Peize Liu
St. Peter's College
University of Oxford

Problem Sheet 4

Quantum Field Theory

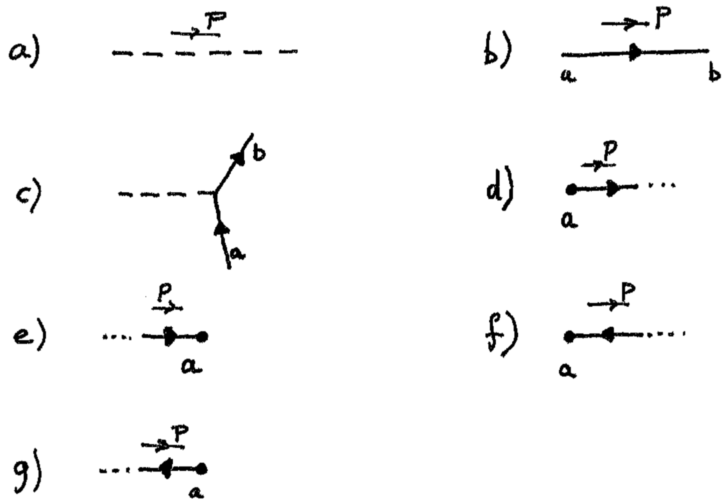
Class Tutor: Krishnendu Ray

7 January, 2022

This Problem Set is all to do with a field theory consisting of a scalar ϕ of mass m and a Dirac fermion ψ of mass m described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\left(\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2\right) + \overline{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi - g\phi\overline{\psi}\psi + \mathcal{L}_{\text{int}}(\phi)$$

Remember that γ^μ are four-by-four matrices and the spinor indices a, b run over $\{1, 2, 3, 4\}$ covering all the components. Fermion lines have an arrow on them which denotes the flow of particle number; the momentum in the propagator is the value of the momentum in the direction of the separate arrow. The Feynman rules are



a) Scalar propagator (internal line)

$$\frac{i}{p^2 - m^2 + i\epsilon}$$

b) Fermion propagator (internal line)

$$i(\not{p} - m + i\epsilon)^{-1}_{ba} = \left(\frac{i}{\not{p} - m + i\epsilon}\right)_{ba} = i\frac{(\not{p} + m)_{ba}}{p^2 - m^2 + i\epsilon}$$

c) Scalar fermion vertex $-ig\delta_{ab}$

d) Initial state fermion (external line) $u^s(p)_a$

e) Final state fermion (external line) $\overline{u}^s(p)_a$

f) Initial state anti-fermion (external line) $\overline{v}^s(p)_a$

g) Final state anti-fermion (external line) $v^s(p)_a$

h) Closed fermion loops have an extra factor -1, and have a Trace over the spinor indices.

Question 1. Gamma matrix manipulations.

Hard-core theorists will derive these results for themselves once in their life! However it is permissible to jump to question 2 and simply use these formulae! The γ^μ matrices are traceless and satisfy

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}\mathbb{I} \tag{2}$$

where $\mathbb{I}$ is the four-by-four identity matrix.

a) By contracting $p^\mu p^\nu$ with (2) show that $\not{p}\not{p} = p^2\mathbb{I}$.

b) By contracting $p^\mu q^\nu$ with (2) and taking the Trace show that

$$\text{Tr}\,\not{p}\not{q} = 4p \cdot q$$

c) Show that the matrix $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$ anticommutes with γ^μ and that $\gamma^5\gamma^5 = -\mathbb{I}$.

d) By considering $\operatorname{Tr}\gamma^5\gamma^5\gamma^\lambda\gamma^\mu\gamma^\nu$, and moving one of the γ^5 s to the right through the other matrices, and finally using the cyclic property of the Trace show that

$$\operatorname{Tr}\gamma^\lambda\gamma^\mu\gamma^\nu = 0$$

Generalise the argument to show that the Trace of any odd number of gamma matrices is zero.

e) Using (2) show that $\operatorname{Tr}\not{p}\not{q}\not{r}\not{s} = \operatorname{Tr}\not{p}\not{q}(2s\cdot r - \not{s}\not{r})$. Repeat the manipulation to "walk" $\not{s}$ through to the left hand end, then put it back at the right hand end by the cyclic property of the trace. Hence show that

$$\operatorname{Tr}\not{p}\not{q}\not{r}\not{s} = 4((p\cdot q)(s\cdot r) - (p\cdot r)(q\cdot s) + (p\cdot s)(q\cdot r))$$

Proof. a) We have, on the LHS:

$$(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)p_\mu p_\nu = \gamma^\mu\gamma^\nu(p_\mu p_\nu + p_\nu p_\mu) = 2\gamma^\mu\gamma^\nu p_\mu p_\nu = 2(\not{p})^2$$

On the RHS:

$$2\eta^{\mu\nu} I p_\mu p_\nu = 2p^2 I$$

Therefore (2) implies that $(\not{p})^2 = p^2 I$.

