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**Problem Sheet 1**

# C2.1: Lie Algebras

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Throughout this sheet we assume that all Lie algebras are over a field $\mathsf{k}$.

**Question 1**

Show that $\mathfrak{sl}_2(\mathbb{C})$ is a simple Lie algebra, i.e. its only ideals are 0 and itself.

*Proof.* $\mathfrak{sl}_2(\mathbb{C})$ is the space of $2\times 2$ traceless matrices. So $\dim\mathfrak{sl}_2(\mathbb{C}) = 3$. We can write down a set of basis matrices:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The commutators between these matrices are given by

$$[A,B] = 2B, \qquad [B,C] = A, \qquad [C,A] = 2C$$

Suppose that $I$ is a non-zero ideal of $\mathfrak{sl}_2(\mathbb{C})$. Let $X = aA + bB + cC \in I\setminus\{0\}$. Then we have

$$[B,[A,X]] = [B, 2bB - 2cC] = -2cA \in I$$

If $c \neq 0$, then $A \in I$. It follows that $B = \dfrac{1}{2}[A,B] \in I$ and $C = \dfrac{1}{2}[C,A] \in I$. Hence $I = \mathfrak{sl}_2(\mathbb{C})$.

If $c = 0$, then $X = aA + bB$. We have $[B,X] = -2aB \in I$. If $a = 0$, then we must have $b \neq 0$ and hence $B \in I$. As above we have $I = \mathfrak{sl}_2(\mathbb{C})$. If $a \neq 0$, then $B \in I$, we also have $I = \mathfrak{sl}_2(\mathbb{C})$.

In conclusion, $\mathfrak{sl}_2(\mathbb{C})$ is a simple Lie algebra. □

**Question 2**

Let $S$ be an $n\times n$ matrix with entries in a field $\mathsf{k}$. Define

$$\mathfrak{gl}_S = \{x \in \mathfrak{g} : x^t S + Sx = 0\}$$

1. Show that $\mathfrak{gl}_S$ is a Lie subalgebra of $\mathfrak{gl}_n$.
2. Let $J_n$ be the $n\times n$-matrix:

$$J_n = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix}$$

   Now let $S$ be the $2n\times 2n$ matrix:

$$S = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}$$

   Find the conditions for a matrix to lie in $\mathfrak{gl}_S$ and hence determine the dimension of $\mathfrak{gl}_S$.

*Proof.* 1. It is trivial linear algebra that $\mathfrak{gl}_S$ is a $\mathsf{k}$-subspace of $\mathfrak{gl}_n$. We need to show that $\mathfrak{gl}_S$ is closed under the commutator.

For $x, y \in \mathfrak{gl}_S$, we have

$$\begin{aligned}
[x,y]^\top S + S[x,y] &= (y^\top x^\top - x^\top y^\top)S + S(xy - yx) \\
&= (y^\top x^\top S - Syx) - (x^\top y^\top S - Sxy) \\
&= y^\top(x^\top S - Sx) + (y^\top S - Sy)x - x^\top(y^\top S - Sy) + (x^\top S - Sx)y \\
&= 0
\end{aligned}$$

Hence $\mathfrak{gl}_S$ is a Lie subalgebra of $\mathfrak{gl}_n$.

2. First we suppose that $\operatorname{char}\Bbbk \neq 2$.

   Note that $J_n$ is symmetric, and $S$ is skew-symmetric. Then

   $$x \in \mathfrak{gl}_S \iff x^\top S + Sx = 0 \iff -(Sx)^\top + Sx = 0 \iff Sx \text{ is symmetric}$$

   Moreover,

   $$S^2 = \begin{pmatrix} -J_n^2 & O \\ O & -J_n^2 \end{pmatrix} = -\mathrm{id}_{2n}$$

   In particular $S$ is invertible, not only as a matrix, but as a linear map $S : x \mapsto Sx$. Let $\mathrm{Sym}_{2n}$ be the $\Bbbk$-subspace of $\mathfrak{gl}_n$ of symmetric matrices. Then we have $\mathfrak{gl}_S = S^{-1}(\mathrm{Sym}_{2n}) = S(\mathrm{Sym}_{2n}) = \{Sx\colon x \text{ is symmetric}\}$. The dimension

   $$\dim \mathfrak{gl}_S = \dim \mathrm{Sym}_{2n} = \frac{2n(2n+1)}{2} = n(2n+1)$$ □

**Question 3**

Classify all Lie algebras $\mathfrak{g}$ with $\dim(\mathfrak{g}) = 3$ and $\mathfrak{z}(\mathfrak{g}) \neq 0$.

