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Problem Sheet 2

Feynman diagrams, Resonances
and Nuclear Reactions

B4: Subatomic Physics

15 November, 2020

51%

Question 2.1

Sketch all the lowest order electromagnetic Feynman diagram(s) for the following processes:

a) $e^- + e^+ \longrightarrow e^- + e^+$

b) $e^- + e^- \longrightarrow e^- + e^-$

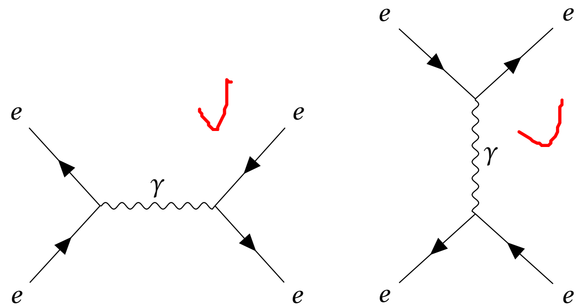
c) $e^- + e^- \longrightarrow e^- + e^- + \mu^+ + \mu^-$

d) $\gamma \longrightarrow e^+ + e^-$ in the presence of matter

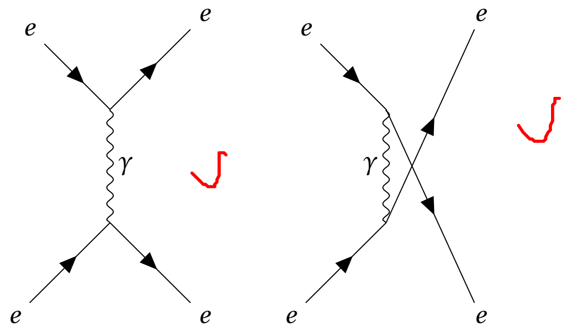
e) $\gamma + \gamma \longrightarrow \gamma + \gamma$.

Sketch some 2$^{\text{nd}}$ and 3$^{\text{rd}}$ order diagrams for case a).

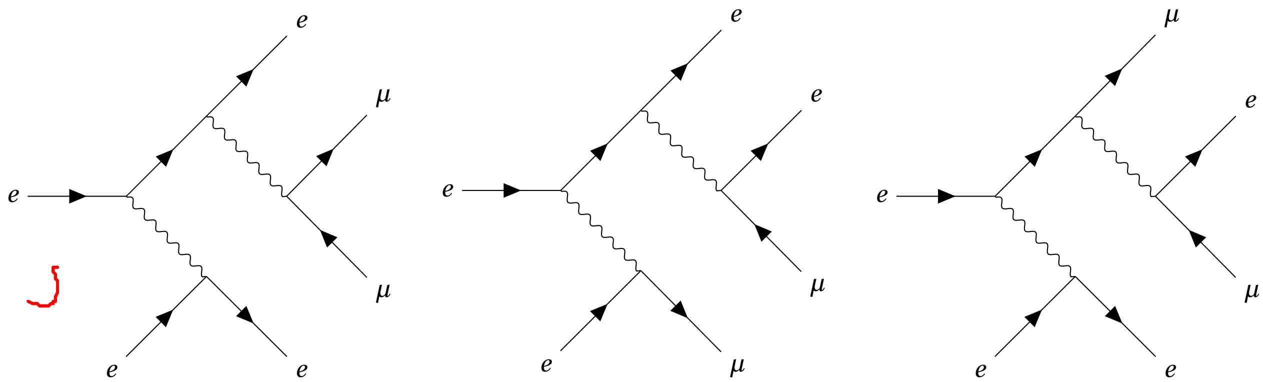
Proof. a) There are two tree level Feynman diagrams, as explained in the lecture notes:

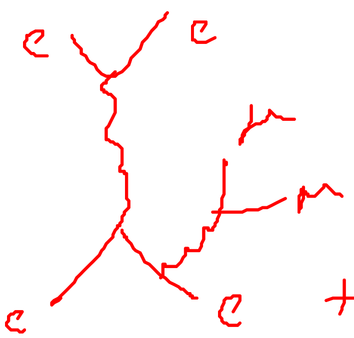


b) There are two tree level Feynman diagrams:



c) I am not sure exactly how many tree level Feynman diagrams there are for the reaction. I can find three of them:

