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Problem Sheet 1

Manifolds and Cobordisms

C3.12: Low-Dimensional Topology & Knot Theory

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Section A: Introductory

Question 1

Let M and N be manifolds with boundary. Show that

$$\partial(M \times N) = (\partial M \times N) \cup (M \times \partial N)$$

Question 2

Construct atlases on the topological manifolds: (a) S^n, (b) $\mathbb{RP}^n$, (c) $\mathbb{CP}^n$.

Section B: Core

Question 3

Recall that an n-dimensional (topological) manifold M is a topological space such that

1. each point of M has a neighbourhood homeomorphic to $\mathbb{R}^n$,
2. M is second countable (i.e., it has a countable basis of open sets), and
3. M is Hausdorff (i.e., different points have disjoint neighbourhoods).

Give examples of topological spaces that satisfy exactly two of conditions (1)-(3) for $n = 1$.

Proof. 1&2: Let X be a "line with two origins". That is, we define $X := (\mathbb{R} \times \{0,1\})/\sim$, where $(x,0) \sim (x,1)$ for all $x \neq 0$.

√

For each $x \in X$, if $x \neq (0,0)$, then $x \in X \setminus \{(0,0)\} \cong \mathbb{R}$. Hence x has a neighbourhood homeomorphic to $\mathbb{R}$. If $x = (0,0)$, $x \in X \setminus \{(0,1)\} \cong \mathbb{R}$, and x still has a neighbourhood homeomorphic to $\mathbb{R}$.

To show that X is second countable, note that $X \setminus \{(0,0)\} \cong \mathbb{R}$ has a countable basis $\mathscr{B}$, and $X \setminus \{(0,1)\} \cong \mathbb{R}$ has a countable basis $\mathscr{B}'$. Then $\mathscr{B} \cup \mathscr{B}'$ is a countable basis of X.

X is not Hausdorff because $(0,0)$ and $(0,1)$ does not have disjoint open neighbourhoods.

1&3: Let $\mathbb{R}'$ be $\mathbb{R}$ endowed with the discrete topology. Let $X := \mathbb{R} \times \mathbb{R}'$.

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For $(x,y) \in X$, $\mathbb{R} \times \{y\}$ is an open neighbourhood of (x,y), which is homeomorphic to $\mathbb{R}$. Since both $\mathbb{R}$ and $\mathbb{R}'$ are Hausdorff, X is also Hausdorff.

Suppose that $\mathscr{B}$ is a countable basis of X. For each $y \in \mathbb{R}'$, the subset $\mathbb{R} \times \{y\}$ is open in X. Then $\mathbb{R} \times \{y\} = \bigcup_i U_i$ for some $\{U_i\} \subseteq \mathscr{B}$. Then there exists some $U_y = I \times \{y\} \in \mathscr{B}$ with $I \subseteq \mathbb{R}$ open. In particular $U_x \neq U_y$ for $x \neq y$. Therefore we have a injection $\mathbb{R}' \to \mathscr{B}$. The basis $\mathscr{B}$ is uncountable. Contradiction. Hence X is not second countable.

2&3: Let $X = \mathbb{R}^2$.

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It is clear that X is Hausdorff (since it is metrisable). X has a countable basis given by

$$\left\{ B\left((q_1, q_2), \frac{1}{n} \right) \subseteq \mathbb{R}^2 : q_1, q_2 \in \mathbb{Q},\ n \in \mathbb{N} \right\}$$

Now suppose that each $x \in X$ has a neighbourhood homeomorphic to $\mathbb{R}$. Let U be a neighbourhood of $0 \in X$ such that $U \cong \mathbb{R}$. Then $U \setminus \{0\} \cong \mathbb{R} \setminus \{*\}$ for some $* \in \mathbb{R}$. Note that $U \setminus \{0\}$ is path-connected whereas $\mathbb{R} \setminus \{*\}$ is not. This is a contradiction. Hence X does not satisfy the condition (1). $\square$

Perfect! very detailed explanations

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Question 4

(a) Construct a cell decomposition of $\mathbb{RP}^n$.

(b) Use this to compute the homology group $H_*(\mathbb{RP}^n;\mathbb{Z})$.