*Proof.* The centre $\mathfrak{z}(\mathfrak{g})$ is an ideal of $\mathfrak{g}$.

- $\dim \mathfrak{z}(\mathfrak{g}) = 1$.

  Let $\{u\}$ be a basis of $\mathfrak{z}(\mathfrak{g})$ and we extend it to a basis $\{u, v, w\}$ of $\mathfrak{g}$. We can write down the Lie brackets

  $$[u, v] = [u, w] = 0, \qquad x := [v, w]$$

  If $x = au + bv + cw$ for $a, b, c \in k$, then:

  $$[u, x] = 0, \qquad [v, x] = c[v, w] = cx, \qquad [w, x] = b[w, v] = -bx$$

  It is clear that $x \neq 0$ for otherwise $\mathfrak{g}$ would be Abelian.

  - $x \in \mathfrak{z}(\mathfrak{g})$:

    We have $x = du$ for some $d \in k \setminus \{0\}$. We can replace $v$ by $d^{-1}v$ and obtain the Lie brackets for basis vectors:

    $$[u, v] = 0, \qquad [v, w] = u, \qquad [w, u] = 0$$

  - $x \notin \mathfrak{z}(\mathfrak{g})$:

    We note that $\mathrm{span}_k\{x\}$ is an ideal of $\mathfrak{g}$. We extend $\{u, x\}$ to a basis $\{u, x, y\}$ of $\mathfrak{g}$. Then $[x, y] \neq 0$ for otherwise $x \in \mathfrak{z}(\mathfrak{g})$. We can rescale $y$ such that $[x, y] = x$. The resulting Lie brackets for basis vectors:

    $$[u, x] = 0, \qquad [x, y] = x, \qquad [y, u] = 0$$

- $\dim \mathfrak{z}(\mathfrak{g}) = 2$.

  Suppose that $\mathfrak{z}(\mathfrak{g})$ has a basis $\{u, v\}$ and it extends to a basis $\{u, v, w\}$ of $\mathfrak{g}$. Then $[w, au + bv + cw] = 0$ and hence $w \in \mathfrak{z}(\mathfrak{g})$. Contradiction. In general, a Lie algebra cannot have a centre of codimensional 1.

- $\dim \mathfrak{z}(\mathfrak{g}) = 3$.

This is the case where $\mathfrak{g}$ is Abelian. If $\{u, v, w\}$ is a basis, then

$$[u, v] = 0, \qquad [v, w] = 0, \qquad [w, u] = 0$$

In summary, there are 3 non-isomorphic 3-dimensional Lie algebras with non-zero kernel. □

**Question 4. The classical groups**

In this course a fundamental role is played by the classical groups. In this question they will be defined, we will calculate their dimensions and we will look at some small dimensional examples. Assume throughout that $\Bbbk$ is a field of characteristic $\neq 2$.

a) (*The special linear group* $\mathfrak{sl}_n$) Recall that $\mathfrak{sl}_n \subseteq \mathfrak{gl}_n$ denotes the subspace of traceless $n \times n$-matrices. Check that $\mathfrak{sl}_n$ is a subalgebra of $\mathfrak{gl}_n$ by verifying $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$ for $n \times n$-matrices $a, b \in \mathfrak{gl}_n$ Calculate the dimension of $\mathfrak{sl}_n$.

b) (*The special orthogonal group* $\mathfrak{so}_n$) Recall the definition of $\mathfrak{gl}_S$ and $J_n$ from Question 2. Consider the matrix

$$S = J_n$$

We define $\mathfrak{so}_n$ to be $\mathfrak{gl}_S$. Find conditions for a matrix to belong to $\mathfrak{so}_n$ and hence calculate its dimension.

c) (*The symplectic group* $\mathfrak{sp}_{2n}$) Consider the matrix

$$S = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}$$

We define $\mathfrak{sp}_{2n}$ to be $\mathfrak{gl}_S$. You already calculated its dimension in question 3. Give an explicit description of $\mathfrak{sp}_2$ in terms of another Lie algebra occuring on the list above.

d) Show that $\mathfrak{so}_2$ is abelian and that $\mathfrak{sl}_2 \cong \mathfrak{so}_3$.