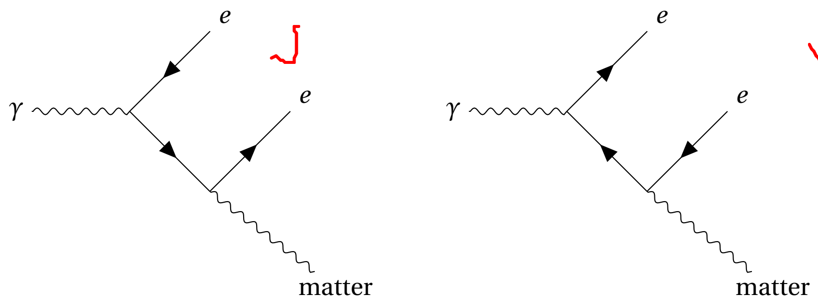




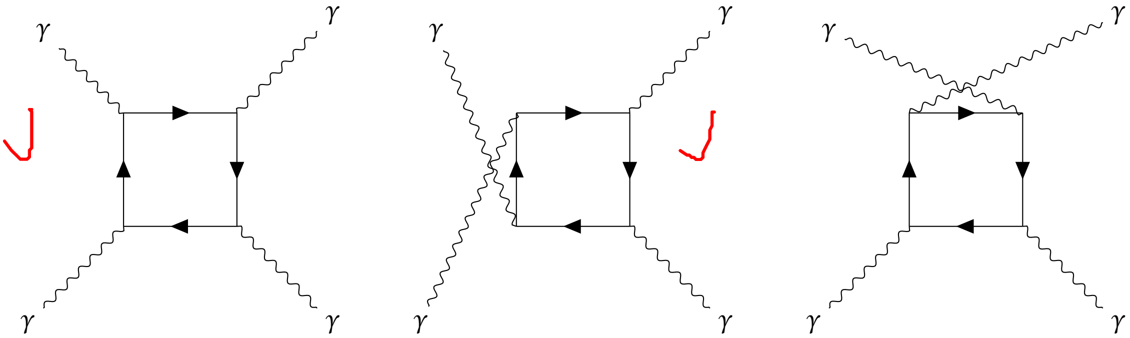
+ others

EM interactions

d) There are two tree level Feynman diagrams:

