_{jm} I_{im}
\end{aligned}
$$

For similar reason as above, we have $J^{(5)}_{jm}I_{im} = J^{(5)}_{jm}J_{im}$. Next we process the time-average angular momentum loss.

$$
\frac{\mathrm{d}L_k}{\mathrm{d}t} = \frac{1}{\tau}\int_0^\tau -\varepsilon^{ijk}\frac{2G}{5c^9}J^{(5)}_{jm}J_{im}\,\mathrm{d}t = -\frac{1}{\tau}\int_0^\tau \varepsilon^{ijk}\frac{2G}{5c^9}\dddot{J}_{jm}\ddot{J}_{im}\,\mathrm{d}t = -\frac{2G}{5c^9}\left\langle \varepsilon^{ijk} J_{jm}\ddot{J}_{im}\right\rangle \qquad \square
$$

Question 2. Desert island GR.

Here we will construct a linear, weak field theory gravity from scratch. Then we will construct GR from scratch! (Well, practically.)

Imagine that it is 1912. Minkowski has formulated the concept of his spacetime geometry (1908). Einstein has had his happy (1907) Equivalence Principle thought, and has just understood that gravity is a Riemannian geometric theory of a curved Minkowski spacetime, and that the name of the game is to relate the derivatives of $g_{\mu\nu}$ to $T_{\mu\nu}$. But he knows nothing more. Let's help him out.

a) Our weak gravity field equation will need, on the left side, a sum of second derivatives of $g_{\mu\nu}$. More conveniently, we use derivatives of the small quantity $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$. Not only is the background spacetime geometry flat Minkowski, our coordinates are very close to Cartesian. So, with $h \equiv h^\rho_\rho$, there are but five combinations that could possibly appear:

$$\Box h_{\mu\nu}, \quad \partial_\mu\partial_\nu h, \quad \left(\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu\right), \quad \eta_{\mu\nu}\Box h, \quad \eta_{\mu\nu}\partial_\rho\partial_\lambda h^{\rho\lambda}$$

$(\Box \equiv \partial^\rho\partial_\rho$. We use the handy notation $\partial^\mu = \partial/\partial x_\mu, \partial_\mu = \partial/\partial x^\mu$, and raise and lower indices on $h_{\mu\nu}$ with $\eta^{\rho\mu}$.) Justify this statement and explain fully.

b) We accordingly search for an equation of the form:

$$\Box h_{\mu\nu} + \alpha\left(\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu\right) + \beta\partial_\mu\partial_\nu h + \eta_{\mu\nu}\left(\gamma\Box h + \delta\partial_\rho\partial_\lambda h^{\rho\lambda}\right) = CT_{\mu\nu}$$

where $\alpha, \beta, \gamma, \delta$ and C are constants to be determined. You remember, of course, the stress tensor $T_{\mu\nu}$, now in Newtonian guise. We demand that $\partial^\mu T_{\mu\nu} = 0$ as an identity. What is the reason for this? Show that $\alpha = -1, \delta = 1, \gamma = -\beta$ follow:

$$\Box h_{\mu\nu} - \left(\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu\right) + \beta\partial_\mu\partial_\nu h - \eta_{\mu\nu}\left(\beta\Box h - \partial_\rho\partial_\lambda h^{\rho\lambda}\right) = CT_{\mu\nu}$$

c) By taking the trace of this last equation and using $T_{00} \gg T_{ii}$ (valid in the Newtonian limit - why?), show that

$$\partial_\rho\partial_\lambda h^{\rho\lambda} = \frac{3\beta - 1}{2}\Box h - \frac{CT_{00}}{2}$$

Be careful with signs and up-down indices.

d) Taking the static Newtonian limit of the (2b) final equation, show that

$$\nabla^2 h_{00} + \frac{1-\beta}{2}\nabla^2 h = \frac{C}{2}T_{00}$$

where ∇^2 is the usual Laplacian operator. Explain why this implies $\beta = 1$ and $C = -16\pi G$:

$$\Box h_{\mu\nu} - \left(\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu\right) + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\left(\Box h - \partial_\rho\partial_\lambda h^{\rho\lambda}\right) = -16\pi G T_{\mu\nu}$$

Compare this with section (9.1) in the notes and comment.

e) Given that the Ricci tensor $R_{\mu\nu}$ and $g_{\mu\nu}R^\rho_\rho$ are the only second rank tensors that are linear in the second derivatives of the metric tensor $g_{\mu\nu}$ when the curvature is weak, explain why the general field equations must take the form

$$R_{\mu\nu} - \frac{g_{\mu\nu}R}{2} = -8\pi G T_{\mu\nu}$$

where $R \equiv R^\rho_\rho$. Notice: not a Bianchi identity in sight. If Einstein could only have seen this in 1912.