*Proof.* a) We note that $\mathrm{tr} : \mathfrak{gl}_n \to \Bbbk$ is a linear map. For $A, B \in \mathfrak{gl}_n$, we have

$$\mathrm{tr}[A, B] = \mathrm{tr}(AB - BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = \sum_{k=1}^{n} \sum_{j=1}^{n} A_{kj} B_{jk} - \sum_{k=1}^{n} \sum_{j=1}^{n} B_{kj} A_{jk} = 0$$

Hence $[A, B] \in \mathfrak{sl}_n$. $\mathfrak{sl}_n$ is an ideal of $\mathfrak{gl}_n$. By first isomorphism theorem, $\dim \mathfrak{sl}_n = \dim \mathfrak{gl}_n - \dim k = n^2 - 1$.

b) We note that $S = S^\top$ and $S^2 = \mathrm{id}_2$. The same argument in Question 2 shows that $x \in \mathfrak{so}_n$ if and only if $Sx$ is skew-symmetric, and therefore

$$\mathfrak{so}_n = S(\mathrm{Skew}_n) = \{J_n x \colon x \text{ is skew-symmetric}\}$$

The dimension is given by

$$\dim \mathfrak{so}_n = \dim \mathrm{Skew}_n = \frac{n(n-1)}{2}$$

c) We use our result in Question 2. For $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$x \in \mathfrak{sp}_2 \iff Sx \text{ is symmetric} \iff \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ -a & -b \end{pmatrix} \text{ is symmetric} \iff x = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \iff x \in \mathfrak{gl}_2$$

Hence $\mathfrak{sp}_2 = \mathfrak{gl}_2$. They are the same subalgebra of $\mathfrak{gl}_2$.

d) We note that $\dim \mathfrak{so}_2 = 1$, which is in fact a field. The Lie algebra is Abelian.

For $\mathfrak{so}_3$, we first write down a basis for $\mathrm{Skew}_3$. The canonical choice is

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which satisfies $[L_i, L_j] = \sum_{k=1}^{3} \varepsilon_{ijk} L_k$.

$J_3$ maps $\{L_1, L_2, L_3\}$ to a basis of $\mathfrak{so}_3$:

$$S_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad S_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

which satisfies $[S_i, S_j] = \sum_{k=1}^{3} \varepsilon_{ijk} S_k$.

Suppose that $\mathrm{k}$ contains a primitive fourth root of unity, i.e. $\mathrm{i} = \sqrt{-1}$. We consider the ladder operators $S_\pm := \mathrm{i} S_2 \pm S_1$, and a rescaled $S_3' = 2\mathrm{i} S_3$. They satisfy the relations

$$[S_3', S_\pm] = \pm 2 S_\pm, \qquad [S_+, S_-] = S_3'$$

Now we have a Lie algebra isomorphism from $\mathfrak{so}_3$ to $\mathfrak{sl}_2$ generated by $S_+ \mapsto B$, $S_- \mapsto C$, and $S_3' \mapsto A$, where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

(*I am unclear how to construct the isomorphism if* $\mathrm{k}$ *does not contain a square root of* $-1$.) □

**Question 5**

Let $\mathrm{k}$ be an arbitrary field.

i) For $1 \leqslant i, j \leqslant n$, let $E_{ij}$ denote the matrix with all entries equal to zero apart from the entry in row $i$ and column $j$, which has entry equal to 1. These are sometimes called "elementary matrices", and they form a basis of $\mathfrak{gl}_n(\mathrm{k})$. Calculate the structure constants of the Lie algebra $\mathfrak{gl}_n(\mathrm{k})$ with respect to this basis, that is, find the scalars $\lambda_{ij,kl}^{rs} \in \mathrm{k}$ where

$$[E_{ij}, E_{kl}] = \sum_{r,s=1}^{n} \lambda_{ij,kl}^{rs} E_{rs}$$

ii) Show that $\mathfrak{sl}_n(\mathrm{k})$ is the derived subalgebra of $\mathfrak{gl}_n(\mathrm{k})$.