b) We have, on the LHS:

$$(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)p_\mu q_\nu = \gamma^\mu\gamma^\nu(p_\mu q_\nu + p_\nu q_\mu) = \not{p}\not{q} + \not{q}\not{p}$$

On the RHS:

$$2\eta^{\mu\nu} I p_\mu q_\nu = 2p\cdot qI$$

Therefore (2) implies that $\not{p}\not{q} + \not{q}\not{p} = (p\cdot q)I$. Taking the trace: $2\operatorname{tr}(\not{p}\not{q}) = \operatorname{tr}(\not{p}\not{q} + \not{q}\not{p}) = 8p\cdot q$. So $\operatorname{tr}(\not{p}\not{q}) = 4p\cdot q$.

c) We have $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ for $\nu \neq \mu$. Therefore

$$\gamma^5\gamma^\mu = \gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu = -\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^\mu\gamma^5$$

Hence $\{\gamma^5, \gamma^\mu\} = 0$. Next,

$$\begin{aligned}(\gamma^5)^2 &= \gamma^5\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 \\ &= -\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 = -\gamma^1\gamma^1\gamma^2\gamma^3\gamma^2\gamma^3 \\ &= \gamma^2\gamma^3\gamma^2\gamma^3 = -\gamma^2\gamma^2\gamma^3\gamma^3 \\ &= -I\end{aligned}$$

d) For odd n,

$$\begin{aligned}\operatorname{tr}(\gamma^{\alpha_1}\cdots\gamma^{\alpha_n}) &= -\operatorname{tr}(-\gamma^{\alpha_1}\cdots\gamma^{\alpha_n}) = -\operatorname{tr}(\gamma^5\gamma^5\gamma^{\alpha_1}\cdots\gamma^{\alpha_n}) \\ &= -(-1)^n\operatorname{tr}(\gamma^5\gamma^{\alpha_1}\cdots\gamma^{\alpha_n}\gamma^5) = \operatorname{tr}(\gamma^5\gamma^5\gamma^{\alpha_1}\cdots\gamma^{\alpha_n}) \\ &= -\operatorname{tr}(\gamma^{\alpha_1}\cdots\gamma^{\alpha_n})\end{aligned}$$

Hence $2\operatorname{tr}(\gamma^{\alpha_1}\cdots\gamma^{\alpha_n}) = 0$. Since $2 \neq 0$, we have $\operatorname{tr}(\gamma^{\alpha_1}\cdots\gamma^{\alpha_n}) = 0$.

e) From (2):

$$\operatorname{tr}(\not{p}\not{q}\not{r}\not{s}) = \operatorname{tr}(\not{p}\not{q}r_\mu s_\nu\gamma^\mu\gamma^\nu) = \operatorname{tr}(\not{p}\not{q}r_\mu s_\nu(2\eta^{\mu\nu}I - \gamma^\nu\gamma^\mu)) = \operatorname{tr}(\not{p}\not{q}(2(r\cdot s)I - \not{s}\not{r}))$$

Using part (b) and repeat the computation 2 more times:

$$\begin{aligned}\operatorname{tr}(\not{p}\not{q}\not{r}\not{s}) &= 8(p\cdot q)(r\cdot s) - \operatorname{tr}(\not{p}\not{q}\not{s}\not{r}) \\ &= 8(p\cdot q)(r\cdot s) - \operatorname{tr}(\not{p}(2(q\cdot s)I - \not{s}\not{q})\not{r}) \\ &= 8(p\cdot q)(r\cdot s) - 8(p\cdot r)(q\cdot s) + \operatorname{tr}(\not{p}\not{s}\not{q}\not{r}) \\ &= 8(p\cdot q)(r\cdot s) - 8(p\cdot r)(q\cdot s) + \operatorname{tr}((2(p\cdot s)I - \not{s}\not{p})\not{q}\not{r}) \\ &= 8(p\cdot q)(r\cdot s) - 8(p\cdot r)(q\cdot s) + 8(p\cdot s)(q\cdot r) - \operatorname{tr}(\not{s}\not{p}\not{q}\not{r}) \\ &= 8(p\cdot q)(r\cdot s) - 8(p\cdot r)(q\cdot s) + 8(p\cdot s)(q\cdot r) - \operatorname{tr}(\not{p}\not{q}\not{r}\not{s})\end{aligned}$$

Hence $\operatorname{tr}(\not{p}\not{q}\not{r}\not{s}) = 4(p\cdot q)(r\cdot s) - 4(p\cdot r)(q\cdot s) + 4(p\cdot s)(q\cdot r)$. □

Question 2. Scalar-fermion scattering.