Proof. a) In general the second-derivative of $h_{\mu\nu}$ is $\partial_\rho\partial_\lambda h_{\mu\nu}$, which is a (coordinate-dependent) type (0,4) tensor. Since $T_{\mu\nu}$ is a symmetric tensor of type (0,2), we need to raise one index of $\partial_\rho\partial_\lambda h_{\mu\nu}$ and contract it to obtain a symmetric type (0,2) tensor.

Since $h_{\mu\nu}$ is symmetric and ∂_ρ is torsion-free, the possible type (1,3) tensors are $\partial_\rho\partial_\lambda h^\mu_\nu$ and $\partial^\rho\partial_\lambda h_{\mu\nu}$. Now we enumerate their possible contractions (with symmetrisations):

$$\partial^\rho\partial_{\{\mu}h_{\nu\}\rho} = \partial_\rho\partial_{\{\mu}h^\rho_{\nu\}} = \frac{1}{2}\left(\partial_\rho\partial_\mu h^\rho_\nu + \partial_\rho\partial_\nu h^\rho_\mu\right), \qquad \partial_\mu\partial_\nu h^\rho_\rho = \partial_\mu\partial_\nu h, \qquad \partial^\rho\partial_\rho h_{\mu\nu} = \Box h_{\mu\nu}$$

We can also raise two indices to obtain a type (2,2) tensor and contract it to a scalar. The possibilities are

$$\partial_\rho \partial_\rho h^{\lambda\lambda} = \partial^\rho \partial_\rho h^\lambda_\lambda = \partial^\rho \partial^\rho h_{\lambda\lambda} = \Box h, \qquad \partial_\rho \partial_\lambda h^{\rho\lambda} = \partial^\rho \partial_\lambda h^\lambda_\rho = \partial^\rho \partial^\lambda h_{\rho\lambda}$$

The scalars acting on Minkowski metric $\eta_{\mu\nu}$ are also candidates of the field equation. In summary, our field tensor would be the linear combination of

$$\Box h_{\mu\nu}, \quad \partial_\mu \partial_\nu h, \quad \left(\partial_\rho \partial_\mu h^\rho_\nu + \partial_\rho \partial_\nu h^\rho_\mu\right), \quad \eta_{\mu\nu} \Box h, \quad \eta_{\mu\nu} \partial_\rho \partial_\lambda h^{\rho\lambda}$$

b) It would be clear to write $g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon \gamma_{\mu\nu}$ for $\varepsilon \ll 1$, so that we know that $h_{\mu\nu} = \varepsilon \gamma_{\mu\nu}$ is of order ε. The linear combinations in (a) is of order ε. So the energy-momentum tensor $T_{\mu\nu}$ is also of order ε. We know that it is divergenceless:

$$\nabla_\mu T^\mu_\nu = 0$$

Accordingly, we need an asymptotic expansion of ∇_μ to order 1. Note that the Christoffel symbol

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma} \left(\partial_\nu g_{\sigma\lambda} + \partial_\lambda g_{\sigma\nu} - \partial_\sigma g_{\nu\lambda}\right) = \frac{1}{2} \eta^{\mu\sigma} \left(\partial_\nu h_{\sigma\lambda} + \partial_\lambda h_{\sigma\nu} - \partial_\sigma h_{\nu\lambda}\right) + \mathrm{O}(\varepsilon^2)$$

It vanishes at order 1. Therefore $\nabla_\mu = \partial_\mu$ at order 1. Therefore at $\mathrm{O}(\varepsilon)$ we have

$$\partial_\mu T^\mu_\nu = \partial^\mu T_{\mu\nu} = 0$$

Next we apply ∂^μ to the left hand side of the equation:

$$\begin{aligned} 0 &= \partial^\mu \left(\Box h_{\mu\nu} + \alpha \left(\partial_\rho \partial_\mu h^\rho_\nu + \partial_\rho \partial_\nu h^\rho_\mu\right) + \beta \partial_\mu \partial_\nu h + \eta_{\mu\nu} \left(\gamma \Box h + \delta \partial_\rho \partial_\lambda h^{\rho\lambda}\right)\right) \\ &= \Box \partial^\mu h_{\mu\nu} + \alpha \Box \partial_\rho h^\rho_\nu + \alpha \partial_\rho \partial_\nu \partial^\mu h^\rho_\mu + \beta \Box \partial_\nu h + \gamma \Box \partial_\nu h + \delta \partial_\rho \partial_\lambda \partial_\nu h^{\rho\lambda} \\ &= (\alpha + 1) \Box \partial^\mu h_{\mu\nu} + (\alpha + \delta) \partial_\rho \partial_\nu \partial^\mu h^\rho_\mu + (\beta + \gamma) \Box \partial_\nu h \end{aligned}$$

