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Problem Sheet 3

B1.2: Set Theory

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We use the first-order language $\mathcal{L} := \{\in, \subseteq; P, \bigcup, \mathcal{P}; \varnothing, \omega\}$, where $\in$ and $\subseteq$ are binary predicates, P is a binary function, $\bigcup$ and $\mathcal{P}$ are unary functions, and $\varnothing$ and ω are constants.

The equality symbol $\doteq$ is used in $\mathcal{L}$ which indicates that two terms have the same value under any model and assignment. The equality symbol $=$ is used in metalanguage which indicates that two strings are equal.

The ZF axioms we shall use in this sheet are listed below:

***ZF1** Extensionality: $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \leftrightarrow x \doteq y)$;*

***ZF2** Empty Set: $\forall x\, \neg x \in \varnothing$;*

***ZF3** Pairs: $\forall x \forall y \forall z(z \in P(x, y) \leftrightarrow (x \doteq z \vee y \doteq z))$;*

***ZF4** Unions: $\forall x \forall y(y \in \bigcup x \leftrightarrow \exists z(y \in z \wedge z \in x))$;*

***ZF5** Comprehension Scheme: Let $\varphi \in \mathrm{Form}(\mathcal{L})$ and $z, w_1, ..., w_k \in \mathrm{Free}(\varphi)$. Then $\forall x \forall w_1 \cdots \forall w_k \exists y \forall z(z \in y \leftrightarrow (z \in x \wedge \varphi))$;*

***ZF6** Power Sets: $\forall x \forall y(y \in \mathcal{P}(x) \leftrightarrow y \subseteq x)$;*

***ZF7** Infinity: $\exists x(\varnothing \in x \wedge \forall y(y \in x \to y^+ \in x))$, where y^+ is defined to be $\bigcup P(y, P(y, y))$;*

***ZF8** Replacement Scheme: Let $\varphi \in \mathrm{Form}(\mathcal{L})$ and $x, y, w_1, ..., w_k, A \in \mathrm{Free}(\varphi)$. Then $\forall A \forall w_1 \cdots \forall w_k(\forall x(x \in A \to \exists! y\, \varphi) \to \exists B \forall y(y \in B \leftrightarrow \exists x(x \in A \wedge \varphi)))$, where $\exists! y\, \varphi$ is defined to be $(\exists y\, \varphi \wedge \forall z \forall y((\varphi \wedge \varphi[z/y]) \to y \doteq z))$.*

***ZF9** Foundation: $\forall x(\neg x \doteq \varnothing \to \exists y(y \in x \wedge \neg \exists z(z \in y \wedge z \in x)))$.*

The predicate $\subseteq$ is introduced for convenience. It satisfies $\forall x \forall y(x \subseteq y \leftrightarrow \forall z(z \in x \to z \in y))$.

The constant ω is the smallest inductive set, whose existence and uniqueness follows from the Axioms of Infinity, Comprehension Scheme, and Extensionality.

Question 1 α

Prove that there is no descending sequence $X_0 \ni X_1 \ni \ldots$ of sets, that is, there is no function f with domain ω such that $f(n^+) \in f(n)$ for all $n \in \omega$.

[*Hint: Apply the Axiom of Foundation to a suitably chosen set.*]

Proof. Suppose for contradiction that there exists a map $f : \omega \to X$ such that $f(n^+) \in f(n)$ for all $n \in \omega$. Consider $\mathrm{im}(f) \subseteq X$. Since $\mathrm{im}(f) \not\doteq \varnothing$, by the Axiom of Foundation there exists $Y \in \mathrm{im}(f)$ such that $Y \cap \mathrm{im}(f) \doteq \varnothing$. As $Y \in \mathrm{im}(f)$, there exists $n \in \omega$ such that $Y \doteq f(n)$. But by definition $f(n^+) \in f(n) \doteq Y$. Then $f(n^+) \in Y \cap \mathrm{im}(f)$. Contradiction. ✓ □

Question 2 α-

Use the Axiom of Foundation to show that, if A is a non-empty set, then $A \neq A \times A$.