2. Draw the Feynman diagram(s) for scattering of a scalar with four-momentum k off a fermion with four-momentum p, spin s, to final states described by k', p', s' respectively.

a) Show that the scattering matrix element can be written

$$\widetilde{M} = \widetilde{M}_1 + \widetilde{M}_2$$

where

$$\widetilde{M}_1 = -ig^2 \frac{\overline{u}^{s'}(p')\left(\not{k}' + 2m\right) u^s(p)}{S - m^2}, \quad \widetilde{M}_2 = -ig^2 \frac{\overline{u}^{s'}(p')\left(-\not{k}' + 2m\right) u^s(p)}{U - m^2}$$

where we define $S = (p+k)^2, T = (p'-p)^2$ and $U = (p'-k)^2$.

b) Use the results $\sum_s u_a^s(p)\overline{u}_b^s(p) = (\not{p} + m)_{ab}$ and $\sum_s v_a^s(p)\overline{v}_b^s(p) = (\not{p} - m)_{ab}$, together with the trace formulae you found in Q.1, to show that

$$A = \sum_{s,s'} \left|\widetilde{M}_1\right|^2 = \frac{g^4}{(S-m^2)^2} \times \left(\left(S + 2m^2\right)^2 + \left(U - 6m^2\right)^2 - T\left(T + 6m^2\right)\right)$$

where $T = (p'-p)^2$ and $U = (p'-k)^2$.

c) In an experiment the initial fermion spin direction is unknown, and the final fermion spin direction is not measured. Explain why $\frac{1}{2}\sum_{s,s'} |\widetilde{M}|^2$ is the correct quantity to insert in the formula for the cross-section.

Proof. The Feynman diagrams at tree level (*from the official solutions*):

a) According to the Feynman rules, we can write down the scattering amplitudes. For the s-channel:

$$\widetilde{M}_1 = (-\mathrm{i}g)^2 \frac{\overline{u}^{s'}(\boldsymbol{p}') \cdot \mathrm{i}(\not{p} + \not{k} + m)u^s(\boldsymbol{p})}{(p+k)^2 - m^2 + \mathrm{i}\varepsilon} = -\mathrm{i}g^2 \frac{\overline{u}^{s'}(\boldsymbol{p}')(\not{p} + \not{k} + m)u^s(\boldsymbol{p})}{S - m^2 + \mathrm{i}\varepsilon}$$

From the lectures we know that

$$(\not{p} - m)_{ab} u_b^s(\boldsymbol{p}) = \sum_{s'} \overline{v}_b^{s'}(\boldsymbol{p}) v_a^{s'}(\boldsymbol{p}) u_b^s(\boldsymbol{p}) = 0$$

which implies that $\not{p} u^s(\boldsymbol{p}) = m u^s(\boldsymbol{p})$. Substituting into the above equation:

$$\widetilde{M}_1 = -\mathrm{i}g^2 \frac{\overline{u}^{s'}(\boldsymbol{p}')(\not{k} + 2m)u^s(\boldsymbol{p})}{S - m^2 + \mathrm{i}\varepsilon}$$

For the u-channel:

$$\widetilde{M}_2 = (-\mathrm{i}g)^2 \frac{\overline{u}^{s'}(\boldsymbol{p}') \cdot \mathrm{i}(\not{p} - \not{k}' + m)u^s(\boldsymbol{p})}{(p-k')^2 - m^2 + \mathrm{i}\varepsilon} = -\mathrm{i}g^2 \frac{\overline{u}^{s'}(\boldsymbol{p}')(-\not{k}' + 2m)u^s(\boldsymbol{p})}{U - m^2 + \mathrm{i}\varepsilon}$$

b) We first note that, when taking complex conjugation,

$$\begin{aligned}(\overline{u}^{s'}(\boldsymbol{p}')\gamma^\mu u^s(\boldsymbol{p}))^\dagger &= u^s(\boldsymbol{p})^\dagger(\gamma^\mu)^\dagger(\gamma^0)^\dagger u^{s'}(\boldsymbol{p}') \\ &= (-1)^a u^s(\boldsymbol{p})^\dagger \gamma^\mu \gamma^0 u^{s'}(\boldsymbol{p}') \qquad (a = 1 \text{ if } \mu \neq 0 \text{ and } a = 0 \text{ if } \mu = 0)\end{aligned}$$

$$= (-1)^{2a} u^s(\boldsymbol{p})^\dagger \gamma^0 \gamma^\mu u^{s'}(\boldsymbol{p}')$$
$$= \overline{u}^s(\boldsymbol{p}) \gamma^\mu u^{s'}(\boldsymbol{p}')$$

Then the conjugate of M_1 is given by

$$M_1^\dagger = \frac{\mathrm{i}g^2}{S - m^2} \overline{u}^s(\boldsymbol{p})^\dagger (\not{k} + 2m) u^{s'}(\boldsymbol{p}')$$