(c) For what n is $\mathbb{RP}^n$ orientable?

(d) Compute $H_*(\mathbb{RP}^n;\mathbb{Z}_2)$.

(e) Compute the cohomology ring $H^*(\mathbb{RP}^n;\mathbb{Z}_2)$.

Proof. (a) We can construct a CW-complex structure of $\mathbb{R}P^n$ inductively as follows: Let $\mathbb{R}P^1 = S^1$. For each n, let $\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup_\varphi e^n$, where the attaching map $\varphi : S^{n-1} \to \mathbb{R}P^{n-1}$ is the double covering map.

(b) For each $k \leqslant n$, $\mathbb{R}P^n$ has exactly one k-cell. The cellcular chain complex is given by

$$\mathbb{Z} \longrightarrow \cdots \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z}$$

For $k \in \mathbb{Z}_+$, $\partial_k : H_n(\mathbb{R}P^k, \mathbb{R}P^{k-1}) \to H_{k-1}(\mathbb{R}P^{k-1}, \mathbb{R}P^{k-2})$ is determined by the degree of the map

$$\partial \mathbb{D}^k = S^{k-1} \xrightarrow{\varphi_k} \mathbb{R}P^{k-1} \xrightarrow{q_k} \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} \cong S^{k-1}$$

which is $\deg \mathrm{id} + \deg(-\mathrm{id}) = 1 + (-1)^k$. Hence $\partial_k = 2$ for even k and $\partial_k = 0$ for odd k.

The cellular chain complex is given by

$$\mathbb{Z} \longrightarrow \cdots \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

Taking the homology, we have

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2 & k = 1, 3, \ldots, n-1 \;\; (n \text{ even}) \\ 0 & \text{otherwise} \end{cases} \qquad H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & k = 0, n \\ \mathbb{Z}/2 & k = 1, 3, \ldots, n-2 \;\; (n \text{ odd}) \\ 0 & \text{otherwise} \end{cases}$$

(c) When n is even, $H_n(\mathbb{R}P^n) = 0$, and $\mathbb{R}P^n$ is non-orientable; when n is odd, $H_n(\mathbb{R}P^n) = \mathbb{Z}$, and $\mathbb{R}P^n$ is orientable.

(d) If working with $\mathbb{Z}/2$-modules, we have the cell~~c~~ular chain complex:

$$\mathbb{Z}/2 \longrightarrow \cdots \longrightarrow \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2$$

Taking the homology, we have

$$H_k(\mathbb{R}P^n; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & 0 \leqslant k \leqslant n \\ 0, & \text{otherwise} \end{cases}$$

(e) By universal coefficient theorem for cohomology, we have the short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}/2}(H_{k-1}(\mathbb{R}P^n; \mathbb{Z}/2), \mathbb{Z}/2) \longrightarrow H^k(\mathbb{R}P^n; \mathbb{Z}/2) \longrightarrow \mathrm{Hom}_{\mathbb{Z}/2}(H_k(\mathbb{R}P^n; \mathbb{Z}/2), \mathbb{Z}/2) \longrightarrow 0$$

Since $H_k(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$ is a projective $\mathbb{Z}/2$-module, the extensions module $\mathrm{Ext}^1_{\mathbb{Z}/2}(H_{k-1}(\mathbb{R}P^n; \mathbb{Z}/2), \mathbb{Z}/2) = 0$. Hence $H^k(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathrm{Hom}_{\mathbb{Z}/2}(H_k(\mathbb{R}P^n; \mathbb{Z}/2), \mathbb{Z}/2) \cong \mathrm{Hom}_{\mathbb{Z}/2}(\mathbb{Z}/2, \mathbb{Z}/2) \cong \mathbb{Z}/2$. The cohomology groups are

$$H^k(\mathbb{R}P^n; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2, & 0 \leqslant k \leqslant n \\ 0, & \text{otherwise} \end{cases}$$

