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Problem Sheet 1

ASO: Group Theory

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Question 1

Let H and K be subgroups of a group G. Show that $HK := \{hk : h \in H, k \in K\}$ is a subgroup of G if and only if $HK = KH$.

Proof. "$\Longleftarrow$": Suppose that $HK = KH$. For $a, b \in HK$, there exists $h_1, h_2 \in H$ and $k_1, k_2 \in K$ such that $a = h_1k_1$, $b = h_2k_2$. Then $ab^{-1} = h_1k_1(h_2k_2)^{-1} = h_1k_1k_2^{-1}h_2^{-1}$. Since K is a subgroup of G, $k_1k_2^{-1} \in K$. Then $k_1k_2^{-1}h_2^{-1} \in KH = HK$. There exists $h' \in H$, $k' \in K$ such that $k_1k_2^{-1}h_2^{-1} = h'k'$. Hence $ab = h_1h'k' \in HK$ since $h_1h' \in H$. By the subgroup test, HK is a subgroup of G.

"$\Longrightarrow$": Suppose that HK is a subgroup of G. For $kh \in KH$ ($k \in K$ and $h \in H$), $h^{-1}k^{-1} \in HK$. Since $HK \leqslant G$, $(h^{-1}k^{-1})^{-1} = kh \in HK$. Hence $KH \subseteq HK$. On the other hand, for $hk \in HK$ ($k \in K$ and $h \in H$), there exists $h'k' \in HK$ ($k' \in K$ and $h' \in H$) such that $hkh'k' = e$. Then $hk = k'^{-1}h'^{-1} \in KH$. Hence $HK \subseteq KH$. We conclude that $HK = KH$. □

Question 2

Let $K \triangleleft G$. Denote $\overline{G} = G/K$ and let $\overline{H} \leqslant \overline{G}$. Show that $H = \{h \in G : hK \in \overline{H}\}$ is a subgroup of G, containing K as a normal subgroup and such that $H/K = \overline{H}$. Show further that if $\overline{H}$ is normal in $\overline{G}$ then H is normal in G.

Proof. For $h_1, h_2 \in H$, $h_1K, h_2K \in \overline{H}$. Since $\overline{H} \leqslant \overline{G}$, $h_1K(h_2K)^{-1} = h_1h_2^{-1}K \in \overline{H}$. Hence $h_1h_2^{-1} \in H$. BY the subgroup test H is a subgroup of G. Since $K \in \overline{H}$, we have $K \subseteq H$. $K \triangleleft G$ implies that $K \triangleleft H$. And it is trivial that $H/K = \{hK : h \in H\} = \overline{H}$. Finally, suppose that $\overline{H} \triangleleft \overline{G}$. For $h \in H$, $g \in G$, $hK \in \overline{H}$, and we have $(gK)^{-1}(hK)(gK) = (g^{-1}hg)K \in \overline{H}$. Hence $g^{-1}hg \in H$. We deduce that H is normal in G. □

Question 3

Identify the following groups from the given presentations.

(i) $G_1 = \langle x \mid x^6 = e \rangle$.

(ii) $G_2 = \langle x, y \mid xy = yx \rangle$.

(iii) $G_3 = \langle x, y \mid x^3y = y^2x^2 = x^2y \rangle$.

(iv) $G_4 = \langle x, y \mid xy = yx,\ x^5 = y^3 \rangle$.

(v) $G_5 = \langle x, y \mid xy = yx,\ x^4 = y^2 \rangle$.

Proof. (i) We claim that $G_1 \cong C_6$. Suppose that C_6 is generated by g. Consider the map that sends x to $g \in C_6$. By universal property of free groups it induces a group epimorphism $\varphi : F(\{x\}) \to C_6$. Since $\varphi(x^6) = g^6 = e$, by universal property of quotient groups, φ induces the group epimorphism $\tilde{\varphi} : G_1 = F(\{x\})/\langle\langle x^6 \rangle\rangle \to C_6$.