Hence

$$\alpha + 1 = 0, \qquad \alpha + \delta = 0, \qquad \beta + \gamma = 0$$

We deduce that $\alpha = -1$, $\beta = \gamma$, and $\delta = 1$. Hence the field equation becomes

$$\Box h_{\mu\nu} - \left(\partial_\rho \partial_\mu h^\rho_\nu + \partial_\rho \partial_\nu h^\rho_\mu\right) + \beta \partial_\mu \partial_\nu h - \eta_{\mu\nu} \left(\beta \Box h - \partial_\rho \partial_\lambda h^{\rho\lambda}\right) = C T_{\mu\nu}$$

c) Technically we need to raise an index before taking trace, which is equivalent to acting $\eta^{\mu\nu}$ on both sides of the equation. On the RHS, we have

$$\eta^{\mu\nu} T_{\mu\nu} = -T_{00} + T^{ii}$$

From the expression of the energy-momentum tensor

$$T_{\mu\nu} = P g_{\mu\nu} + \left(\rho + \frac{P}{c^2}\right) U_\mu U_\nu$$

In the Newtonian limit, $c \to \infty$. So the 4-velocity is given by

$$U = \frac{\mathrm{d}t}{\mathrm{d}\tau} \left(-c\partial_t + v^i \partial_i\right) \approx -c\partial_t \implies U^\flat \approx c\,\mathrm{d}t$$

We have $T \approx \rho c^2 \mathrm{d}t^2$ or $T_{00} \gg T_{ii}$. Hence $\eta^{\mu\nu} T_{\mu\nu} \approx -T_{00}$.

On the LHS, we have

$$\begin{aligned} \eta^{\mu\nu} \left(\Box h_{\mu\nu} - \left(\partial_\rho \partial_\mu h^\rho_\nu + \partial_\rho \partial_\nu h^\rho_\mu\right) + \beta \partial_\mu \partial_\nu h - \eta_{\mu\nu} \left(\beta \Box h - \partial_\rho \partial_\lambda h^{\rho\lambda}\right)\right) &= \Box h - 2\partial_\rho \partial_\lambda h^{\rho\lambda} + \beta \Box h - 4\left(\beta \Box h - \partial_\rho \partial_\lambda h^{\rho\lambda}\right) \\ &= (1 - 3\beta) \Box h + 2\partial_\rho \partial_\lambda h^{\rho\lambda} \end{aligned}$$

Hence

$$(1 - 3\beta) \Box h + 2\partial_\rho \partial_\lambda h^{\rho\lambda} = -C T_{00} \implies \partial_\rho \partial_\lambda h^{\rho\lambda} = \frac{3\beta - 1}{2} \Box h - \frac{C}{2} T_{00}$$

d) In the static Newtonian limit, $\partial_0 = 0$ and hence $\Box = \nabla^2$. We take the (0,0) component of the equation in (b):

$$\nabla^2 h_{00} + \left(\beta \nabla^2 h - \partial_\rho \partial_\lambda h^{\rho\lambda}\right) = C T_{00}$$

Substituting the result of (c):

$$\nabla^2 h_{00} + \left(\beta \nabla^2 h - \left(\frac{3\beta - 1}{2} \nabla^2 h - \frac{C}{2} T_{00}\right)\right) = C T_{00} \implies \nabla^2 h_{00} + \frac{1-\beta}{2} \nabla^2 h = \frac{C}{2} T_{00}$$

Next we shall relate the classical gravitational potential φ with the perturbation in metric $h_{\mu\nu}$. From the geodesic equation:

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\nu\lambda} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau} = 0$$

In the Newtonian limit, $\tau \approx t$ and $\left|\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\right| \ll \frac{\mathrm{d}x^0}{\mathrm{d}\tau} \approx c$. We have