For $k, \ell \in \mathbb{N}$ such that $k+\ell \leqslant n$, let ω_k, ω_ℓ be generators of $H^k(\mathbb{R}P^n;\mathbb{Z}/2)$ and $H^\ell(\mathbb{R}P^n;\mathbb{Z}/2)$ respectively. From the intersection theory, these forms arises naturally from the inclusions $\mathbb{R}P^{n-k} \subseteq \mathbb{R}P^n$ and $\mathbb{R}P^{n-\ell} \subseteq \mathbb{R}P^n$. Since all spaces involved are $\mathbb{Z}/2$-orientable, we have

$$\omega_k \smile \omega_\ell = \omega_{\mathbb{R}P^{n-k}} \smile \omega_{\mathbb{R}P^{n-\ell}} = \omega_{\mathbb{R}P^{n-k} \cap \mathbb{R}P^{n-\ell}} = \omega_{k+\ell} \in H^{k+\ell}(\mathbb{R}P^n;\mathbb{Z}/2)$$

Let x be a generator of $H^1(\mathbb{R}P^n;\mathbb{Z}/2)$. The the cohomology ring is given by

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$$H^\bullet(\mathbb{R}P^n;\mathbb{Z}/2) \cong \mathbb{Z}/2[x]/\langle x^{n+1}\rangle, \qquad |x| = 1$$

□

Again, everything is explained with perfect clarity

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Question 5

Show that the 1-jet spaces $J^1(\mathbb{R}, M) = \mathbb{R} \times TM$ and $J^1(M,\mathbb{R}) = T^*M \times \mathbb{R}$.

Proof. By definition,

$$\mathrm{J}^1(\mathbb{R}, M) = \bigsqcup_{t\in\mathbb{R}} \mathrm{J}^1_t(\mathbb{R}, M) = \bigsqcup_{t\in\mathbb{R}} \mathrm{C}^\infty(\mathbb{R}, M)/\{f \sim g \iff f(t) = g(t) \wedge f'(t) = g'(t)\}$$

For $q \in M$ and $v \in \mathrm{T}_q M$, let $f_{t,q,v} \in \mathrm{C}^\infty(\mathbb{R}, M)$ such that $f(t) = q$ and $f'(t) = v$. By definition $f_{t,q,v} = g_{t,q',v'} \in J^1_t(\mathbb{R}, M)$ if and only if $q = q'$ and $v = v'$. Then $(t, q, v) \mapsto (t, f_{t,q,v})$ is a well-defined map from $\mathbb{R} \times \mathrm{T}M$ to $\mathrm{J}^1(\mathbb{R}, M)$ with inverse given by $(t, f_{t,q,v}) \mapsto (t, q, v)$. It is clear that the following diagram commutes

given two fibre bundles $E \to B$, $E' \to B$ with homeomorphic fibres, you can always construct a fibre-preserving bijection $E \longrightarrow E'$, but unless this map and its inverse are continuous, you can't deduce that $E \simeq E'$ as fibre bundles.

$$\begin{tikzcd} \mathbb{R} \times \mathrm{T}M \arrow[rr] \arrow[dr, "\pi_t"'] & & \mathrm{J}^1(\mathbb{R}, M) \arrow[dl, "\pi_t"] \\ & \mathbb{R} & \end{tikzcd}$$

in particular, $(t,q,v) \mapsto (t, f_{t,q,v})$ is not evidently continuous (even though it is, in fact, continuous)

So this **defines** the 1-jet space $\mathrm{J}^1(\mathbb{R}, M)$ as a trivial vector bundle over $\mathbb{R}$. In this sense we have $\mathrm{J}^1(\mathbb{R}, M) = \mathbb{R} \times \mathrm{T}M$.

By definition,

$$\mathrm{J}^1(M,\mathbb{R}) = \bigsqcup_{q\in M} \mathrm{J}^1_q(M,\mathbb{R}) = \bigsqcup_{q\in M} \mathrm{C}^\infty(M,\mathbb{R})/\{f \sim g \iff f(q) = g(q) \wedge \mathrm{d}f|_q = \mathrm{d}g|_q\}$$