It remains to prove the injectivity. For $x^n \in \ker \tilde{\varphi}$, $\tilde{\varphi}(x^n) = g^n = e \Longrightarrow n \in 6\mathbb{Z} \Longrightarrow x^n = e$. Hence $\ker \tilde{\varphi} = \{e\}$. We conclude that φ is a group isomorphism.

(ii) We claim that $G_2 \cong \mathbb{Z}^2$. Consider the map which sends x to $(0, 1)$ and y to $(0, 1)$. It induces a group epimorphism $\varphi : F(\{x, y\}) \twoheadrightarrow \mathbb{Z}^2$. Since $\varphi(xy(yx)^{-1}) = (1, 0) + (0, 1) - (1, 0) - (0, 1) = 0$, φ induces the group epimorphism $\tilde{\varphi} : G_2 \twoheadrightarrow \mathbb{Z}^2$.

It remains to prove the injectivity. Since $xy = yx$ is a relation, we can verify that x, y, x^{-1} and y^{-1} commutes with each other. Inductively we have $x^my^n = y^nx^m$ for all $m, n \in \mathbb{Z}$. Suppose that $w \in F(\{x, y\})$ such that $\varphi(w) = 0$. w is a finite string with alphabet $\{x, y\}$. For each substring y^mx^n in w, we insert the relation $y^{-m}x^ny^mx^{-n}$ between y^m and x^n. Inductively we have that $w = x^iy^j \in G_2$.

Therefore $\tilde{\varphi}(w) = (i,0)+(0,j) = (i,j) = (0,0) \implies i = 0, j = 0 \implies w = 0$. We conclude that $\tilde{\varphi}$ is a group isomorphism.

Remark. Using the same idea, we can prove that $\langle X_1 \mid R_1\rangle \times \langle X_2 \mid R_2\rangle \cong \langle X_1 \sqcup X_2 \mid R_1 \sqcup R_2 \sqcup [X_1, X_2]\rangle$, where $[X_1, X_2] := \{x_1x_2x_1^{-1}x_2^{-1} : x_1 \in X_1, x_2 \in X_2\}$ is the commutator of X_1 and X_2.

(iii) We claim that $G_3 \cong \{e\}$. The relation $x^3y = x^2y$ implies that $x = e$. Hence $x^3y = y^2x^2$ implies that $y = y^2$, which implies that $y = e$. Then G_3 is the trivial group.

(iv) As we have shown in part (ii), the relation $xy = yx$ implies that G_4 is Abelian. Then it is a finitely-presented $\mathbb{Z}$-module. There is an exact sequence:

$$\mathbb{Z}^2 \xrightarrow{\ \varphi\ } \mathbb{Z} \xrightarrow{\ \pi\ } G_4 \longrightarrow \{e\}$$

The group homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}^2$ sends 1 to $(4,-2)$, which corresponds to the relations in G_4. This gives the presentation matrix:

$$\varphi = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

We can put the matrix of φ into Smith normal form by repeatedly applying Euclidean algorithm:

$$\begin{pmatrix} 5 \\ -3 \end{pmatrix} \sim \begin{pmatrix} -1 \\ -3 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 3 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Hence $\operatorname{im}\varphi = \mathbb{Z}$. By exactness and first isomorphism theorem, we have

$$G_4 = \operatorname{im}\pi \cong \mathbb{Z}^2/\ker\pi \cong \mathbb{Z}^2/\operatorname{im}\varphi = \mathbb{Z}^2/\mathbb{Z} \cong \mathbb{Z}.$$

(v) The proceduce is the same as in part (iv). The relation $xy = yx$ implies that G_5 is Abelian. We can write down its presentation matrix and put it into Smith normal form:

$$\varphi = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \sim \begin{pmatrix} 2 \\ -2 \end{pmatrix} \sim \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Hence $\operatorname{im}\varphi = 2\mathbb{Z}$. We have

$$G_5 \cong \operatorname{coker}\varphi = \mathbb{Z}^2/\operatorname{im}\varphi = \mathbb{Z}^2/2\mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Remark. Part (iv) and (v) are special cases of the Structure Theorem for finitely generated modules over a PID.