Hence

$$\begin{aligned}
A = \sum_{s,s'} |M_1|^2 = \sum_{s,s'} M_1 M_1^\dagger &= \frac{g^4}{(S-m^2)^2} \overline{u}^{s'}(\boldsymbol{p}')(\not{k} + 2m) u^s(\boldsymbol{p}) \overline{u}^s(\boldsymbol{p})^\dagger (\not{k} + 2m) u^{s'}(\boldsymbol{p}') \\
&= \frac{g^4}{(S-m^2)^2} \operatorname{tr}\Big(\overline{u}^{s'}(\boldsymbol{p}')(\not{k} + 2m) u^s(\boldsymbol{p}) \overline{u}^s(\boldsymbol{p})^\dagger (\not{k} + 2m) u^{s'}(\boldsymbol{p}')\Big) \\
&= \frac{g^4}{(S-m^2)^2} \operatorname{tr}\Big((\not{k} + 2m) u^s(\boldsymbol{p}) \overline{u}^s(\boldsymbol{p})^\dagger (\not{k} + 2m) u^{s'}(\boldsymbol{p}') \overline{u}^{s'}(\boldsymbol{p}')\Big) \\
&= \frac{g^4}{(S-m^2)^2} \operatorname{tr}\big((\not{k} + 2m)(\not{p} + m)(\not{k} + 2m)(\not{p}' + m)\big) \\
&= \frac{g^4}{(S-m^2)^2} \operatorname{tr}\big(\not{k}\not{p}\not{k}\not{p}' + 2m^2 \not{k}\not{p} + m^2 \not{k}\not{k} + 2m^2 \not{k}\not{p}' + 2m^2 \not{p}\not{k} + 4m^2 \not{p}\not{p}' + 2m^2 \not{k}\not{p}' + 4m^4\big) \\
&= \frac{g^4}{(S-m^2)^2} \big(8(k\cdot p)(k\cdot p') - 4k^2(p\cdot p') + 16m^2(k\cdot p) + 16m^2(k\cdot p') + 4m^2k^2 + 16m^2(p\cdot p') + 16m^4\big)
\end{aligned}$$

Note that $p^2 = k^2 = m^2$. We have $p\cdot k = S/2 - m^2$, $p\cdot p' = m^2 - T/2$, and $p'\cdot k = m^2 - U/2$. Substituting into the above equation:

$$\begin{aligned}
A &= \frac{g^4}{(S-m^2)^2} \big(8(k\cdot p)(k\cdot p') + 16m^2(k\cdot p) + 16m^2(k\cdot p') + 12m^2(p\cdot p') + 20m^4\big) \\
&= \frac{g^4}{(S-m^2)^2} \big(2(S - 2m^2)(2m^2 - U) + 8m^2(S - 2m^2) + 8m^2(2m^2 - U) + 6m^2(2m^2 - T) + 20m^4\big) \\
&= \frac{g^4}{(S-m^2)^2} \big((S + 2m^2)^2 + (U - 6m^2)^2 - T(T + 6m^2)\big)
\end{aligned}$$

□

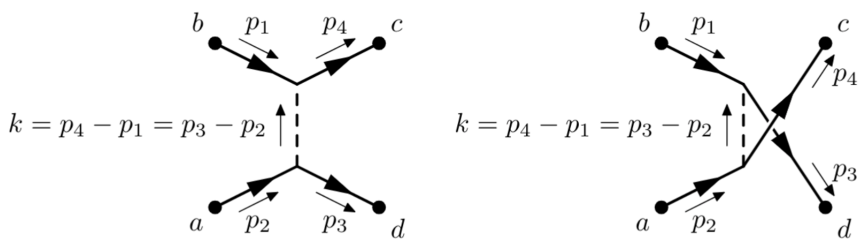
Question 3. Fermion-Fermion scattering.

Draw the Feynman diagram(s) for scattering of two fermions, with spin and four-momenta s_1, p_1 and s_2, p_2 respectively, to two final state fermions with spin and four-momenta s_3, p_3 and s_4, p_4.

a) Use the Feynman rules to write down the matrix element — but take care that these are identical fermions so the matrix element must be antisymmetric under $1 \leftrightarrow 2$, or $3 \leftrightarrow 4$.

b) Calculate $\sum_{s_1,s_2,s_3,s_4} |\widetilde{M}|^2$.