For $t \in \mathbb{R}$ and $\omega \in \mathrm{T}^*_q M$, let $f_{q,t,\omega} \in \mathrm{C}^\infty(M,\mathbb{R})$ such that $f(q) = t$ and $\mathrm{d}f|_q = \omega$. By definition $f_{q,t,\omega} = g_{q,t',\omega'} \in \mathrm{J}^1_q(M,\mathbb{R})$ if and only if $t = t'$ and $\omega = \omega'$. Then $(q, \omega, t) \mapsto (q, f_{q,t,\omega})$ is a well-defined map from $\mathrm{T}^*M \times \mathbb{R}$ to $\mathrm{J}^1(M,\mathbb{R})$ with inverse given by $(q, f_{q,t,\omega}) \mapsto (q, \omega, t)$. As both bundles are defined fibre-wise on M, it is clear that the following diagram commutes

same "issue" as above

$$\begin{tikzcd} \mathrm{T}^*M \times \mathbb{R} \arrow[rr] \arrow[dr, "\pi_q"'] & & \mathrm{J}^1(M,\mathbb{R}) \arrow[dl, "\pi_q"] \\ & M & \end{tikzcd}$$

the topology of jet spaces is tricky to work with, and you were not expected to prove the isomorphism of fibre bundles rigorously. I just wanted to point out that showing a fibre-preserving bijection is not enough in general

So this **defines** the 1-jet space $\mathrm{J}^1(M,\mathbb{R})$ as a vector bundle over M. In this sense we have $\mathrm{J}^1(M,\mathbb{R}) = \mathrm{T}^*M \times \mathbb{R}$. □

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Question 6

Prove that the cobordism group Ω_0^{SO} of compact oriented 0-manifolds is $\mathbb{Z}$.

Proof. Note that the 0-manifolds are discrete countable topological spaces, of which the compact ones are the finite topological spaces. Note that if X is a compact 0-manifold with cardinality n, then $H_0(X) \cong \mathbb{Z}^n$. By choosing the generators of H_0 at each connected component of X, we see that X has 2^n distinct orientations. Let $\varphi_X : X \to \{\pm 1\}$ be the map which choose the orientation at each point. If $X = \{x_1, ..., x_n\}$, then let $\phi_X := \sum_{i=1}^n \varphi_X(x_i)$.

- $\phi_{-X} = -\phi_X$: Obvious by definition.

- $\phi_{X\sqcup Y} = \phi_X + \phi_Y$: Obvious by definition.
- Any two oriented compact 0-manifolds X and Y are cobordant if and only if $\phi_X = \phi_Y$.

  " $\Longrightarrow$ ": Suppose that M is an oriented cobordism from X to Y. Then $\partial M = -X \sqcup Y$. Note that M is a compact orientable 1-manifold with boundary. Then M is a disjoint union of compact intervals and circles S^1. The boundary $\partial M = \{a_i, b_i : i = 1, ..., m\}$, with $\varphi_{\partial M}(a_i) = -1$ and $\varphi_{\partial M}(b_i) = 1$. Then $\phi_{\partial M} = \phi_{-X\sqcup Y} = 0$. Hence $\phi_X = -\phi_{-X} = -\phi_{-X\sqcup Y} + \phi_Y = \phi_Y$.

  " $\Longleftarrow$ ": Suppose that $\phi_X = \phi_Y$. Then $\phi_{-X\sqcup Y} = 0$. Let $x_1, ..., x_n \in -X \sqcup Y$ be the points with positive orientation, and $y_1, ..., y_n \in -X \sqcup Y$ be the point with negative orientation. Then we define $M = [0,1] \times \{1, ..., n\}$, which is a compact oriented 1-manifold with boundary $\{0,1\} \times \{1, ..., n\}$. We then define $\alpha : -X \sqcup Y \to \partial M$ by $\alpha(x_i) = (0, i)$ and $\alpha(y_i) = (1, i)$. Then M is a cobordism from $\varnothing$ to $-X \sqcup Y$, and hence a cobordism from X to Y.

perfect! A

Now we can conclude that ϕ defines a group isomorphism from Ω_0^{SO} to $\mathbb{Z}$. □

Question 7

Let $\psi : M \to N$ be diffeomorphism of smooth n-manifolds. Show that the cobordisms W_ψ and $W_{\psi'}$ are equivalent if and only if ψ and ψ' are pseudo-isotopic.