Proof. Suppose that $A \doteq A \times A$. For $x_0 \in A$, there exists $x_1, y_1 \in A$ such that $x_0 \doteq \langle x_1, y_1 \rangle = \{\{x_1\}, \{x_1, y_1\}\}$. Hence $x_0 \ni \{x_1\} \ni x_1$. Recursively, we can construct a descending chain of sets:

$$x_0 \ni \{x_1\} \ni x_1 \ni \{x_2\} \ni x_2 \ni \cdots$$

contradicting the result of Question 1. ✓ □

Using Qn.1 is a short-cut approach to this, but formally speaking you would need something as complicated as the oddity of natural numbers to define the recursion you mentioned.
It is actually expected that you make a tricky construction (∪{A, ∪A}), and apply the axiom of foundation directly to it.

Question 3 α

Prove that a subset of a finite set is finite.

[*Hint: First show, by induction, that, for $n \in \omega$, every subset of n is equinumerous with some natural number.*]

Proof. We shall use induction on $n \in \omega$ to show that for all $n \in \omega$ and $x \subseteq n$, x is a finite set.

Base case: For $0 = \varnothing$, the only subset is 0 itself. So it is finite by definition.

Induction case: Suppose that this is true for $n \in \omega$. Then for $x \subseteq n^+$: if $n \notin x$, then $x \subseteq n$ and hence x is finite by induction hypothesis; if $n \in x$, then $x \doteq (x \cap n) \cup \{n\}$, where $x \cap n \subseteq n$ is finite by induciton hypothesis. There exist $m \in \omega$ and bijection $f : x \cap n \to m$. Let $\tilde{f} : x \to m^+$ defined by

$$\tilde{f}(z) := \begin{cases} f(z), & z \in x \cap n; \\ m & z \doteq n. \end{cases}$$

It is immediate that $\tilde{f}$ is a bijection. Hence $x \sim m^+$ is a finite set.

Now suppose that a is a finite set. There exist $n \in \omega$ and bijection $f : a \to n$. For $b \subseteq a$, $\mathrm{im}(f|_b) \subseteq n$ is finite. Since $f|_b$ is a bijection between b and $\mathrm{im}(f|_b)$, we conclude that b is finite. ✓ □

Question 4 α

Prove that the following properties of a set X are equivalent:

(1) $\omega \preccurlyeq X$ (i.e. there is an injective function $f : \omega \to X$)

(2) there exists a function $g : X \to X$ which is injective but not surjective.

[*Hint: For (2)⇒(1) use the Recursion Theorem, and induction to verify that the function you define is indeed injective.*]

Proof. (1)⟹(2): Let $h : \omega \to \omega$ defined by $n \mapsto 2 \cdot n$. Consider the function $g : X \to X$ defined by

$$g(x) := \begin{cases} f \circ h \circ f^{-1}(x), & x \in \mathrm{im}(f); \\ x, & x \in X \setminus \mathrm{im}(f) \end{cases}$$

It is straightforward that g is injective but not surjective, as $f(1) \notin \mathrm{im}(g)$. ✓

(2)⟹(1): Fix $a_0 \in X \setminus \mathrm{im}(g)$. By Recursion Theorem, there exists a unique function $f : \omega \to X$ such that $f(0) \doteq a_0$ and $f(n^+) \doteq g \circ f(n)$ for each $n \in \omega$. We claim that f is injective. It suffices to prove that $f|_n : n \to X$ is injective for all $n \in \omega$. We use induction on n:

Base case: $f|_0 : \varnothing \to X$ is injective vacuously.

