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Problem Sheet 1

Quantum Field Theory

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Question 1. Dirac delta function

The Dirac delta function $\delta(x)$ is an infinite spike of weight 1 located at $x = 0$. It is really a distribution, not a regular function, and strictly only makes sense inside integrals - which means that the relationships discussed below are only supposed to be true inside integrals. For a function $f(x)$ that is sufficiently well-behaved in the region of $x = 0$,

$$I_f[\delta(x)] = \int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$$

By considering integrals of this form show that

a) $I_f[\delta(ax)] = |a|^{-1} I_f[\delta(x)]$;

b) $I_f[\delta(g(x))] = \sum_{n=1}^{N} \left|g'(x_n)\right|^{-1} I_f[\delta(x - x_n)]$, where $g(x)$ has zeroes at $\{x = x_n, n = 1 \ldots N\}$;

c) $I_f[\delta(x)] = \lim_{\epsilon->0} K I_f\left[\frac{\epsilon}{x^2+\epsilon^2}\right]$, and find the value of K. What limitations on $f(x)$ are necessary?

d) $I_f[\delta(x)] = \lim_{\epsilon->0} \hat{K} I_f\left[\epsilon^{-1} e^{-x^2/\epsilon^2}\right]$, and find the value of $\hat{K}$. What limitations on $f(x)$ are necessary?

Note that the results in c) and d) are often used in physics, but that the true Dirac delta function does not have the same limitations.

Proof. The commonly used domain for distributions is $\mathscr{D}(\mathbb{R}) = C_c^\infty(\mathbb{R})$, the space of compactly-supported smooth functions on an open set $U \subseteq \mathbb{R}$. Then all distributions are infinitely differentiable in the sense of distributions.

For clarity we use T_g to denote the regular distribution induced by the function g. That is, $T_g[f] := I_f[g]$. Likewise we denote $\delta[f] := I_f[\delta(x)]$.

It is worth noting that expressions like $\delta(g(x))$ is not well-defined in an obvious sense. One possible definition is via mollification. We need the following lemma:

Lemma 1. Approximating the δ-Function

Let ρ_ε be the *standard mollifiers* in $\mathbb{R}$. Then T_{ρ_ε} converges to δ in $\mathscr{D}'(\mathbb{R})$ as $\varepsilon \searrow 0$.

Proof. We know that the standard mollifiers are given by $\rho_\varepsilon(x) = \dfrac{1}{\varepsilon}\rho\left(\dfrac{x}{\varepsilon}\right)$, and $\displaystyle\int_{\mathbb{R}} \rho(x)\,\mathrm{d}x = 1$. For $f \in C_c^\infty(\mathbb{R})$, we have

$$T_{\rho_\varepsilon}[f] = \int_{\mathbb{R}} \frac{1}{\varepsilon}\rho\left(\frac{x}{\varepsilon}\right) f(x)\,\mathrm{d}x = \int_{\mathbb{R}} \rho(y) f(\varepsilon y)\,\mathrm{d}y$$

Note that $f(\varepsilon y) \to f(0)$ as $\varepsilon \searrow 0$, and $|\rho(y) f(\varepsilon y)|$ is integrable. By Dominated Convergence Theorem,

$$\lim_{\varepsilon \searrow 0} T_{\rho_\varepsilon}[f] = \int_{\mathbb{R}} \rho(y) f(0)\,\mathrm{d}y = f(0) = \delta[f] \qquad \square$$

In this way we **define** $\delta(g(x))$ to be the distributional limit of $T_{\rho_\varepsilon \circ g}$ as $\varepsilon \searrow 0$, provided it exists.

a)

$$\begin{aligned} I_f[\delta(ax)] &= \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \frac{1}{\varepsilon}\rho\left(\frac{ax}{\varepsilon}\right) f(x)\,\mathrm{d}x = |a|^{-1} \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \rho(x) f\left(\frac{\varepsilon x}{a}\right)\mathrm{d}x \\ &\overset{\text{(DCT)}}{=\!=} |a|^{-1} \int_{\mathbb{R}} \lim_{\varepsilon \searrow 0} \rho(x) f\left(\frac{\varepsilon x}{a}\right)\mathrm{d}x = |a|^{-1} \int_{\mathbb{R}} \rho(x) f(0)\,\mathrm{d}x = |a|^{-1} f(0) \\ &= |a|^{-1} I_f[\delta(x)] \end{aligned}$$

b) We assume that $g(x)$ *is piecewise invertible and the zeros* $x_1, \ldots, x_N$ *are simple.* Let $I_1, \ldots, I_m$ be the intervals on which g is invertible, with inverses $g_1^{-1},\dots,g_m^{-1}$.