Proof. The Feynman diagrams at tree level (*from the official solutions, the momentum on the scalar propagator of the right hand side diagram should be* $p_3 - p_1 = p_4 - p_2$.):



a) For the t-channel, from Feynman rules we can write down the scattering amplitude:

$$M_1 = (-\mathrm{i}g)^2 \frac{\mathrm{i}\overline{u}^{s_4}(\boldsymbol{p_4}) u^{s_1}(\boldsymbol{p_1}) \overline{u}^{s_3}(\boldsymbol{p_3}) u^{s_2}(\boldsymbol{p_2})}{(T - m^2 + \mathrm{i}\varepsilon}$$

For the u-channel:

$$M_2 = (-\mathrm{i}g)^2 \frac{\mathrm{i}\overline{u}^{s_4}(\boldsymbol{p_4})u^{s_2}(\boldsymbol{p_2})\overline{u}^{s_3}(\boldsymbol{p_3})u^{s_1}(\boldsymbol{p_1})}{U - m^2 + \mathrm{i}\varepsilon}$$

Since the total amplitude is antisymmetric under the exchange of labels of identical fermions, we have

$$M = M_1 - M_2 = -\mathrm{i}g^2 \left(\frac{\overline{u}^{s_4}(\boldsymbol{p_4})u^{s_1}(\boldsymbol{p_1})\overline{u}^{s_3}(\boldsymbol{p_3})u^{s_2}(\boldsymbol{p_2})}{T - m^2} - \frac{\overline{u}^{s_4}(\boldsymbol{p_4})u^{s_2}(\boldsymbol{p_2})\overline{u}^{s_3}(\boldsymbol{p_3})u^{s_1}(\boldsymbol{p_1})}{U - m^2} \right)$$

b) For simplicity we write $u^a := u^{s_a}(\boldsymbol{p_a})$, $a \in \{1, 2, 3, 4\}$. Then

$$\begin{aligned}
\sum_{s_1,\ldots,s_4} |M_1|^2 &= \sum_{s_1,\ldots,s_4} \frac{g^4}{(T-m^2)^2} \overline{u}^4 u^1 \overline{u}^3 u^2 \overline{u}^2 u^3 \overline{u}^1 u^4 \\
&= \frac{g^4}{(T-m^2)^2} \sum_{s_1,\ldots,s_4} \overline{u}^1 u^4 \overline{u}^4 u^1 \overline{u}^3 u^2 \overline{u}^2 u^3 \\
&= \frac{g^4}{(T-m^2)^2} \sum_{s_1,\ldots,s_4} \mathrm{tr}\big(u^1 \overline{u}^1 u^4 \overline{u}^4\big)\, \mathrm{tr}\big(u^3 \overline{u}^3 u^2 \overline{u}^2\big) \\
&= \frac{g^4}{(T-m^2)^2} \mathrm{tr}\Big((\not{p}_1 + m)(\not{p}_4 + m)\Big)\, \mathrm{tr}\Big((\not{p}_3 + m)(\not{p}_2 + m)\Big) \\
&= \frac{16g^4}{(T-m^2)^2} (p_1 \cdot p_4 + m^2)(p_2 \cdot p_3 + m^2)
\end{aligned}$$

Similarly, we have

$$\sum_{s_1,\ldots,s_4} |M_2|^2 = \frac{16g^4}{(U-m^2)^2} (p_1 \cdot p_3 + m^2)(p_2 \cdot p_4 + m^2)$$

The cross term

$$\begin{aligned}
\sum_{s_1,\ldots,s_4} M_1 M_2^\dagger &= \frac{g^2}{(T-m^2)(U-m^2)} \sum_{s_1,\ldots,s_4} \overline{u}^4 u^1 \overline{u}^3 u^2 \overline{u}^1 u^3 \overline{u}^2 u^4 \\
&= \frac{g^2}{(T-m^2)(U-m^2)} \sum_{s_1,\ldots,s_4} \overline{u}^4 u^1 \overline{u}^1 u^3 \overline{u}^3 u^2 \overline{u}^2 u^4 \\
&= \frac{g^2}{(T-m^2)(U-m^2)} \sum_{s_1,\ldots,s_4} \mathrm{tr}\big(u^1 \overline{u}^1 u^3 \overline{u}^3 u^2 \overline{u}^2 u^4 \overline{u}^4\big) \\
&= \frac{g^2}{(T-m^2)(U-m^2)} \mathrm{tr}\Big((\not{p}_1 + m)(\not{p}_3 + m)(\not{p}_2 + m)(\not{p}_4 + m)\Big) \\
&= \frac{g^2}{(T-m^2)(U-m^2)} \left(\mathrm{tr}\Big(\not{p}_1 \not{p}_3 \not{p}_4 \not{p}_2\Big) + m^2 \sum_{i \neq j} \mathrm{tr}\Big(\not{p}_i \not{p}_j\Big) + 4m^4 \right) \\
&= \frac{4g^2}{(T-m^2)(U-m^2)} \left((p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) + m^2 \sum_{i \neq j} (p_i \cdot p_j) + m^4 \right)
\end{aligned}$$

Using the Mandelstam invariants:

$$p_1 \cdot p_2 = p_3 \cdot p_4 = \frac{S}{2} - m^2, \qquad p_2 \cdot p_3 = p_1 \cdot p_4 = m^2 - \frac{T}{2}, \qquad p_2 \cdot p_4 = p_1 \cdot p_3 = m^2 - \frac{U}{2}$$