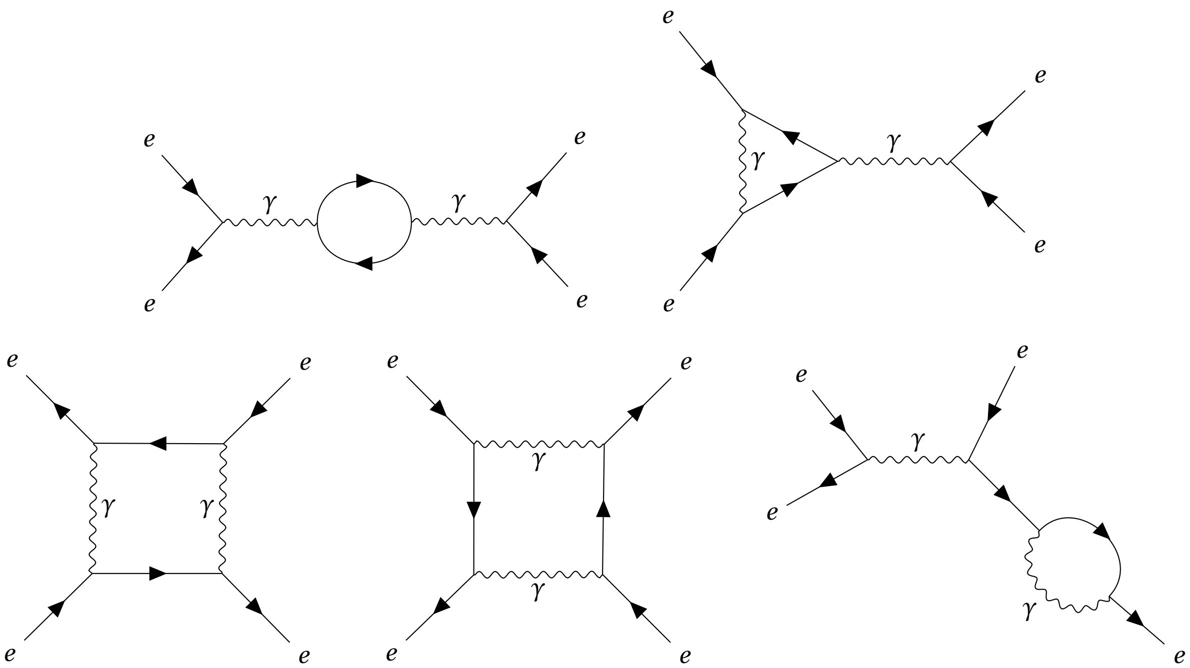


e) The four photons are indistinguishable. So the different Feynman diagrams corresponds to the different labelling of the vertices of a square. The number of different labelling is the number of orbits of S_4 acted on by D_8, which is 3.

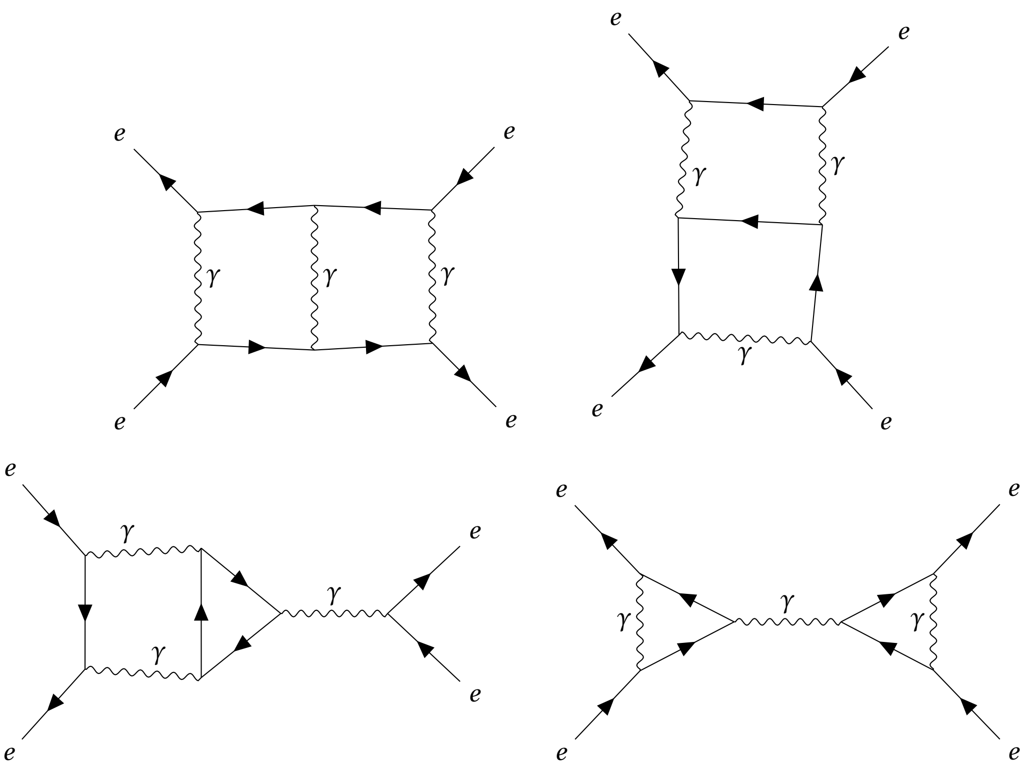


For the Bhabha scattering in (a), the second order Feynman diagrams have 4 vertices, and the third order Feynman diagrams have 6 vertices. There are too many of them so I just sketch some:

Second order:



Third Order:



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□

Question 2.2

Two particles A and $\overline{A}$ collide with $\sqrt{s} = 5\text{GeV}$, annihilate and create a pair of particles $B\overline{B}$ due to an interaction which has a massless exchange particle and a coupling factor of 0.1 at each vertex. You can assume that all the particles are massless on the energy scale of the collision.

a) Draw a Feynman diagram for this process at tree level.

b) How would the cross-section change if the coupling factor at both vertices would be 0.3? What problem would we have if the coupling factor becomes about 1?