$$\begin{aligned}I_f[\delta(g(x))] &= \lim_{\varepsilon\searrow 0}\int_{\mathbb{R}}\rho_\varepsilon(g(x))f(x)\,\mathrm{d}x = \lim_{\varepsilon\searrow 0}\sum_{i=1}^{m}\int_{I_i}\frac{1}{\varepsilon}\rho\left(\frac{g(x)}{\varepsilon}\right)f(x)\,\mathrm{d}x\\ &\overset{y=g(x)/\varepsilon}{=\!=\!=} \lim_{\varepsilon\searrow 0}\sum_{i=1}^{m}\int_{g(I_i)/\varepsilon}\rho(y)\cdot\frac{f\circ g_i^{-1}(\varepsilon y)}{\left|g'\circ g_i^{-1}(\varepsilon y)\right|}\,\mathrm{d}y\end{aligned}$$

For sufficiently well-behaved g[1], $\rho(y)\cdot\dfrac{f\circ g_i^{-1}(\varepsilon y)}{\left|g'\circ g_i^{-1}(\varepsilon y)\right|}\mathbf{1}_{g(I_i/\varepsilon)}$ is integrable. Note that

$$\lim_{\varepsilon\searrow 0}\rho(y)\cdot\frac{f\circ g_i^{-1}(\varepsilon y)}{\left|g'\circ g_i^{-1}(\varepsilon y)\right|}\mathbf{1}_{g(I_i/\varepsilon)} = \begin{cases}\rho(y)\cdot\dfrac{f(x_j)}{|g'(x_j)|}\mathbf{1}_{\mathbb{R}} & 0=g(x_j)\in g(I_i),\ x_j\in\{x_1,\dots,x_N\}\\ 0 & 0\notin g(I_i)\end{cases}$$

By Dominated Convergence Theorem, we have

$$I_f[\delta(g(x))] = \sum_{j=1}^{N}\int_{\mathbb{R}}\rho(y)\cdot\frac{f(x_j)}{|g'(x_j)|}\,\mathrm{d}y = \sum_{j=1}^{N}\frac{f(x_j)}{|g'(x_j)|} = \frac{I_f[\delta(x-x_j)]}{|g'(x_j)|}$$

c) The condition that $f\in C_c^\infty(\mathbb{R})$ is too strong for this part. Following the proof of Lemma 1 we give two conditions on f:

- $\dfrac{f(x)}{1+x^2}\in L^1(\mathbb{R})$;
- f is continuous at 0.

The proof is essentially identical to Lemma 1.

$$\lim_{\varepsilon\searrow 0}I_f\left[\frac{\varepsilon}{x^2+\varepsilon^2}\right] = \lim_{\varepsilon\searrow 0}\int_{\mathbb{R}}f(x)\frac{\varepsilon}{x^2+\varepsilon^2}\,\mathrm{d}x = \lim_{\varepsilon\searrow 0}\int_{\mathbb{R}}f(\varepsilon x)\frac{1}{x^2+1}\,\mathrm{d}x$$

By assumption $\dfrac{f(\varepsilon x)}{1+x^2}$ is integrable. By Dominated Convergence Theorem,

$$\lim_{\varepsilon\searrow 0}I_f\left[\frac{\varepsilon}{x^2+\varepsilon}\right] = \int_{\mathbb{R}}\lim_{\varepsilon\searrow 0}f(\varepsilon x)\frac{1}{x^2+1}\,\mathrm{d}x = f(0)\int_{\mathbb{R}}\frac{1}{1+x^2}\,\mathrm{d}x = f(0)\arctan x\Big|_{-\infty}^{+\infty} = \pi f(0) = \pi I_f[\delta(x)]$$

Hence we take $K=1/\pi$ and obtain that

$$I_f[\delta(x)] = \lim_{\varepsilon\searrow 0}\frac{1}{\pi}I_f\left[\frac{\varepsilon}{x^2+\varepsilon^2}\right]$$

d) The conditions on f:

- $f(x)\,\mathrm{e}^{-x^2}\in L^1(\mathbb{R})$;
- f is continuous at 0.