Proof. Suppose that W_ψ and $W_{\psi'}$ are equivalent. Let $\Phi : W_\psi \to W_{\psi'}$ be an equivalence of cobordism. We claim that $\Psi = (\psi' \times \mathrm{id}_I) \circ \Phi : M \times I \to N \times I$ is a pseudo-isotopy from ψ' to ψ. To see this, we can compute:

$$\Psi(x,0) = (\psi' \times \mathrm{id}_I) \circ (\phi_0')^{-1} \circ \phi_0(x,0) = (\psi' \times \mathrm{id}_I) \circ (\phi_0')^{-1}(x) = (\psi' \times \mathrm{id}_I)(x,0) = (\psi'(x), 0)$$
$$\Psi(x,1) = (\psi' \times \mathrm{id}_I) \circ (\phi_1')^{-1} \circ \phi_1(x,1) = (\psi' \times \mathrm{id}_I) \circ (\phi_0')^{-1}(\psi(x)) = (\psi' \times \mathrm{id}_I)((\psi')^{-1} \circ \psi(x), 1) = (\psi(x), 1)$$

√ Hence ψ and ψ' are pseudo-isotopic.

Conversely, suppose that $\Psi : M \times I \to N \times I$ is a pseudo-isotopy from ψ to ψ'. We claim that $\Phi := (\psi^{-1} \times \mathrm{id}_I) \circ \Psi : M \times I \to M \times I$ is an equivalence of cobordism from $W_{\psi'}$ to W_ψ.

$$\Phi(x,0) = (\psi^{-1} \times \mathrm{id}_I)(\psi(x), 0) = (x, 0) = \phi_0^{-1} \circ \phi_0'(x,0)$$
$$\Phi(x,1) = (\psi^{-1} \times \mathrm{id}_I)(\psi'(x), 1) = (\psi^{-1} \circ \psi'(x), 1) = \phi_1^{-1} \circ \phi_1'(x,1)$$

clear and to the point

√ Hence $\Phi|_{M\times\{i\}} = \phi_i^{-1} \circ \phi_i'$ for $i \in \{0,1\}$. The cobordisms W_ψ and $W_{\psi'}$ are equivalent. □

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Question 8

Let W be a cobordism from M_0 to M_1, and suppose that W, M_0, and M_1 are simply-connected. Show that the following are equivalent:

1. the embedding $e_0 : M_0 \hookrightarrow W$ is a homotopy equivalence,
2. $H_*(W, M_0) = 0$,
3. $H^*(W, M_1) = 0$,
4. $H_*(W, M_1) = 0$,
5. $e_1 : M_1 \hookrightarrow W$ is a homotopy equivalence.

Proof. Suppose that $\dim W = m + 1$ and $\dim M_0 = \dim M_1 = m$.

$1 \Longrightarrow 2$: Consider the long exact sequence of relative homology:

$$\longrightarrow H_n(M_0) \xrightarrow{(e_0)_n} H_n(W) \xrightarrow{\pi_n} H_n(W, M_0) \xrightarrow{\delta_n} H_{n-1}(M_0) \xrightarrow{(e_0)_{n-1}} H_{n-1}(W) \longrightarrow$$

Since the embedding $e_0 : M_0 \to W$ is a homotopy equivalence, it induces isomorphisms of homology groups $(e_0)_n : H_n(M_0) \to H_n(W)$ for each $n \in \mathbb{N}$. By exactness at $H_n(W)$ and at $H_{n-1}(M_0)$, we must have $\pi_n = 0$ and $\delta_n = 0$. By exactness at $H_n(W, M_0)$, we deduce that $H_n(W, M_0) \cong 0$.

$2 \Longrightarrow 3$: We need to use a generalised version of the Lefschetz duality. First we note that W is orientable since it is simply-connected. We quote Theorem 3.43 of *Hatcher*:

> *Suppose that W is a compact orientable $(m+1)$-manifold with $\partial W = M_0 \cup M_1$, where M_0, M_1 are compact m-manifolds such that $\partial M_0 = \partial M_1 = M_0 \cap M_1$. Then cap product with the fundamental class $[W] \in H_m(W, \partial W)$ gives isomorphisms $D_W : H^k(W, M_1) \to H_{m-k}(W, M_0)$ for all $k \in \mathbb{N}$.*

So $H_n(W, M_0) = 0$ for all $n \in \mathbb{N}$ implies that $H^n(W, M_1) = 0$ for all $n \in \mathbb{N}$.