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + c^2 \Gamma^i_{00} = 0$$

where

$$\Gamma^i_{00} \approx \eta^{i\sigma} \left(\partial_0 h_{0\sigma} - \frac{1}{2} \partial_\sigma h_{00}\right) \approx -\frac{1}{2} \partial_i h_{00}$$

Comparing with the classical Newton second law:

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\partial_i \varphi$$

We find that (up to a constant)

$$\varphi = \frac{c^2}{2} h_{00}$$

Substituting back to the equation in (d):

$$\frac{2}{c^2} \nabla^2 \varphi + \frac{1-\beta}{2} \nabla^2 h = \frac{C}{2} \rho c^2$$

Recall that the Newtonian gravity satisfies the Poisson equation:

$$\nabla^2 \varphi = -4\pi G \rho$$

We find that $\beta = 1$ and $C = -\frac{16\pi G}{c^4}$. We have:

$$\Box h_{\mu\nu} - \left(\partial_\rho \partial_\mu h^\rho_\nu + \partial_\rho \partial_\nu h^\rho_\mu\right) + \partial_\mu \partial_\nu h - \eta_{\mu\nu} \left(\Box h - \partial_\rho \partial_\lambda h^{\rho\lambda}\right) = -16\pi G T_{\mu\nu}$$

This is exactly Equation (512) in the notes. In this way, we construct a linearised theory from scratch and obtain the same equation as does in directly linearise the Einstein field equation.

e) We assume that the Einstein tensor is homogeneous in the second derivatives of the metric. Then the field equation takes the form

$$R_{\mu\nu} + a g_{\mu\nu} R = b T_{\mu\nu}$$

for some constants a, b. We can expand the LHS as in Section 9.1 of the notes, and then compare the coefficients with the equation we obtain in (d). In this way we can determine the constants $a = -\frac{1}{2}$ and $b = -\frac{8\pi G}{c^4}$. $\Box$

Question 3. Coordinate sinuosities, the speed of gravitational radiation, and the harmonic gauge.

a) Recall the linear fully covariant curvature tensor:

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2} \left(\frac{\partial^2 h_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 h_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 h_{\lambda\kappa}}{\partial x^\mu \partial x^\nu} + \frac{\partial^2 h_{\mu\kappa}}{\partial x^\nu \partial x^\lambda}\right)$$

For a plane wave of the form $h_{\mu\nu} = A_{\mu\nu}\exp\left(ik_\rho x^\rho\right)$ travelling in vacuum, show that

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left(-k_\kappa k_\mu h_{\lambda\nu} + k_\kappa k_\lambda h_{\mu\nu} + k_\mu k_\nu h_{\lambda\kappa} - k_\nu k_\lambda h_{\mu\kappa}\right)$$

and that the linear vacuum field equation is

$$k_\kappa k^\rho \overline{h}_{\rho\mu} + k_\mu k^\rho \overline{h}_{\rho\kappa} - k^2 h_{\mu\kappa} = 0$$

where $\overline{h}_{\mu\nu} = h_{\mu\nu} - \eta_{\mu\nu}h/2$ and $k^2 = k^\rho k_\rho$. We do not yet assume that $k^2 = 0$, but shall try to deduce this.

b) Show that if $k^2 \neq 0$ then $R_{\lambda\mu\nu\kappa} = 0$. Yikes! *No curvature. A mere coordinate sinuosity propagating at the speed of thought.*

c) Finally, show that if we consider only disturbances propagating at the speed of light, then we must have $k^\rho \overline{h}_{\rho\sigma} = 0$. In other words, the harmonic gauge condition must be satisfied. You want gravitational radiation to travel at the speed of light and to actually produce curvature? No choice: use a harmonic gauge.

Proof. a) For the plane wave metric $h_{\mu\nu} = A_{\mu\nu}\exp(\mathrm{i}k_\rho x^\rho)$, we have

$$\partial_\lambda h_{\mu\nu} = \partial_\lambda\left(A_{\mu\nu}\exp(\mathrm{i}k_\rho x^\rho)\right) = A_{\mu\nu}\exp(\mathrm{i}k_\rho x^\rho)\partial_\lambda\left(\mathrm{i}k_\rho x^\rho\right) = \mathrm{i}k_\lambda h_{\mu\nu}$$