$$\lim_{\varepsilon\searrow 0}I_f\left[\varepsilon^{-1}\mathrm{e}^{-x^2/\varepsilon^2}\right] = \lim_{\varepsilon\searrow 0}\int_{\mathbb{R}}f(x)\varepsilon^{-1}\mathrm{e}^{-x^2/\varepsilon^2}\,\mathrm{d}x = \lim_{\varepsilon\searrow 0}\int_{\mathbb{R}}f(\varepsilon x)\,\mathrm{e}^{-x^2}\,\mathrm{d}x$$

By assumption $f(\varepsilon x)\,\mathrm{e}^{-x^2}$ is integrable. By Dominated Convergence Theorem,

$$\lim_{\varepsilon\searrow 0}I_f\left[\varepsilon^{-1}\mathrm{e}^{-x^2/\varepsilon^2}\right] = \int_{\mathbb{R}}\lim_{\varepsilon\searrow 0}f(\varepsilon x)\,\mathrm{e}^{-x^2}\,\mathrm{d}x = f(0)\int_{\mathbb{R}}\mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}f(0) = \sqrt{\pi}I_f[\delta(x)]$$

[1] We essentially invoke the DCT unjustified. Perhaps the best way is to treat the identity in (b) as **definition**.

Hence we take $K = 1/\pi$ and obtain that

$$I_f[\delta(x)] = \lim_{\varepsilon \searrow 0} \frac{1}{\sqrt{\pi}} I_f\left[\varepsilon^{-1} \mathrm{e}^{-x^2/\varepsilon^2}\right] \qquad \square$$

Question 2. Lorentz transformations

The four vectors $p^\mu = (E, \boldsymbol{p})$ and $p'^\mu = (E', \boldsymbol{p}')$ are related by a Lorentz transformation Λ. We frequently will have to deal with integrals of the form

$$\int F(p) d^4 p = \int F(p) dE d^3 \boldsymbol{p}$$

Find the 4×4 matrix representation of Λ explicitly for a boost β along the z-axis and the Jacobian for the change of variables $p^\mu \to p'^\mu$. Hence show that the 4-volume element $d^4 p$ is Lorentz invariant. Now, by making a suitable Lorentz invariant choice of $F(p)$, show that $(2E_{\boldsymbol{p}})^{-1} d^3 \boldsymbol{p}$ is Lorentz invariant if $E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$

Proof. The Lorentz boost along the z-axis is given by

$$\Lambda(\beta) = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

where $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$.

Note that $\Lambda(\beta)$ is a linear transformation on the Minkowski spacetime $(\mathbb{R}^4, \eta)$. The Jacobian is given by the determinant:

$$\det J = \det \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} = \gamma^2 - \beta^2\gamma^2 = 1$$

The volume form is given by

$$\mathrm{d}^4 p' = \det J \cdot \mathrm{d}^4 p = \mathrm{d}^4 p$$

so it is invariant under a Lorentz boost.

We consider the distribution $F(p) = F(E, \boldsymbol{p})$ defined by

$$F(p) = \delta(p^\mu p_\mu - m^2)\theta_0(E)$$

where θ_0 is the Heaviside step function. The delta measure is clearly Lorentz invariant as $p^\mu p_\mu$ is, and $\theta_0(E)$ is invariant under proper orthochronous Lorentz transformations. Then we have the Lorentz invariant scalar

$$\begin{aligned}\int F(p)\,\mathrm{d}E\,\mathrm{d}^3\boldsymbol{p} &= \int \delta(E^2 - \boldsymbol{p}^2 - m^2)\theta_0(E)\,\mathrm{d}E\,\mathrm{d}^3\boldsymbol{p} \\ &= \int \left(\frac{\delta(E - \sqrt{\boldsymbol{p}^2 + m^2})}{2E} - \frac{\delta(E + \sqrt{\boldsymbol{p}^2 + m^2})}{2E} \right) \theta_0(E)\,\mathrm{d}E\,\mathrm{d}^3\boldsymbol{p} \\ &= \int \frac{\delta(E - E_{\boldsymbol{p}})}{2E}\,\mathrm{d}E\,\mathrm{d}^3\boldsymbol{p} = \int \frac{1}{2E_{\boldsymbol{p}}}\,\mathrm{d}^3\boldsymbol{p}\end{aligned}$$

Therefore the 3-form $\dfrac{1}{2E_{\boldsymbol{p}}}\mathrm{d}^3\boldsymbol{p}$ is Lorentz invariant. $\square$