*Proof.* i) The elements of the elementary matrix $E_{ij}$ is given by $(E_{ij})_{\mu\nu} = \delta_{i\mu}\delta_{j\nu}$. Then we have compute the commutator:

$$\begin{aligned}
[E_{ij}, E_{k\ell}]_{\mu\nu} &= \sum_{\kappa=1}^{n} (E_{ij})_{\mu\kappa}(E_{k\ell})_{\kappa\nu} - \sum_{\kappa=1}^{n} (E_{k\ell})_{\mu\kappa}(E_{ij})_{\kappa\nu} \\
&= \sum_{\kappa=1}^{n} \delta_{i\mu}\delta_{j\kappa}\delta_{k\kappa}\delta_{\ell\nu} - \sum_{\kappa=1}^{n} \delta_{k\mu}\delta_{\ell\kappa}\delta_{i\kappa}\delta_{j\nu} \\
&= \delta_{i\mu}\delta_{jk}\delta_{\ell\nu} - \delta_{k\mu}\delta_{i\ell}\delta_{j\nu} \\
&= (E_{i\ell}\delta_{jk} - E_{kj}\delta_{i\ell})_{\mu\nu}
\end{aligned}$$

Hence $[E_{ij}, E_{k\ell}] = E_{i\ell}\delta_{jk} - E_{kj}\delta_{i\ell}$.

In terms of the structure constants, we have

$$\lambda_{ijk\ell}^{rs} = \delta_i^r \delta_\ell^s \delta_{jk} - \delta_k^r \delta_j^s \delta_{i\ell}$$

ii) We have proven in Question 4.(a) that $[A,B] \in \mathfrak{sl}_n$ for $A,B \in \mathfrak{gl}_n$. Hence $[\mathfrak{gl}_n, \mathfrak{gl}_n] \subseteq \mathfrak{sl}_n$. On the other hand, we can write down an explicit basis of $\mathfrak{sl}_n$:

$$\mathscr{B} = \left\{E_{ij} \colon i \neq j,\ 1 \leqslant i,j \leqslant n\right\} \cup \{E_{ii} - E_{nn} \colon 1 \leqslant i \leqslant n-1\}$$

It is clear that $\mathscr{B}$ is linearly independent. $\mathscr{B}$ spans $\mathfrak{sl}_n$ because $\dim \mathfrak{sl}_n = n^2 - 1 = \operatorname{card} \mathscr{B}$.

From (i) we know that:

$$E_{ij} = [E_{ik}, E_{kj}] \text{ (for any } 1 \leqslant k \leqslant n), \qquad E_{ii} - E_{nn} = [E_{in}, E_{ni}]$$

Hence $\mathscr{B} \subseteq [\mathfrak{gl}_n, \mathfrak{gl}_n]$. We conclude that $[\mathfrak{gl}_n, \mathfrak{gl}_n] = \mathfrak{sl}_n$. $\mathfrak{sl}_n$ is the derived subalgebra of $\mathfrak{gl}_n$. □

**Question 6**

i) Show that $\mathfrak{sl}_n(\mathbb{C})$ is simple, that is, show that $\mathfrak{sl}_n(\mathbb{C})$ has no proper nontrivial ideals.

(Hint: It might be easier show that $\mathfrak{gl}_n(\mathbb{C})$ has no non-trivial ideals contained in $\mathfrak{sl}_n$.)

ii) Now suppose $\mathsf{k}$ is an arbitrary field. Is $\mathfrak{sl}_n(\mathsf{k})$ always a simple Lie algebra?