The type (0,4) Riemann curvature tensor is given by

$$R_{\lambda\mu\nu\kappa} = \partial_\kappa\partial_{[\mu}g_{\lambda]\nu} - \partial_\nu\partial_{[\mu}g_{\lambda]\kappa} = -k_\kappa k_{[\mu}h_{\lambda]\nu} + k_\nu k_{[\mu}h_{\lambda]\kappa} = \frac{1}{2}\left(-k_\kappa k_\mu h_{\lambda\nu} + k_\kappa k_\lambda h_{\mu\nu} + k_\nu k_\mu h_{\lambda\kappa} - k_\nu k_\lambda h_{\mu\kappa}\right)$$

The Ricci curvature tensor is given by

$$R_{\mu\kappa} = g^{\lambda\nu}R_{\lambda\mu\nu\kappa} \approx \eta^{\lambda\nu}\left(-k_\kappa k_{[\mu}h_{\lambda]\nu} + k_\nu k_{[\mu}h_{\lambda]\kappa}\right) = \frac{1}{2}\left(k_\lambda k_\kappa h^\lambda_\mu + k_\lambda k_\mu h^\lambda_\kappa - k_\kappa k_\mu h - k^2 h_{\mu\kappa}\right)$$

For the trace-reversed metric perturbation $\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$, we find that

$$k_\kappa k^\rho \overline{h}_{\rho\mu} + k_\mu k^\rho \overline{h}_{\rho\kappa} = k_\kappa k^\rho h_{\rho\mu} + k_\mu k^\rho h_{\rho\kappa} - k_\mu k_\kappa h$$

Therefore

$$R_{\mu\kappa} = k_\kappa k^\rho \overline{h}_{\rho\mu} + k_\mu k^\rho \overline{h}_{\rho\kappa} - k^2 h_{\mu\kappa}$$

The vacuum field equation $R_{\mu\kappa} = 0$ is simply

$$k_\kappa k^\rho \overline{h}_{\rho\mu} + k_\mu k^\rho \overline{h}_{\rho\kappa} - k^2 h_{\mu\kappa} = 0$$

b) We have a gauge freedom in that the geometry is invariant under the infinitesimal diffeomorphism ξ, that is, $g \mapsto g + \mathscr{L}_\xi\eta$ or $h_{\mu\nu} \mapsto h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$. As in the notes we can choose the harmonic gauge $\Box\xi_\mu = -\partial^\rho\overline{h}_{\mu\rho}$, which gives $\partial^\rho\overline{h}_{\mu\rho} = 0$. Hence we have

$$k^2 h_{\mu\kappa} = 0$$

If $k^2 \neq 0$, then $h_{\mu\kappa} = 0$, $g_{\mu\kappa} = \eta_{\mu\nu}$ and $R_{\lambda\mu\nu\kappa} = 0$. The spacetime is flat.

c) From (b) we know that $k^2 = 0$ for curvature-producing gravitational plane wave. Therefore the field equation becomes

$$k_\kappa k^\rho \overline{h}_{\rho\mu} + k_\mu k^\rho \overline{h}_{\rho\kappa} = 0$$

(I think this equation is insufficient to determine that $k^\rho\overline{h}_{\rho\mu} = 0$...)

□

Question 4. Radiation from a parabolic fly by.

The Peters-Mathews formula for the time-averaged gravitational wave luminosity of a binary system in an elliptical orbit (with semi-major axis a, masses m_1 and m_2, $M \equiv m_1 + m_2$, eccentricity e) is given by (c is now back in the equation):

$$\langle L_{GW} \rangle = \frac{32}{5} \frac{G^4}{c^5} \frac{m_1^2 m_2^2 M}{a^5} \left[\frac{1 + (73/24)\epsilon^2 + (37/96)\epsilon^4}{\left(1-\epsilon^2\right)^{7/2}} \right]$$

It's derivation is outlined in the notes (§9.6), or you may take it on perfect good faith from your humble instructor, however startling it may seem. Using this result, show that the total gravitational wave energy emitted by a single parabolic encounter between two bodies is

$$E_{GW} = \frac{85\pi\sqrt{2}}{24} \frac{G^{7/2} M^{1/2} m_1^2 m_2^2}{c^5 b^{7/2}}$$

where b is radius of closest approach. Recall that for a parabolic orbit, the radius r and aximuth ϕ are related by $r(1+\cos\phi) = L$, where $L = a\left(1-\epsilon^2\right)$ is the "semi-latus rectum," a constant. A parabola corresponds to the $\epsilon \to 1$ limit, with $a\left(1-\epsilon^2\right) = L$ finite. You may find the material in §6.8.1 useful.

Proof. □