Question 3. Simple Harmonic Oscillator

(See Gasiorowicz Chapter 6 for annihilation and creation operators in the SHO) This problem combines a review of the harmonic oscillator with that of complex coordinates. In ordinary classical mechanics, consider a two-dimensional harmonic oscillator with Lagrangian

$$L = \frac{1}{2}m\left(\dot{q}_1{}^2 + \dot{q}_2{}^2\right) - \frac{1}{2}m\omega_0^2\left(q_1^2 + q_2^2\right)$$

where to get you in the field-theory mood I've labeled x and y as q_1 and q_2. Now rewrite these coordinates as

$$z \equiv \left(q_1 + i q_2\right)/\sqrt{2}, \quad z^* \equiv \left(q_1 - i q_2\right)/\sqrt{2}$$

a) Find the classical equations of motion in terms of q_1 and q_2.

b) Then rederive them in terms of z and z^*, not by just plugging into the preceding, but by rewriting L in terms of z and z^*, and then minimising the action under variations $z \to z + \delta z$ and $z^* \to z^* + \delta z^*$. Treat these variations as independent.

c) Find the Hamiltonian in terms of z, z^* and the corresponding canonical momenta.

d) Now write the quantum Hamiltonian using lowering and raising operators defined in the usual way; $a_1, a_1^\dagger$ for q_1 and $a_2, a_2^\dagger$ for q_2. Then rewrite in terms of

$$A \equiv (a_1 + i a_2)/\sqrt{2}, \quad B \equiv (a_1 - i a_2)/\sqrt{2}$$

e) Find the commutations relation for $A, A^\dagger, B$ and $B^\dagger$ from those for a_1 etc., and check that indeed they are an independent pair of raising/lowering operators.

Proof. a) Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1,2$$

Substituting L into the equation:

$$m\ddot{q}_i + m\omega_0^2 q_i = 0, \quad i = 1,2$$

The classical equations of motion are given by

$$\ddot{q}_1 + \omega_0^2 q_1 = 0, \qquad \ddot{q}_2 + \omega_0^2 q_2 = 0$$

b) We have

$$q_1 = \frac{z + z^*}{\sqrt{2}}, \qquad q_2 = \frac{z - z^*}{\sqrt{2}\mathrm{i}}$$

which gives

$$q_1^2 + q_2^2 = \left(\frac{z + z^*}{\sqrt{2}}\right)^2 + \left(\frac{z - z^*}{\sqrt{2}\mathrm{i}}\right)^2 = 2zz^*$$

In the new coordinates the Lagrangian is given by

$$L = m\dot{z}\dot{z}^* - m\omega_0^2 zz^*$$

Substituting into the Euler-Lagrange equations we obtain the equations of motion:

$$\ddot{z} + \omega_0^2 z = 0, \qquad \ddot{z}^* + \omega_0^2 z^* = 0$$

How do argue that z and z^* are independent?

c) The canonical momenta are given by

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}^*, \qquad p_z^* = \frac{\partial L}{\partial \dot{z}^*} = m\dot{z}$$

The Hamiltonian is given by the Legendre transformation:

$$H = \dot{z}p_z + \dot{z}^* p_z^* - L = 2m\dot{z}\dot{z}^* - (m\dot{z}\dot{z}^* - m\omega_0^2 zz^*) = m\dot{z}\dot{z}^* + m\omega_0^2 zz^* = \frac{p_z p_z^*}{m} + m\omega_0^2 zz^*$$

d) The classical Hamiltonian in the original coordinates is given by

$$H = \frac{p_1^2 + p_2^2}{2m} + \frac{1}{2}m\omega_0^2(q_1^2 + q_2^2)$$

In the canoncial quantisation, the phase space $\mathbb{R}^2$ corresponds to the Hilbert space $\mathrm{L}^2(\mathbb{R}^2)$. The quantum Hamiltonian operator is given by

$$\widehat{H} = \frac{\widehat{p}_1^2 + \widehat{p}_2^2}{2m} + \frac{1}{2}m\omega_0^2(\widehat{q}_1^2 + \widehat{q}_2^2)$$

with the canonical commutation relation

$$[\widehat{q}_i, \widehat{p}_j] = \mathrm{i}\hbar\delta_{ij}$$

From now on we only consider the quantum system and drop the hats for simplicity. The ladder operators are defined by

$$a_i = \frac{1}{\sqrt{2m\omega_0\hbar}}\left(m\omega_0 q_i + \mathrm{i}p_i\right), \qquad a_i^\dagger = \frac{1}{\sqrt{2m\omega_0\hbar}}\left(m\omega_0 q_i - \mathrm{i}p_i\right), \qquad i = 1,2$$