Then

$$\sum |M_1|^2 = 4g^4 \left(\frac{4m^2 - T}{m^2 - T} \right)^2, \qquad \sum |M_2|^2 = 4g^4 \left(\frac{4m^2 - U}{m^2 - U} \right)^2$$

and

$$\begin{aligned}
\sum M_1 M_2^\dagger &= \frac{g^4}{(T-m^2)(U-m^2)} \left((2m^2 - U)^2 + (2m^2 - T)^2 - (2m^2 - S)^2 + 8m^2(S - m^2) + 4m^4 \right) \\
&= \frac{g^4}{(T-m^2)(U-m^2)} \left((2m^2 - U)^2 + (2m^2 - T)^2 - (2m^2 - U - T)^2 + 8m^2(S - m^2) + 4m^4 \right) \\
&= 2g^4 \frac{4m^2 S - TU}{(T-m^2)(U-m^2)}
\end{aligned}$$

Finally, the total scattering amplitude is given by

$$A = \sum_{s_1,\dots,s_4} |M_1 - M_2|^2 = \sum_{s_1,\dots,s_4} |M_1|^2 + \sum_{s_1,\dots,s_4} |M_2|^2 - 2 \sum_{s_1,\dots,s_4} M_1 M_2^\dagger$$
$$= 4g^4 \left(\left(\frac{4m^2 - T}{m^2 - T} \right)^2 + \left(\frac{4m^2 - U}{m^2 - U} \right)^2 - \frac{4m^2 S - TU}{(T - m^2)(U - m^2)} \right)$$

□

Question 4. Scalar decay.

Just for this part assume that the scalar has a different mass $\mu > 2m$ so that decay to a fermion anti-fermion pair, with spin and four-momenta s_1, p_1 and s_2, p_2 respectively, is kinematically allowed.

a) Draw the Feynman diagram for the decay and write down an expression for the matrix element $\widetilde{M}$.

b) Working in the rest frame of the scalar, calculate $\sum_{s_1,s_2} |\widetilde{M}|^2$. Explain why this is the relevant quantity for calculating the total decay rate..

c) The total decay rate Γ is given by

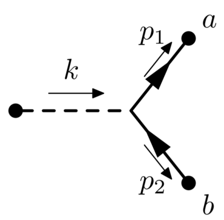
$$\Gamma = \frac{1}{2\mu} \int \frac{d^3\mathbf{p}_1}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}_1}} \int \frac{d^3\mathbf{p}_2}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}_2}} \sum_{s_1,s_2} |\widetilde{M}|^2 (2\pi)^4 \delta^4 (P - p_1 - p_2)$$

where $P = (\mu, 0, 0, 0)$. Find Γ.

Proof. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi - \mu^2 \phi^2 \right) + i\overline{\psi}(\not{\partial} - m)\psi + g\overline{\psi}\phi\psi + \mathcal{L}_{\text{int}}[\phi]$$

a) The Feynman diagram of the scalar decay (*from the official solution*):



From the Feynman rules, the S-matrix element is given by

$$M = -ig\overline{u}^{s_1}(\boldsymbol{p_1}) v^{s_2}(\boldsymbol{p_2})$$

b) Scattering amplitude:

$$\sum_{s_1,s_2} |M|^2 = g^2 \sum_{s_1,s_2} \overline{u}^{s_1}(\boldsymbol{p_1}) v^{s_2}(\boldsymbol{p_2}) \overline{v}^{s_2}(\boldsymbol{p_2}) u^{s_1}(\boldsymbol{p_1})$$
$$= g^2 \sum_{s_1,s_2} \operatorname{tr}(u^{s_1}(\boldsymbol{p_1}) \overline{u}^{s_1}(\boldsymbol{p_1}) v^{s_2}(\boldsymbol{p_2}) \overline{v}^{s_2}(\boldsymbol{p_2}))$$
$$= g^2 \operatorname{tr}\Big((\not{p}_1 + m)(\not{p}_2 - m) \Big)$$
$$= 4g^2 (p_1 \cdot p_2 - m^2)$$

The conservation of momentum $k = p_1 + p_2$ implies that

$$p_1 \cdot p_2 = \frac{1}{2}(k^2 - p_1^2 - p_2^2) = \frac{\mu^2}{2} - m^2$$

So the amplitude is given by

$$\sum_{s_1,s_2} |M|^2 = 2g^2(\mu^2 - 4m^2)$$

(*I don't think working in the rest frame of the scalar is relevant...*)

The equation given in (c) implies that the total decay rate is proportional to the amplitude.