Induction case: Suppose that $f|_n$ is injective but $f|_{n^+}$ is not. If $n \doteq 0$, then this is impossible because $n^+ \doteq 1 \doteq \{\varnothing\}$ is a singleton. If $n > 0$, then there exists $m \in n$ such that $f(n) \doteq f(m)$. If $m \doteq 0$, then $f(n) \doteq g(f(n-1)) \doteq a_0$. This is impossible as $a_0 \notin \mathrm{im}(g)$. If $m > 0$, then $g(f(n-1)) \doteq f(n) \doteq f(m) \doteq g(f(m-1))$ implies that $f(n-1) \doteq f(m-1)$, as g is injective. But this contradiction that $f|_n$ is injective. ✓ □

Question 5 α

Suppose κ, λ, μ are cardinals. Prove (no need to check obvious bijections)

(i) $(\kappa + \lambda) + \mu = \kappa + (\lambda + \mu)$

(ii) $(\kappa \cdot \lambda) \cdot \mu = \kappa \cdot (\lambda \cdot \mu)$

(iii) $\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu$

(iv) $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$

(v) $\kappa^{\lambda \cdot \mu} = \left(\kappa^\lambda\right)^\mu$

(vi) $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$

Proof. Let A, B, C be disjoint sets such that $\mathrm{card}\,(A) = \kappa$, $\mathrm{card}\,(B) = \lambda$, and $\mathrm{card}\,(C) = \mu$.

(i) By definition, $(\kappa + \lambda) + \mu = \mathrm{card}\,((A \cup B) \cup C)$ and $\kappa + (\lambda + \mu) = \mathrm{card}\,(A \cup (B \cup C))$. By thr Axiom of Extensionality, $(A \cup B) \cup C \doteq A \cup (B \cup C)$. Hence $(\kappa + \lambda) + \mu = \kappa + (\lambda + \mu)$. ✓

(ii) By definition, $(\kappa \cdot \lambda) \cdot \mu = \mathrm{card}\,((A \times B) \times C)$ and $\kappa \cdot (\lambda \cdot \mu) = \mathrm{card}\,(A \times (B \times C))$. The map $f : (A \times B) \times C \to A \times (B \times C)$ given by $\langle \langle a, b \rangle, c \rangle \mapsto \langle a, \langle b, c \rangle \rangle$ is obviously a bijection. Hence $(\kappa \cdot \lambda) \cdot \mu = \kappa \cdot (\lambda \cdot \mu)$. ✓

(iii) By definition, $\kappa\cdot(\lambda+\mu) = \operatorname{card}(A\times(B\cup C))$ and $\kappa\cdot\lambda+\kappa\cdot\mu = \operatorname{card}(A\times B\cup A\times C)$. By Axiom of Extensionality $A\times(B\cup C) \doteq A\times B\cup A\times C$. Hence $\kappa\cdot(\lambda+\mu) = \kappa\cdot\lambda+\kappa\cdot\mu$.

(iv) By definition, $\kappa^{\lambda+\mu} = \operatorname{card}\left(A^{B\cup C}\right)$ and $\kappa^{\lambda}\cdot\kappa^{\mu} = \operatorname{card}\left(A^B\times A^C\right)$. For $f : B\to A$ and $g : C\to A$, we define $f*g : B\cup C\to A$ by

$$(f*g)(x) := \begin{cases} f(x), & x\in B; \\ g(x), & x\in C. \end{cases}$$

We claim that the map $\langle f,g\rangle \mapsto f*g$ is a bijection from $A^B\times A^C$ to $A^{B\cup C}$. It is trivially injective. For $h : B\cup C\to A$, we have $h \doteq h|_B\times h|_C$, where $h|_B : B\to A$ and $h|_C : C\to A$. Hence the map is surjective.

It should be h = h|B ∪ h|C

(v) By definition, $\kappa^{\lambda\cdot\mu} = \operatorname{card}\left(A^{B\times C}\right)$ and $\left(\kappa^{\lambda}\right)^{\mu} = \operatorname{card}\left(\left(A^B\right)^C\right)$. For $f : B\times C\to A$ and $c\in C$, we define $f_c : B\to A$ by $b\mapsto f(\langle b,c\rangle)$. We denote the map $c\mapsto f_c$ by θ_f. We claim that the map $f\mapsto\theta_f$ is a bijection from $A^{B\times C}$ to $\left(A^B\right)^C$.