Then

$$a_i^\dagger a_i = \frac{1}{2m\omega_0\hbar}\left(m^2\omega_0^2 q_i^2 + p_i^2 + \mathrm{i}m\omega_0[q_i, p_i]\right) = \frac{1}{\hbar\omega_0}\left(\frac{1}{2}m\omega_0 q_i^2 + \frac{p_i^2}{2m}\right) - \frac{1}{2}$$

Therefore the Hamiltonian operator is given by

$$H = \hbar\omega_0\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right)$$

Using

$$a_1 = \frac{A+B}{\sqrt{2}}, \qquad a_2 = \frac{A-B}{\sqrt{2}\mathrm{i}}$$

We have

$$H = \hbar\omega_0\left(\frac{(A^\dagger + B^\dagger)(A+B)}{2} + \frac{(A^\dagger - B^\dagger)(A-B)}{2} + 1\right) = \hbar\omega_0\left(A^\dagger A + B^\dagger B + 1\right)$$

e) The commutation relations for the ladder operators are given by

$$[a_i, a_i^\dagger] = \frac{1}{2m\omega_0\hbar}\cdot 2m\omega_0\mathrm{i}[q_i, p_i] = 1, \qquad [a_i, a_j^\dagger] = 0, \qquad [a_i, a_j] = 0$$

Now we verify these relations for our new ladder operators:

$$[A, A^\dagger] = \left[\frac{a_1 + \mathrm{i}a_2}{\sqrt{2}}, \frac{a_1^\dagger - \mathrm{i}a_2^\dagger}{\sqrt{2}}\right] = \frac{1}{2}\left([a_1, a_1^\dagger] + [a_2, a_2^\dagger]\right) = 1$$

$$[B, B^\dagger] = \left[\frac{a_1 - \mathrm{i}a_2}{\sqrt{2}}, \frac{a_1^\dagger + \mathrm{i}a_2^\dagger}{\sqrt{2}}\right] = \frac{1}{2}\left([a_1, a_1^\dagger] + [a_2, a_2^\dagger]\right) = 1$$

$$[A,B] = \left[\frac{a_1 + \mathrm{i}a_2}{\sqrt{2}}, \frac{a_1 - \mathrm{i}a_2}{\sqrt{2}}\right] = 0$$

$$[A,B^\dagger] = \left[\frac{a_1 + \mathrm{i}a_2}{\sqrt{2}}, \frac{a_1^\dagger + \mathrm{i}a_2^\dagger}{\sqrt{2}}\right] = \frac{1}{2}\left([a_1,a_1^\dagger] - [a_2,a_2^\dagger]\right) = 0$$

Therefore $A, A^\dagger$ and $B, B^\dagger$ are indeed independent pairs of ladder operators. ✓ □

Question 4. Dirac wave equation

(See the notes on Relativistic Quantum Mechanics) A plane wave solution for the Dirac equation takes the form

$$\psi(t,\boldsymbol{x}) = e^{-\frac{i}{\hbar}(Et - \boldsymbol{p}\cdot\boldsymbol{x})}\psi_{\boldsymbol{p}}$$

where $\psi_{\boldsymbol{p}}$ is a 4-component Dirac spinor satisfying the eigenvalue equation

$$\left(\boldsymbol{p}.\boldsymbol{\alpha} + m\beta - EI_4\right)\psi_{\boldsymbol{p}} = 0 \tag{1}$$

with

$$I_4 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \alpha_i = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$$

where 0 and I represent 2×2 null and identity matrices respectively, and σ_i are the Pauli sigma matrices. Show explicitly that the determinant condition for solutions to (1) is

$$\left(E^2 - (\boldsymbol{p}^2 + m^2)\right)^2 = 0$$

and thus that two eigenspinors $\psi_{\boldsymbol{p}}$ have $E = \sqrt{\boldsymbol{p}^2 + m^2}$ and two have $E = -\sqrt{\boldsymbol{p}^2 + m^2}$. Show that the eigenspinors $\psi_{\boldsymbol{p}}$ can be chosen so that they are also eigenspinors of the spin projection operator

$$h = \begin{pmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{p} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\boldsymbol{p} \end{pmatrix}$$

Find all four independent eigenspinors of definite h for $\boldsymbol{p} = (0,0,p)$.