$3 \Longrightarrow 4$: By the universal coefficient theorem for cohomology, we have the short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(W, M_1), \mathbb{Z}) \longrightarrow H^n(W, M_1) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_n(W, M_1), \mathbb{Z}) \longrightarrow 0$$

Since $H^n(W, M_1) = 0$ for all $n \in \mathbb{N}$, we have $\mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(W, M_1), \mathbb{Z}) = 0$ and $\mathrm{Hom}_{\mathbb{Z}}(H_n(W, M_1), \mathbb{Z}) = 0$ for $n \in \mathbb{N}$. Since M_1 and W are compact, they have finitely generated homology groups. From the long exact sequence of relative homology, it is easy to prove that the relative homology groups $H_n(W, M_1)$ are also finitely generated (the proof is similar to a step in Question 10 of Sheet 4 of *C3.1 Algebraic Topology*).

By the structure theorem for finitely generated Abelian groups, $H_n(W, M_1) = \mathbb{Z}^{k_n} \oplus T_n$, where T_n is the torsion subgroup of $H_n(W, M_1)$. Then by elementary homological algebra,

$$0 = \mathrm{Ext}^1_{\mathbb{Z}}(H_n(W, M_1), \mathbb{Z}) = T_n, \qquad 0 = \mathrm{Hom}_{\mathbb{Z}}(H_n(W, M_1), \mathbb{Z}) = \mathbb{Z}^{k_n}$$

Hence $H_n(W, M_1) = 0$ for all $n \in \mathbb{N}$ as claimed.

$4 \Longrightarrow 5$: For this step we need some homotopy theory, which I think is not covered in *C3.1 Algebraic Topology*.

Since W and M_1 are smooth, by Proposition 1.6.5 they have handle decompositions. In particular they are CW-complexes. By the relative Hurewicz Theorem, the Hurewicz map h is a morphism between the long exact sequences of relative homotopy groups and relative homology groups:

$$\begin{tikzcd}
{} \arrow[r] & \pi_{n+1}(W, M_1) \arrow[r] \arrow[d, "h"] & \pi_n(M_1) \arrow[r] \arrow[d, "h"] & \pi_n(W) \arrow[r] \arrow[d, "h"] & \pi_n(W, M_1) \arrow[r] \arrow[d, "h"] & {} \\
{} \arrow[r] & H_{n+1}(W, M_1) \arrow[r] & H_n(M_1) \arrow[r] & H_n(W) \arrow[r] & H_n(W, M_1) \arrow[r] & {}
\end{tikzcd}$$

Since (M_1, W) is a pair of simply connected spaces, and $H_n(W, M_1) = 0$ for all $n \in \mathbb{N}$, then $h : \pi_n(W, M_1) \to H_n(W, M_1)$ is an isomorphism, and hence $\pi_n(W, M_1) = 0$ for all $n \in \mathbb{N}$. Therefore the embedding $e_1 : M_1 \to W$ induces isomorphisms of homotopy groups $\pi_n(M_1) \to \pi_n(W)$ for each $n \in \mathbb{N}$. By Whitehead's Theorem, e_1 is a homotopy equivalence.

$5 \Longrightarrow 1$: We can simply swap the labels of M_0 and M_1. Then the above sequence of arguments $1 \Longrightarrow 2 \Longrightarrow 3 \Longrightarrow 4 \Longrightarrow 5$ becomes $5 \Longrightarrow 4 \Longrightarrow 3' \Longrightarrow 2 \Longrightarrow 1$, where $3'$ is the statement that $H^\bullet(W, M_0) = 0$. This finishes the proof. □

I'm running out of compliments, but very well done!

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Section C: Optional

Question 9

Let $f : M \to N$ be a submersion.

(a) Show that if f is proper; i.e., $f^{-1}(K)$ is compact for every $K \subseteq N$ compact, then M is a fibre bundle over N with fibre $f^{-1}(\{y\})$ for $y \in N$.

(b) Give a counterexample when f is not proper.