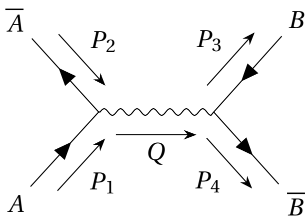
c) How would the cross-section change if the exchange particle has a mass of $90\text{GeV}/c^2$ (assume that the original coupling factor still applies)?

d) For the case of the massless exchange particle: how would the cross-section change if the collision energy increases by a factor 2?

e) How would the cross-section change if the final state particles would exist in three flavours which are experimentally indistinguishable?

f) If a particle C decays into $D\overline{D}$ via the original interaction above (without the exchange particle), how would the decay rate to the same final state change for a particle C', which has twice the mass of C (you can assume that D is massless)?

Solution. a) We use the s-channel Feynman diagram, which is



b) First we derive the cross-section of the scattering. This is very similar to the $e^+e^- \to \mu^+\mu^-$ process, which is well-discussed in the lecture notes. I will follow the derivation as an exercise.

The massless exchange particle has 4-momentum $Q = P_1 + P_2$, where $(P_1+P_2)^2c^2 = s$. So the propagator is

$$\frac{1}{Q^\mu Q_\mu - m^2c^2} = \frac{c^2}{s}$$

Let g be the vertex factor. Then the matrix element M_{if} is given by

$$|M_{if}|^2 = \left|\frac{g^2\hbar^3c^3}{s}\right|^2 \frac{1}{V^2}$$

where V is a fixed volume for normalisation. By Fermi's Golden Rule, the transition rate is given by

$$R_{if} = \frac{2\pi}{\hbar}|M_{if}|^2\frac{\mathrm{d}N}{\mathrm{d}E}$$

Next we calculate the density of states $\mathrm{d}N/\mathrm{d}E$:

$$\frac{\mathrm{d}N}{\mathrm{d}E} = \frac{\mathrm{d}N}{\mathrm{d}p_B}\frac{\mathrm{d}p_B}{\mathrm{d}E} = \frac{4\pi V p_B^2}{(2\pi\hbar)^3}\frac{\mathrm{d}p_B}{\mathrm{d}E}$$

In the ultra-relativistic limit $E_B \gg m_Bc^2$, we have $E_B \approx p_Bc$. So

$$\frac{\mathrm{d}p_B}{\mathrm{d}E} = \frac{1}{2}\frac{\mathrm{d}p_B}{\mathrm{d}E_B} = \frac{1}{2c}$$

Hence

$$R_{if} = \frac{2\pi}{\hbar}|M_{if}|^2\frac{4\pi V p_B^2}{(2\pi\hbar)^3}\frac{1}{2c} = \frac{2\pi}{\hbar}\left|\frac{g^2\hbar^3c^3}{s}\right|^2\frac{1}{V^2}\frac{4\pi V p_B^2}{(2\pi\hbar)^3}\frac{1}{2c}$$

The incoming particles have flux $j = 2v_AV = 2c^2p_A/E_AV \approx 2c/V$. Hence the total cross-section is given by

$$\sigma = \frac{R_{if}}{j} = \frac{V}{2c}\frac{2\pi}{\hbar}\left|\frac{g^2\hbar^3c^3}{s}\right|^2\frac{1}{V^2}\frac{4\pi V p_B^2}{(2\pi\hbar)^3}\frac{1}{2c} = \frac{1}{16\pi}\hbar^2c^2\frac{g^4}{s}$$

Hence $\sigma \propto g^4$. If the coupling constant change from 0.1 to 0.3, then the cross-section will be increased by 81 times.