(vi) By definition, $(\kappa\cdot\lambda)^{\mu} = \operatorname{card}\left((A\times B)^C\right)$ and $\kappa^{\mu}\cdot\lambda^{\mu} = \operatorname{card}\left(A^C\times B^C\right)$. For $f : C\to A$ and $g : C\to B$, we define $f\times g$ by $(f\times g)(x) := \langle f(x), g(x)\rangle$. We claim that the map $\langle f,g\rangle\mapsto f\times g$ is a bijection from $(A\times B)^C$ to $A^C\times B^C$. □

Question 6 α

(a) Let A, X, Y be sets such that $X \preccurlyeq A$. Prove that $X^Y \preccurlyeq A^Y$. Deduce that, for cardinals κ,λ,μ, if $\kappa\leqslant\lambda$ then $\kappa^{\mu}\leqslant\lambda^{\mu}$.

(b) Now let A, B, X, Y be sets with $X\preccurlyeq A$ and $Y\preccurlyeq B$. Prove that, apart from exceptional case(s), $X^Y\preccurlyeq A^B$.

[*You need to show that the map you give from X^Y to A^B is really injective.*]

What are the exceptional cases?

Proof. (a) Since $X\preccurlyeq A$, there exists an injection $f : X\to A$. For each $g : Y\to X$, we claim that $g\mapsto f\circ g$ is an injection from X^Y to A^Y. For $g_1, g_2\in X^Y$ such that $f\circ g_1 \doteq f\circ g_2$, we have $g_1\doteq g_2$ (injections are left-invertible). Hence $g\mapsto f\circ g$ is injective. We conclude that $X^Y\preccurlyeq A^Y$. Hence $\kappa^{\mu}\leqslant\lambda^{\mu}$ for cardinals κ,λ,μ.

X → A

(b) $X^Y\preccurlyeq A^B$ is true unless $(A\doteq\varnothing\wedge B\not\doteq\varnothing\wedge Y\doteq\varnothing)$. First suppose that $A\not\doteq\varnothing$. Suppose that $f : X\to Y$ and $g : A\to B$ are injections. We shall prove that $A^Y\preccurlyeq A^B$. Fix $a_0\in A$. For $h : Y\to A$, we define $\theta_h : B\to A$ By

Y → B

$$\theta_h(x) := \begin{cases} h(y), & x\doteq g(y); \\ a_0, & x\notin\operatorname{im}(g). \end{cases}$$

θ_h is well-defined since g is injective. It follows immediately that the map $h\mapsto\theta_h$ is injective. Hence $A^Y\preccurlyeq A^B$. We have proven in (a) that $X^Y\preccurlyeq A^Y$. By transitivity of $\preccurlyeq$, we have $X^Y\preccurlyeq A^B$.

If $A\doteq\varnothing$ and $B\doteq\varnothing$, then $X\doteq\varnothing$ and $Y\doteq\varnothing$. Here $X^Y\doteq\varnothing^{\varnothing}\doteq\{\varnothing\}\doteq\varnothing^{\varnothing}\doteq A^B$. The only map $f : \{\varnothing\}\to\{\varnothing\}$, $\varnothing\mapsto\varnothing$ is bijective. Hence $X^Y\preccurlyeq A^B$.

If $A\doteq\varnothing$, $B\not\doteq\varnothing$ and $Y\not\doteq\varnothing$, then $X\doteq\varnothing$. Here $X^Y\doteq\varnothing^Y\doteq\varnothing\doteq\varnothing^B\doteq A^B$. The only map $\varnothing : \varnothing\to\varnothing$ is bijective. Hence $X^Y\preccurlyeq A^B$.