Proof. We compute the determinant for the eigenvalue equation:

$$\det\left(\boldsymbol{p}\cdot\boldsymbol{\alpha} + m\beta - EI_4\right) = \det\begin{pmatrix} -\boldsymbol{\sigma}\cdot\boldsymbol{p} - EI_2 & mI_2 \\ mI_2 & \boldsymbol{\sigma}\cdot\boldsymbol{p} - EI_2 \end{pmatrix}$$

Since mI_2 and $\boldsymbol{\sigma}\cdot\boldsymbol{p} - EI_2$ commute, we have in fact

$$\det\left(\boldsymbol{p}\cdot\boldsymbol{\alpha} + m\beta - EI_4\right) = \det\left((-\boldsymbol{\sigma}\cdot\boldsymbol{p} - EI_2)(\boldsymbol{\sigma}\cdot\boldsymbol{p} - EI_2) - m^2 I_2\right) = \det\left(-(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2 + (E^2 - m^2)I_2\right)$$

Note that

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2 = \sum_{i=1}^{3} p_i^2\sigma_i^2 = \sum_{i=1}^{3} p_i^2 I_2 = \boldsymbol{p}^2 I_2$$

We have

$$\det\left(\boldsymbol{p}\cdot\boldsymbol{\alpha} + m\beta - EI_4\right) = \det\left((E^2 - \boldsymbol{p}^2 - m^2)I_2\right) = (E^2 - \boldsymbol{p}^2 - m^2)^2$$ ✓

which is the characteristic polynomial of the operator $(\boldsymbol{p}\cdot\boldsymbol{\alpha} + m\beta)$. Note that $\boldsymbol{p}\cdot\boldsymbol{\alpha} + m\beta$ is symmetric and hence diagonalisable. The eigenvalues are $E_\pm = \pm\sqrt{\boldsymbol{p}^2 + m^2}$, each of which corresponds to an eigenspace of dimension two. ✓

For $h = \mathrm{diag}\{\boldsymbol{\sigma}\cdot\boldsymbol{p}, \boldsymbol{\sigma}\cdot\boldsymbol{p}\}$, it is clear that it commutes with $\boldsymbol{p}\cdot\boldsymbol{\alpha} = \mathrm{diag}\{-\boldsymbol{\sigma}\cdot\boldsymbol{p}, \boldsymbol{\sigma}\cdot\boldsymbol{p}\}$ and β. Hence h and $\boldsymbol{p}\cdot\boldsymbol{\alpha} + m\beta$ are simultaneously diagonalisable. In order words, we can choose a basis of eigenspinors $\{\psi_{\boldsymbol{p}}\}$ such that the spin projection operator is diagonalised.

For $\boldsymbol{p} = (0,0,p)$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we have

$$\boldsymbol{p}\cdot\boldsymbol{\alpha} + m\beta - E_\pm I_4 = \begin{pmatrix} -p-E_\pm & 0 & m & 0 \\ 0 & p-E_\pm & 0 & m \\ m & 0 & p-E_\pm & 0 \\ 0 & m & 0 & -p-E_\pm \end{pmatrix}$$

We can directly read out the eigenspaces:

$$V_\pm = \operatorname{span}\left\{(m,0,p+E_\pm,0)^\top, (0,m,0,-p+E_\pm)^\top\right\}$$

Since

$$\boldsymbol{\sigma}\cdot\boldsymbol{p} = p\,\mathrm{diag}\{1,-1,1,-1\}$$

it is automatically diagonalised in the basis chosen above. In summary the eigenspinors are given by

$$\psi^1 = \begin{pmatrix} m \\ 0 \\ p+\sqrt{p^2+m^2} \\ 0 \end{pmatrix}, \quad \psi^2 = \begin{pmatrix} 0 \\ m \\ 0 \\ -p+\sqrt{p^2+m^2} \end{pmatrix}, \quad \psi^3 = \begin{pmatrix} m \\ 0 \\ p-\sqrt{p^2+m^2} \\ 0 \end{pmatrix}, \quad \psi^4 = \begin{pmatrix} 0 \\ m \\ 0 \\ -p-\sqrt{p^2+m^2} \end{pmatrix}$$

□

Question 5. γ matrices

(1) Show that the two sets of γ matrices

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

and

$$\overline{\gamma}^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \overline{\gamma}^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

are unitarily equivalent ie there exists a unitary matrix U such that $\overline{\gamma}^\mu = U^\dagger \gamma^\mu U$.