For higher degree Feynman diagrams, $\sigma \propto g^{2n}$ where n is the number of vertices in the diagram. If $g \sim 1$, then we cannot neglect the higher order perturbations, so the perturbation theory fails for the problem.

c) If the exchange particle is massless, the magnitude of the propagator is

$$\frac{1}{Q^\mu Q_\mu} = \frac{c^2}{s} = 0.04\ c^2/\mathrm{GeV}$$

If the exchange particle has mass $m = 90$ GeV$/c^2$, then the magnitude of the propagator becomes

$$\frac{1}{Q^\mu Q_\mu - mc^2} = \frac{1}{25-90^2}\ c^2/\mathrm{GeV} = -1.24\times10^{-4}\ c^2/\mathrm{GeV}$$

This corresponds to the matrix element and hence the cross-section decreased to $\frac{1}{104329}$ of the original magnitude. (*What is the physical significance of this result?*)

d) Since $\sigma \propto s^{-1} \propto E_A^{-2}$, if the collision energy increases by 2 times, then the cross-section will be decreased to 1/4 of the original magnitude.

e) *I am not sure what will change. If the final state particles are experimentally indistinguishable, then why would the result change?* 3 flavours ⟹ 3x more available phase space

$\sigma' = 3\sigma$

f)

□

(c) If you have a A+B->C+D where the interaction can take place via two types of propagators, you would expect the cross-section to be much larger for the interaction which involves a much lighter exchange particle.

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Question 2.3

The scattering process $e^+e^- \to e^+e^-$ is called Bhabha scattering after the Indian physicist Homi J. Bhabha (1909 – 1966), who performed the first calculation to determine the cross-section for this process. (For this problem you can ignore any possible weak interaction processes.)

a) Which diagrams contribute at tree level to this process? What different processes do these diagrams describe? What Mandelstam variables do the amplitudes associated with these diagrams depend on?

b) How do the amplitudes associated with these diagrams contribute to the calculation of the differential cross-section at lowest order?

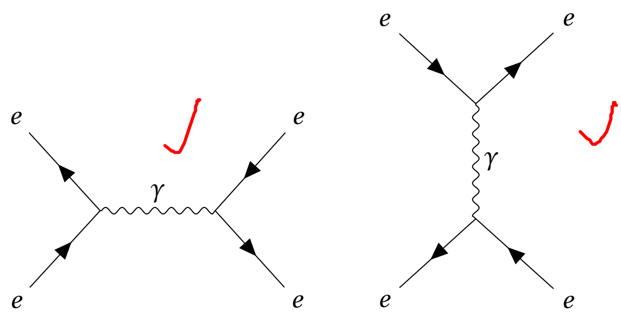
c) Write down the propagator factors for the two contributing diagrams using Mandelstam variables and using our simplified Feynman rules. Neglect the electron mass relative to the collision energy and only consider the contribution from photon exchange. (As usual the actual result is different from what you get here because we are ignoring spins, but it is obtained along similar lines.)

d) Express your result as a function of p and $\cos\theta$ and hence find how the matrix element squared depends on these variables. (For this you will have to express the Mandelstam variables using p and $\cos\theta$.)

e) You will find that the differential cross-section diverges for small scattering angles. Why is this the case? Why did we not observe this when we looked at the annihilation process $e^+e^- \to \mu^+\mu^-$?

f) What is the contribution from interference of the involved diagrams?

Solution. a) There are two tree level Feynman diagrams.



The s-channel corresponds to the annihilation and creation process; the t-channel corresponds to the scattering process. For the s-channel, the propagator is

$$\frac{1}{Q^2} = \frac{1}{(P_1+P_2)^2} = \frac{c^2}{s}$$

For the t-channel, the propagator is

$$\frac{1}{Q^2} = \frac{1}{(P_3-P_1)^2} = \frac{1}{t}$$

b) For the s-channel, the differential cross-section $\frac{d\sigma}{d\Omega} \propto s^{-1}$ as derived in Question 2. For the t-channel, the differential cross-section $\frac{d\sigma}{d\Omega} \propto \frac{s}{t^2}$ (t^{-2} comes from the propagator and s comes from the density of states).