Assume that $(A\doteq\varnothing\wedge B\not\doteq\varnothing\wedge Y\doteq\varnothing)$. Then $X\doteq\varnothing$. $X^Y\doteq\{\varnothing\}$ and $A^B\doteq\varnothing$. There are no maps from $\{\varnothing\}$ to $\varnothing$. So $X^Y\not\preccurlyeq A^B$. □

Question 7 α-

Calculate the cardinalities of the following sets, simplifying your answers as far as possible: your answer in each case should be a cardinal from the list $\aleph_0, 2^{\aleph_0}, 2^{2^{\aleph_0}}, \ldots$

(i) The set of all finite sequences of natural numbers

[*Note that the axioms given so far do not prove that a countable union of countable sets is countable. Use unique factorization of non-zero natural numbers into powers of primes.*]

(ii) The set of functions $f : \mathbb{R} \to \mathbb{R}$

(iii) The set of continuous functions $f : \mathbb{R} \to \mathbb{R}$

[*Hint: a continuous function is determined by its values on* $\mathbb{Q}$.]

(iv) The set of equivalence relations on ω.

[*Hint: To get a lower bound think about partitions of* ω.]

Proof. (i) The set of all finite sequences of natural numbers has cardinality $\aleph_0$.

First we define a finite sequence on X to be a map $f : n \to X$ for some $n \in \omega$. We denote the set of all maps $n \to \omega$ by ω^n. Then the set of all finite sequences is $\bigcup_{n\in\omega} \omega^n$. We claim that it is countably infinite.

Let $\varphi := \forall m(m \in \omega \to \forall p((p \in \omega \wedge n \doteq p \cdot m) \to (p \doteq 1 \vee p \doteq n)))$. Let $\Pi := \{n \in \omega : \varphi\}$. Then Π is the set of all prime natural numbers. It is trivial that Π is infinite. Since $\Pi \subseteq \omega$, it is countably infinite. In particular there exists a bijection $\omega \to \Pi$, $n \mapsto p_n$.

Since $f : n \to \omega$ is uniquely determined by the values $f(0), ..., f(n-1)$. There is a natural bijection between ω^n and $\underbrace{\omega \times \cdots \times \omega}_{n \text{ times}}$, so that every $f : n \to \omega$ is identified with $\langle f(0), ..., f(n-1)\rangle$. Let $F : \bigcup_{n\in\omega} \omega^n \to \omega$ given by $\langle f(0), ..., f(n-1)\rangle \mapsto \sum_{i=0}^{i-1} f(i) \cdot p_i$. By the Fundamental Theorem of Arithmetics, F is injective. Hence $\bigcup_{n\in\omega} \omega^n \preccurlyeq \omega$. It is trivial that $\bigcup_{n\in\omega} \omega^n$ is infinite. Hence $\bigcup_{n\in\omega} \omega^n$ is countably infinite.

Sum of prime multiples is not unique, and you need product of prime powers instead! Additionally, f(i) can be zero, so you will not be able to differentiate ⟨1,1⟩ and ⟨1,1,0⟩ in this way. You in fact need Π(p_i^(f(i) + 1))

(ii) The set of functions $f : \mathbb{R} \to \mathbb{R}$ has cardinality $2^{2^{\aleph_0}}$.

Firstly, $\operatorname{card}(\mathbb{R} \times \mathbb{R}) = 2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0 + \aleph_0} = 2^{\aleph_0} = \operatorname{card}(\mathbb{R})$, since $\aleph_0 + \aleph_0 = \aleph_0$.

Secondly, $\mathbb{R}^{\mathbb{R}} \subseteq \mathcal{P}(\mathbb{R} \times \mathbb{R})$. Then $\operatorname{card}(\mathbb{R}^{\mathbb{R}}) \leqslant \operatorname{card}(\mathcal{P}(\mathbb{R} \times \mathbb{R})) = \operatorname{card}(\mathcal{P}(\mathbb{R})) = 2^{\operatorname{card}(\mathbb{R})} = 2^{2^{\aleph_0}}$.

Thirdly, $2^{2^{\aleph_0}} = \operatorname{card}(2^{\mathbb{R}}) \preccurlyeq \operatorname{card}(\mathbb{R}^{\mathbb{R}})$, since $2 \leqslant 2^{\aleph_0}$.

Finally, by Schröder-Bernstein Theorem, we have $\operatorname{card}(\mathbb{R}^{\mathbb{R}}) = 2^{2^{\aleph_0}}$.