*Proof.* i) Suppose that $I$ is a non-zero proper ideal of $\mathfrak{gl}_n(\mathbb{C})$ such that $I \subseteq \mathfrak{sl}_n(\mathbb{C})$. Let $D$ be the $\mathbb{C}$-subspace of $\mathfrak{gl}_n(\mathbb{C})$ generated by all diagonal matrices. we consider the maps $\varphi_i := \operatorname{ad} E_{ii} \in \operatorname{End}_{\mathbb{C}}(\mathfrak{gl}_n(\mathbb{C}))$ for $i = 1, \ldots, n$. We have

$$\varphi_i(E_{jk}) = [E_{ii}, E_{jk}] = \begin{cases} E_{ik}, & i = j \neq k; \\ -E_{ji}, & i = k \neq j; \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{\varphi_1, \ldots, \varphi_n\}$ is a family of diagonalisable and pairwise commutative operators on $\mathfrak{gl}_n(\mathbb{C})$. They can be simultaneously diagonalised. That is, $\mathfrak{gl}_n(\mathbb{C})$ has the eigenspace decomposition

$$\mathfrak{gl}_n(\mathbb{C}) = D \oplus \bigoplus_{i \neq j} \mathbb{C} E_{ij}$$

If $I \cap D \neq \varnothing$, then there exists

$$A = \sum_{i=1}^{n} a_i E_{ii} \in I \cap D$$

Since $\mathrm{id}_n \notin I$, we may assume that $1 \leqslant \mu < \nu \leqslant n$ such that $a_\mu \neq a_\nu$. Then

$$[A, E_{\mu\nu}] = \sum_{i=1}^{n} a_i [E_{ii}, E_{\mu\nu}] = (a_\mu - a_\nu) E_{\mu\nu} \in I$$

Hence $E_{\mu\nu} \in I$.

If $I \cap D = \varnothing$, since $I$ is an ideal, the family $\{\varphi_1, \ldots, \varphi_n\}$ restricts to a family of operators on $I$, and we have

$$I = \bigoplus_{i \neq j} (I \cap \mathbb{C} E_{ij})$$

As $I \neq \{0\}$, there exists $1 \leqslant \mu < \nu \leqslant n$ such that $E_{\mu\nu} \in I$.

In both cases we have some $E_{\mu\nu} \in I$. For $j \neq \mu$, we have $E_{\mu j} = [E_{\mu\nu}, E_{\nu j}] \in I$. And therefore $E_{\mu\mu} - E_{jj} = [E_{\mu j}, E_{j\mu}] \in I$. We deduce that $D \cap \mathfrak{sl}_n(\mathbb{C}) \subseteq I$.

Finally, for any $1 \leqslant i < j \leqslant n$, we have

$$E_{ij} = \frac{1}{2}[E_{ii} - E_{jj}, E_{ij}] \in I$$

We conclude that $\mathfrak{sl}_n(\mathbb{C}) \subseteq I$. This proves that $\mathfrak{sl}_n(\mathbb{C})$ is a simple Lie algebra.

ii) Take $\mathsf{k} = \mathbb{Z}/2$. We note that $\mathrm{id}_2 \in \mathfrak{sl}_2(\mathbb{Z}/2)$ because $\operatorname{tr}\mathrm{id}_2 = 2 = 0$. We note that $[\mathrm{id}_2, A] = 0$ for any $A \in \mathfrak{sl}_2(\mathbb{Z}/2)$. So $\mathrm{id}_2 \in \mathfrak{z}(\mathfrak{sl}_2(\mathbb{Z}/2))$. Therefore $\mathfrak{z}(\mathfrak{sl}_2(\mathbb{Z}/2))$ is a non-zero ideal of $\mathfrak{sl}_2(\mathbb{Z}/2)$. It is proper because

$$\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \neq 0$$

So $\mathfrak{z}(\mathfrak{sl}_2(\mathbb{Z}/2)) \neq \mathfrak{sl}_2(\mathbb{Z}/2)$. We conclude that $\mathfrak{sl}_2(\mathbb{Z}/2)$ is not a simple Lie algebra. $\square$