✗ c) The propagator factor for the s-channel is $\frac{4\pi\alpha_{\mathrm{EM}}c^2}{s}$ and for the t-channel is $\frac{4\pi\alpha_{\mathrm{EM}}}{t}$. Yes, if we only have the s channel or only the t channel.

d) In the centre-of-mass frame, the 4-momenta of the particles are given by

$$P_1 = (E/c, 0, 0, p), \quad P_2 = (E/c, 0, 0, -p), \quad P_3 = (E/c, 0, p\sin\theta, p\cos\theta), \quad P_4 = (E/c, 0, -p\sin\theta, -p\cos\theta)$$

in which we have used the conservation of 4-momenta implicitly.

Hence the Mandelstam variables are given by

$$s = (P_1+P_2)^2c^2 = 4E^2 = 4p^2c^2 + 4m_e^2c^4 \approx 4p^2c^2$$
$$t = (P_1-P_3)^2 = p^2(1-\cos\theta)^2 = 4p^2\sin^4\frac{\theta}{2}$$

(c) $\sigma \propto (A_s + A_t)^2$

For the s-channel, the propagator factor is

$$A_s \propto \frac{\pi \alpha_{\text{EM}}}{p^2 + m_e^2 c^2} \approx \frac{\pi \alpha_{\text{EM}}}{p^2}$$ ✓

For the t-channel, the propagator factor is

$$A_t \propto \frac{\pi \alpha_{\text{EM}}}{p^2 \sin^4 \frac{\theta}{2}} = \frac{4\pi \alpha_{\text{EM}}}{p^2 (1 - \cos\theta)^2}$$ ✓

✓

e) We observe that when $\theta \to 0$, the propagator factor of the t-channel diverges. So the differential cross-section diverges. This corresponds to the case where the incoming and outgoing electrons have unchanged momenta, which implies that the virtual exchange photon has zero energy. A discussion of the underlying physics for this problem may be found at `http://www.sciforums.com/threads/low-angle-divergence-in-bhabha-and-moller-scattering.104687/`.

For the $e^+ e^- \to \mu^+ \mu^-$ scattering process, we will not have the divergence. Since $E_e = E_\mu$ and $m_e \neq m_\mu$, we shall never have $p_e = p_\mu$. The virtual photon always has a nonzero energy.

f) Because of (c) the interference term is $2A_s A_t$

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Question 2.4

The nucleus $^5_3$Li was observed (Heydenburg & Ramsey, Phys. Rev. 60(1941) 42) as a resonance in the elastic scattering of protons from $^4_2$He at a proton energy of about 2 MeV. The resonance has a width of about 0.5 MeV and has spin 3/2.

(The spin of $^4_2$He is 0 and its mass is 3728.4 MeV/c^2)

a) What is the lifetime of $^5_3$Li?

b) Why is in this process the elastic cross-section equal to the total cross-section?

$[m(^2\text{H}) = 1876.1\text{MeV}/c^2, m(^3\text{He}) = 2809.4\text{MeV}/c^2]$

c) Estimate the cross-section at the resonance energy in barn.

Solution. a) The lifetime is given by $\tau = \hbar/\Gamma$, where $\Gamma = 0.5$ MeV is the width of the Lorentzian. So $\tau = 1.32 \times 10^{-21}$ s. ✓

b) Conside the possible scattering processes:

$$^4_2\text{He} + ^1_1\text{H} \longrightarrow ^4_2\text{He} + ^1_1\text{H}$$ ✓
$$^4_2\text{He} + ^1_1\text{H} \longrightarrow ^3_2\text{He} + ^2_1\text{H}$$
$$^4_2\text{He} + ^1_1\text{H} \longrightarrow ^2_2\text{He} + ^3_1\text{H}$$ → what about this reaction?