c) Total decay rate:

$$\begin{aligned}
\Gamma &= \frac{1}{2\mu}\int\frac{\mathrm{d}^3\boldsymbol{p}_1}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}_1}}\int\frac{\mathrm{d}^3\boldsymbol{p}_2}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}_2}}\sum_{s_1,s_2}|M|^2(2\pi)^4\delta^4(P-p_1-p_2)\\
&= \frac{g^2(\mu^2-4m^2)}{(2\pi)^2\mu}\int\mathrm{d}^3\boldsymbol{p}_1\frac{1}{2E_{\boldsymbol{p}_1}}\int\mathrm{d}^3\boldsymbol{p}_2\frac{1}{2E_{\boldsymbol{p}_2}}\delta(\mu-E_{\boldsymbol{p}_1}-E_{\boldsymbol{p}_2})\delta^3(\boldsymbol{p}_1+\boldsymbol{p}_2)\\
&= \frac{g^2(\mu^2-4m^2)}{(2\pi)^2\mu}\int\mathrm{d}^3\boldsymbol{p}_1\frac{1}{(2E_{\boldsymbol{p}_1})^2}\delta(\mu-2E_{\boldsymbol{p}_1})\\
&= \frac{g^2(\mu^2-4m^2)}{4(2\pi)^2\mu}\int_0^\infty\mathrm{d}k\,4\pi k^2\frac{\delta(\mu-2\sqrt{k^2+m^2})}{k^2+m^2}\\
&= \frac{g^2(\mu^2-4m^2)}{4(2\pi)^2\mu}\left(\frac{4\pi k^2}{k^2+m^2}\cdot\frac{\sqrt{k^2+m^2}}{2k}\right)\Bigg|_{k=\frac{1}{2}\sqrt{\mu^2-4m^2}}\\
&= \frac{g^2(\mu^2-4m^2)^{3/2}}{8\pi\mu^2}
\end{aligned}$$

□

Question 5

Suppose that we start with a theory with $+\mathcal{L}_{\text{Int}}\,(\phi)=0$ ie no scalar self interaction terms.

a) Write down the Lagrangian including counterterms.

b) Draw the Feynman diagram for the one fermion loop correction to the scalar two point function and show that the 1PI part can be written

$$i\Pi(k) = -4g^2\int_0^1 dx\int\frac{d^4p}{(2\pi)^4}\left(\frac{1}{p^2-M^2}+\frac{2M^2}{(p^2-M^2)^2}\right)$$

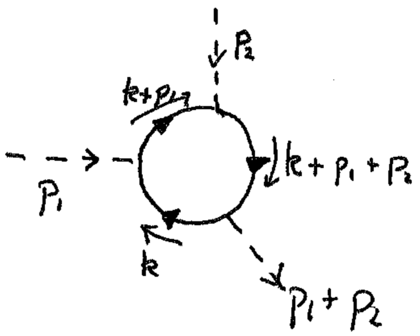
where $M^2=m^2-k^2x(1-x)$. Calculate the divergent parts of δ_m and δ_Z for the scalar field.

c) Draw the Feynman diagram for the one loop correction to the fermion two point function and show that the 1PI part can be written

$$i\Sigma(k) = g^2\int_0^1 dx\int\frac{d^4p}{(2\pi)^4}\frac{\not{k}(1-x)+m}{(p^2-M^2)^2}$$

where $M^2=m^2-k^2x(1-x)$. Calculate the divergent parts of δ_m and δ_Z for the fermion field. Compare these results with those for the scalar.

d) Consider the 1PI contribution to the scalar three point function.



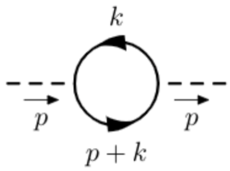
Write down the expression for this graph using the Feynman rules. Now focus on the leading powers of k; and using the Trace rules from Q.1 show that the k^3 term in the numerator vanishes. Hence show that this graph has a divergent part const $mg^3\log\left(\Lambda^2/m^2\right)$ (it is not necessary to find the full result). What is the implication of this result for $\mathcal{L}$?

e) Repeat the previous part for the scalar four point function and show that it has a divergent part const $g^4 \log\left(\Lambda^2/m^2\right)$.

Proof. a) The Lagrangian density with counterterms:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi(1+\delta_Z) - \frac{1}{2}(m^2+\delta_m)\phi^2 + \overline{\psi}(\mathrm{i}\partial\!\!\!/(1+\delta_{Z'}) - (m+\delta_{m'}))\psi - (g+\delta_g)\phi\overline{\psi}\psi$$

b) Feynman diagram (*from the official solutions*):



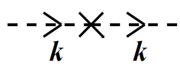
From Feynman rules the amputated Green's function is given by

$$\begin{aligned}
\Gamma(p) &= (-1)(-\mathrm{i}g)^2 \,\mathrm{tr}\left(\frac{\mathrm{i}(k\!\!\!/ + m)}{k^2-m^2+\mathrm{i}\varepsilon}\frac{\mathrm{i}(p\!\!\!/ + k\!\!\!/ + m)}{(p+k)^2-m^2+\mathrm{i}\varepsilon}\right) \\
&= -\frac{g^2}{(k^2-m^2+\mathrm{i}\varepsilon)((p+k)^2-m^2+\mathrm{i}\varepsilon)}\,\mathrm{tr}\big((k\!\!\!/ + m)(p\!\!\!/ + k\!\!\!/ + m)\big) \\
&= -\frac{4g^2(k\cdot(p+k)+m^2)}{(k^2-m^2+\mathrm{i}\varepsilon)((p+k)^2-m^2+\mathrm{i}\varepsilon)}
\end{aligned}$$