(iii) The set of continuous functions $f : \mathbb{R} \to \mathbb{R}$ has cardinality $2^{\aleph_0}$.

From analysis we know that a continuous function $f : \mathbb{R} to \mathbb{R}$ is uniquely determined by its values on a dense subset of $\mathbb{R}$. Since $\mathbb{Q} \subseteq \mathbb{R}$ is dense, The set of continuous functions $f : \mathbb{R} \to \mathbb{R}$ has cardinality $\operatorname{card}(\mathbb{R}^{\mathbb{Q}})$.

By Corollary 10.12, $\operatorname{card}(\mathbb{Q}) = \aleph_0$, and $\aleph_0 \cdot \aleph_0 = \aleph_0$. Then $\operatorname{card}(\mathbb{R}^{\mathbb{Q}}) = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0}$.

This only shows the cardinality is not greater than card(R^Q) though

(iv) The set of equivalence relations on ω has cardinality $2^{\aleph_0}$.

Let S be the set of equivalence relations on ω. Since $S \subseteq \mathcal{P}(\omega \times \omega)$, $\operatorname{card}(S) \leqslant \operatorname{card}(\mathcal{P}(\omega \times \omega)) = \operatorname{card}(\mathcal{P}(\omega)) = 2^{\aleph_0}$.

For $a \subseteq \omega$, we define an equivalence relation R_a on ω: for $m, n \in \omega$,

$$(\langle m, n\rangle \in R_a \leftrightarrow ((m \in a \wedge n \in a) \vee (\neg m \in a \wedge \neg n \in a))$$

The map $a \mapsto R_a$ is clearly an injection from $\mathcal{P}(\omega)$ to S. Hence $\operatorname{card}(\mathcal{P}(\omega)) = 2^{\aleph_0} \leqslant \operatorname{card}(S)$.

No, this is not an injection, because R_a = R_(ω \ a). The correct way is to only consider a ⊆ ω that does not contain 0, so you map P(ω \ {0}) into S

Finally, by Schröder-Bernstein Theorem, we have $\operatorname{card}(S) = 2^{\aleph_0}$. □

Question 8 α

Let $f : X \to Y$ be surjective. Prove that $\mathcal{P}(Y) \preccurlyeq \mathcal{P}(X)$.

[*You should not assume there exists an injective map* $g : Y \to X$ *as the axioms we have so far do not suffice to prove this.*]

Proof. The map $f : X \to Y$ induces $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$, which is clearly injective as f is surjective. □

Question 9 α

(a) Let κ be any cardinal number and $n \in \omega$. Prove that (for cardinal addition)

(i) $\kappa + 0 = \kappa$ and $\kappa \cdot 0 = 0$

(ii) $\kappa \cdot n^+ = \kappa \cdot n + \kappa$

(b) We now have two definitions of addition and multiplication for elements of ω. Prove that they agree.

Proof. Let A be a set disjoint from ω with cardinality κ.

(a) (i) Since $A \cup \varnothing = A$, $\kappa + 0 = \kappa$. Since $A \times \varnothing = \varnothing$, $\kappa \times 0 = 0$. ✓

(ii) $\kappa \cdot n^+ = \kappa \cdot (n+1) = \kappa \cdot n + \kappa \cdot 1$. There is a natural bijection from A to $A \times \{\varnothing\}$ given by $a \mapsto \langle a, \varnothing \rangle$. Hence $\kappa \cdot 1 = \kappa$ and $\kappa \cdot n^+ = \kappa \cdot n + \kappa$. ✓

(b) The cardinal addition satisfies that $n + 0 = n$ and $n + m^+ = (n+m)^+ = n + m + 1$ for $n, m \in \omega$. By Proposition 6.1, the binary operation with these proeprties is unique. So it agrees with usual addition on ω.

The cardinal multiplication satisfies that $n \cdot 0 = 0$ and $m \cdot n^+ = m \cdot n + m$ for $n, m \in \omega$. By Proposition 6.1, the binary operation with these proeprties is unique. So it agrees with usual multiplication on ω. ✓

□