Calculate the stationary energies:

$$m(^4_2\text{He}) + m(^1_1\text{H}) = 3728.4 + 938 = 4666.4 \text{ MeV}$$ ✓
$$m(^3_2\text{He}) + m(^2_1\text{H}) = 2809.4 + 1876.1 = 4685.5 \text{ MeV}$$ ✓

By conservation of energy, a proton with kinetic energy 2 MeV cannot trigger the reaction $^4_2\text{He} + ^1_1\text{H} \longrightarrow ^3_2\text{He} + ^2_1\text{H}$, and also the reactions with higher energies. So there is only the elastic scattering process. The elastic cross-section equals to the total cross-section. ✓

c) The Breit-Wigner resonance formula with spin factors is

$$\sigma(i \to R \to f) = \frac{\pi}{k_i^2} \frac{2j+1}{(2s_1+1)(2s_2+1)} \frac{\Gamma_i \Gamma_f}{(E_R - E_i)^2 + \Gamma^2/4}$$

At resonance energy $E_i = E_R$ and $\Gamma_i = \Gamma_f = \Gamma$. Hence

$$\sigma_{\max} = \frac{4\pi}{k_i^2} \frac{2j+1}{(2s_1+1)(2s_2+1)}$$ ✓

$k^2 = \frac{2\mu E}{\hbar^2}$

The spins $j = 3/2$, $s_1 = 1/2$ and $s_2 = 0$. Hence

$$\sigma_{\max} = \frac{8\pi}{k_i^2} = \frac{4\pi\hbar^2}{m_p E_i} = 2.6 \times 10^{-28}\ \text{m}^2 = 2.6\ \text{barn}$$

$\simeq$ ✓

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□

Question 2.5

When thermal neutrons hit a nucleus they can get captured to form a compound nucleus, which subsequently decays dominantly by photon emission, the probability of which is virtually independent of the resonance or incidence energy (we can therefore take the total width of the resonance independent of the neutron energy).

a) The neutron width Γ_n is proportional to the transition rate for the compound nucleus to decay by emitting a neutron. Show that the density of states available to the neutron is proportional to k. Hence show that the cross section for the process $n + N \to N' + \gamma$ has a $1/v$ dependence at low energies, where v is the velocity of the neutron.

b) ^{113}Cd has a cross section for capture by the $n + N \to N' + \gamma$ reaction which depends on energy as follows:

E(eV)	0.01	0.02	0.03	0.05	0.08	0.10	0.15	0.20	0.25	0.30	0.40	0.5
σ(b)	3250	2500	2300	2400	2800	3400	6800	6000	2400	1100	320	140

The cross sections are accurate to $\pm 5\%$. The maximum cross section is 7750 ± 150b. The spin of ^{113}Cd is $I = \frac{1}{2}$ and that of the compound nucleus is $J = 1$.

With the aid of a graph of $\sigma\sqrt{E}$ vs E determine E_R, Γ_n^R and Γ from the data assuming that $\Gamma_f = \Gamma_\gamma \approx \Gamma$. Here Γ_n^R is the value of Γ_n at the resonance energy E_R. Using the given data determine the maximum cross section for neutron scattering off ^{113}Cd. Sketch the energy dependence of the $n + N \to N + n$ cross section.

c) What does the fact that the compound nucleus displays a narrow resonance and dominantly decays by photon emission tell us about the way the energy of the incoming neutron is absorbed by the nucleus?

Solution. a) We use the non-relativistic relation $E = \hbar^2 k^2 / 2m$. The density of states

$$\frac{dN}{dE} = \frac{dN}{dk}\frac{dk}{dE} = \frac{Vk^2}{2\pi^2\hbar}\frac{dk}{dE} = \frac{Vk^2}{2\pi^2\hbar}\frac{m}{\hbar^2 k} \propto k$$ ✓

The Breit-Wigner formula

$$\sigma(i \to R \to f) = \frac{\pi}{k_i^2}\frac{2j+1}{(2s_1+1)(2s_2+1)}\frac{\Gamma_n\Gamma_f}{(E_R - E_i)^2 + \Gamma^2/4}$$ ✓

We know that Γ_n is proportional to the transition rate R_{if}, which is proportional to the density of states by Fermi's Golden Rule. The Breit-Wigner formula also contains a k^{-2} term. So we have $\sigma \propto k^{-1} \propto v^{-1}$. ✓

b) We put $E = \hbar^2 k_i^2 / 2m$, $j = 1$, $s_1 = 1/2$ and $s_2 = 0$. So

$$\sigma = \frac{3\pi\hbar^2}{4mE}\frac{\Gamma_n\Gamma}{(E_R - E)^2 + \Gamma^2/4}$$