(2) Show that there are no three-dimensional matrix representations of the algebra $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$ for 4-dimensional space-time i.e. $\mu,\nu \in \{0,1,2,3\}$.

Proof. (1) It is easier to work with Kronecker products. We rewrite the γ-matrices as

$$\gamma^0 = \sigma_1 \otimes I_2, \qquad \overline{\gamma}^0 = \sigma_3 \otimes I_2, \qquad \gamma^i = \overline{\gamma}^i = -\mathrm{i}\sigma_2 \otimes \sigma_i$$

We seek the unitary transformation of the form $U = \alpha\otimes\beta$, with $\alpha,\beta \in M_{2\times 2}(\mathbb{C})$. We may further demand that α,β are unitary. The condition that $\overline{\gamma}^\mu = U^\dagger\gamma^\mu U$ implies that

$$\sigma_3 \otimes I_2 = \alpha^\dagger \sigma_1 \otimes I_2, \qquad \sigma_2 \otimes \sigma_i = \alpha^\dagger \sigma_2 \alpha \otimes \beta^\dagger \sigma_i \beta$$

It is tempting to set $\beta = I_2$ and see if it works. We have now

$$\sigma_3 = \alpha^\dagger \sigma_1 \alpha, \qquad \sigma_2 = \alpha^\dagger \sigma_2 \alpha$$

$\{I_2,\sigma_1,\sigma_2,\sigma_3\}$ is a basis of $M_{2\times 2}(\mathbb{C})$. The second equation will hold trivially if α commutes with σ_2. In this simple case we have

$$\alpha = mI_2 + n\sigma_2, \qquad m, n \in \mathbb{C}$$

We substitute this into the first equation:

$$\begin{aligned}
\sigma_3 &= (\overline{m}I_2 + \overline{n}\sigma_2)\sigma_1(mI_2 + n\sigma_2) \\
&= |m|^2\sigma_1 + |n|^2\sigma_2\sigma_1\sigma_2 + \overline{n}m\sigma_2\sigma_1 + \overline{m}n\sigma_1\sigma_2 \\
&= \left(|m|^2 - |n|^2\right)\sigma_1 + \mathrm{Re}(\overline{m}n)\{\sigma_1, \sigma_2\} + \mathrm{Im}(\overline{m}n)\mathrm{i}[\sigma_1, \sigma_2] \\
&= \left(|m|^2 - |n|^2\right)\sigma_1 + 2\mathrm{i}\,\mathrm{Im}(\overline{m}n)\sigma_3
\end{aligned}$$

By comparing the coefficients we have

$$|m|^2 - |n|^2 = 0, \qquad \mathrm{Im}(\overline{m}n) = -\frac{1}{2}\mathrm{i}$$

We also note that α is unitary:

$$I_2 = \alpha^\dagger\alpha = (\overline{m}I_2 + \overline{n}\sigma_2)(mI_2 + n\sigma_2) = \left(|m|^2 + |n|^2\right)I_2 + 2\,\mathrm{Re}(\overline{m}n)\sigma_2$$

which implies that

$$|m|^2 + |n|^2 = 1, \qquad \mathrm{Re}(\overline{m}n) = 0$$

Hence $|m| = |n| = \frac{1}{\sqrt{2}}$, and $\overline{m}n = -\frac{\mathrm{i}}{2}$. We choose $m = \frac{1}{\sqrt{2}}$ and $n = -\frac{\mathrm{i}}{\sqrt{2}}$. Then $\alpha = \frac{1}{\sqrt{2}}(I_2 - \mathrm{i}\sigma_2)$. The unitary transformation is given by[2]

$$U = \frac{1}{\sqrt{2}}(I_2 - \mathrm{i}\sigma_2) \otimes I_2$$

In the matrix form,

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} I_2 & -I_2 \\ I_2 & I_2 \end{pmatrix}$$

(2) The one-line answer is that the γ-matrices generate the 16-dimensional Clifford algebra $\mathrm{Cl}_{1,3}(\mathbb{R})$ and hence cannot have a **faithful** representation in $\mathrm{GL}_3(\mathbb{R})$, which has dimension 9. □

[2]We essentially reach this result by guessing. A rigorous proof might require writing down more variables and constraint equations.