The contribution to fermion self-energy:

$$\begin{aligned}
\mathrm{i}\Pi(k) &= \int_{M^4}\frac{\mathrm{d}^4p}{(2\pi)^4}\Gamma(p) = -\frac{4g^2}{(2\pi)^4}\int_{M^4}\mathrm{d}^4p\,\frac{p\cdot(p+k)+m^2}{(p^2-m^2+\mathrm{i}\varepsilon)((p+k)^2-m^2\mathrm{i}\varepsilon)} \\
&= -\frac{4g^2}{(2\pi)^4}\int_{M^4}\mathrm{d}^4p\int_0^1\mathrm{d}s\,\frac{p\cdot(p+k)+m^2}{((p^2-m^2\mathrm{i}\varepsilon)(1-s)+((p+k)^2-m^2\mathrm{i}\varepsilon)s)^2} \\
&= -4g^2\int_0^1\mathrm{d}s\int_{M^4}\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{p\cdot(p+k)+m^2}{((p^2+sk)^2-(m^2-s(1+s)k^2)+\mathrm{i}\varepsilon)^2} \\
&= -4g^2\int_0^1\mathrm{d}s\int_{M^4}\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{(q-sk)\cdot(q+(1-s)k)+M^2+s(1+s)k^2}{(q^2-M^2+\mathrm{i}\varepsilon)^2} \qquad (q := p+sk,\ M^2 := m^2-s(1-s)k^2) \\
&= -4g^2\int_0^1\mathrm{d}s\int_{M^4}\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{q^2+M^2+2(1-2s)q\cdot k}{(q^2-M^2+\mathrm{i}\varepsilon)^2} \\
&= -4g^2\int_0^1\mathrm{d}s\int_{M^4}\frac{\mathrm{d}^4q}{(2\pi)^4}\left(\frac{1}{(q^2-M^2+\mathrm{i}\varepsilon)}+\frac{2M^2}{(q^2-M^2+\mathrm{i}\varepsilon)^2}\right) \qquad \text{(linear term vanishes due to symmetry)} \\
&= -4\mathrm{i}g^2\int_0^1\mathrm{d}s\int_{E^4}\frac{\mathrm{d}^4\ell}{(2\pi)^4}\left(-\frac{1}{q^2-M^2}+\frac{2M^2}{(q^2-M^2)^2}\right) \qquad \text{(Wick's rotation: } \ell := (\mathrm{i}q^0,q^1,q^2,q^3)) \\
&= 4\mathrm{i}g^2\int_0^1\mathrm{d}s\,(I_0(M^2)+2M^2I_1(M^2)) \\
&= 4\mathrm{i}g^2\int_0^1\mathrm{d}s\left(\frac{1}{16\pi^2}\left(\Lambda^2-M^2\log\frac{\Lambda^2}{M^2}\right)+\frac{2M^2}{16\pi^2}\left(1-\log\frac{\Lambda^2}{M^2}\right)\right) \\
&= \frac{\mathrm{i}g^2}{4\pi^2}\int_0^1\mathrm{d}s\left(\Lambda^2+2(m^2-s(1-s)k^2)-3(m^2-s(1-s)k^2)\log\frac{\Lambda^2}{m^2-s(1-s)k^2}\right) \qquad = \\
&= \frac{\mathrm{i}g^2}{4\pi^2}\left(\Lambda^2-3\left(m^2-\frac{1}{6}k^2\right)\log\frac{\Lambda^2}{m^2}\right)
\end{aligned}$$

Another diagram contributing to the self-energy at $\mathcal{O}(g^2)$ is given by (*from Srednicki p. 319, Figure 51.1*)



It contributes $\mathrm{i}k^2\delta_Z - \mathrm{i}\delta_m$ to the self energy. For $\Pi(k)$ to be finite at $\mathcal{O}(g^2)$, we must have

$$\frac{\mathrm{i}g^2}{4\pi^2}\left(\Lambda^2 - 3\left(m^2 - \frac{1}{6}k^2\right)\log\frac{\Lambda^2}{m^2}\right) + \mathrm{i}k^2\delta_Z - \mathrm{i}\delta_m < \infty$$

Therefore we have

$$\delta_Z = -\frac{g^2}{8\pi^2}\log\frac{\Lambda^2}{m^2} + \text{finite terms}; \qquad \delta_m = \frac{g^2}{4\pi^2}\left(\Lambda^2 - 3m^2\log\frac{\Lambda^2}{m^2}\right) + \text{finite terms.}$$

c)

d)

e)

□