Let $\Gamma_n = K\sqrt{E}$ for some constant K. Then

$$\sigma\sqrt{E} = K'\frac{\Gamma}{(E_R - E)^2 + \Gamma^2/4}$$

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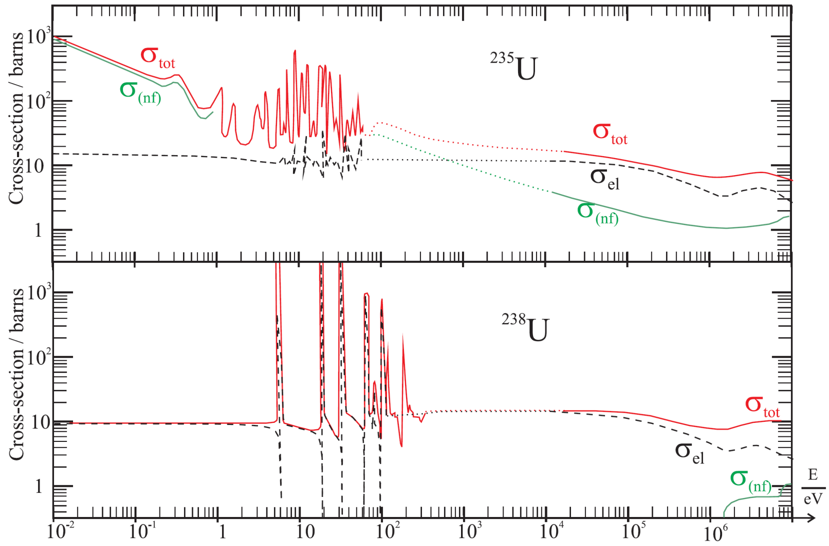
□

Question 2.6

The figure below shows the total (σ_{tot}), elastic (σ_{el}) and induced fission $(\sigma_{(\text{nf})})$ cross-sections for scattering of neutrons on $^{238}_{92}$U and $^{235}_{92}$U versus the neutron kinetic energy E.



a) Why is the total cross-section for neutron scattering off ${}^{238}_{92}$U independent of the energy at low energies ($i1e$V), but decreases with energy for the ${}^{235}_{92}$U in the same energy range?

b) The total cross-section displays a series of resonances in the range of about 1 to 10^4eV which above 100eV are too intricate to be resolved (dotted lines up to 10^4eV).

   Assuming that the scattering process of the neutron from the Uranium nucleus is like the classical scattering on a hard ball with a cross-section $^1\sigma = \pi R^2$, show for ${}^{238}_{92}$U that the order of magnitude of the total cross-section away from the resonances is given by this formula with $R = R_{\rm U}$, the radius of the Uranium nucleus, whereas at the resonances it is given by $1/k^2$, where k is the wavenumber of the neutron.

c) What is the mean free path for neutrons in natural uranium at $E = 2$MeV and for thermal neutrons at $T = 300°$C?

d) Let p be the probability that a neutron with kinetic energy E induces fission in a single collision with a nucleus in natural uranium. Let q be the probability that a neutron induces fission in natural uranium before being lost in an inelastic collision. Find expressions for p and q in terms of the cross-sections $\sigma_{\rm tot}$,$\sigma_{\rm (nf)}$, and $\sigma_{\rm el}$. From the plots obtain a crude estimate for p and q at the energies given in part b).

   Assuming that fission of a uranium nucleus produces on average 2.5 free neutrons find an expression for the neutron multiplication factor η in terms of one of the above probabilities and obtain a prediction for its value for the two energies.

[Natural U contains $c = 0.7\%$ of ${}^{235}_{92}$U by number and the remainder ${}^{238}_{92}$U. It's density is $\rho = 19.1$ g/ cm^3, the atomic mass 238.0u, and the nuclear radius $R_{\rm U} = 11.7$fm.]