□

Question 4

Let G be a non-Abelian group of order 8. We know then that all elements have order 1, 2 or 4 and there exist elements of order 4. Let a be an element of order 4, set $A = \langle a\rangle$ and let $b \in G\backslash A$. Show that $b^{-1}ab = a^{-1}$ and that either $b^2 = e$ or $b^2 = a^2$. Use this to prove that up to isomorphism there are just five groups of order 8.

Proof. If G has an element of order 8, then $G \cong C_8$ is a cyclic group and is Abelian. If G has only elements of order 1 and 2, then by Prelim Group Theory Sheet 4 Question 5 we know that $G \cong C_2^3$ is Abelian.

Suppose that G is non-Abelian. We have $G = \langle a\rangle \cup b\langle a\rangle$. We obtain a complete list of elements in G:

$$G = \{e, a, a^2, a^3, b, ba, ba^2, ba^3\}.$$

Consider the element $ab \in G$. Clearly $ab \notin \langle a\rangle$ because $b \notin \langle a\rangle$. $ab \neq ba$ because G is non-Abelian. If $ab = ba^2$, then

$$a^3b = a^2ba = aba^4 = ab \implies a^2 = e,$$

which is a contradiction. It could only be the case that

$$ab = ba^3 = ba^{-1} \implies b^{-1}ab = a^{-1}.$$

The order of b is 2 or 4. If $b^2 = e$, then $G = \langle a, b \mid a^4, b^2, abab^{-1}\rangle \cong D_8$. If $b^2 \neq e$, then b has order 4, and b^2 has order 2. Since a and a^3 has order 2, $b^2 \neq a, a^3$. If $b^2 = ba$, then $b = a$, which is impossible. If $b^2 = ba^2$, then $b = a^2$ has order 2, which is impossible. If $b^2 = ba^3$, then $b = a^3$, which is impossible. The only possibility is that $b^2 = a^2$. The map $a \mapsto \boldsymbol{i}$, $b \mapsto \boldsymbol{j}$, $ba \mapsto \boldsymbol{k}$, $a^2 \mapsto -1$ induces a group isomorphism $G \to Q_8$. We deduce that any non-Abelian group of order 8 is isomorphism to D_8 and Q_8. Moreover, we know that the Abelian groups of order 8 are C_8, $C_4 \times C_2$ and C_2^3. We conclude that there are five different groups of order 8 up to isomorphism. □

Question 5

Let $G = \langle x, y \mid x^2 = e = y^2\rangle$ and let D_∞ denote the isometry group of $\mathbb{Z}$, the *infinite dihedral group*.

(i) Show that G is infinite.

(ii) Let $z = xy$. Show that every element of G can be uniquely written as z^k or yz^k where $k \in \mathbb{Z}$. Show that $G = \langle y, z \mid y^2 = e, yzy = z^{-1}\rangle$.

(iii) Show that $y(n) = -n$ and $z(n) = n + 1$ are elements of D_∞. Deduce that $G \cong D_\infty$.

Proof. (i) We claim that $(xy)^n$ are distinct elements in G for different $n \in \mathbb{Z}$. Suppose that there are $n_1, n_2 \in \mathbb{Z}$ such that $(xy)^{n_1} = (xy)^{n_2}$. Then we have $(xy)^m = e$ in G, where $m := n_1 - n_2$. Since $G = F(\{x, y\})/\langle\langle x^2, y^2\rangle\rangle$ and $(xy)^m$ is a reduced word in $F(\{x, y\})$, $(xy)^m \in \langle\langle x^2, y^2\rangle\rangle$. But

$$\langle\langle x^2, y^2\rangle\rangle = \langle\{w^{-1}x^2w,\ w^{-1}y^2w :\ w \in F(\{x, y\})\}\rangle$$

Every non-empty word in $\langle\langle x^2, y^2\rangle\rangle$ must contain the substring x^2 or y^2. Therefore $m = 0$ and $(xy)^{n_1} = (xy)^{n_2}$. We conclude that G is infinite.

(ii) We can perform a sequence of Tietze transformations:

$$\begin{aligned} G &= \langle x, y \mid x^2,\ y^2\rangle \\ &\cong \langle x, y, z \mid x^2,\ y^2,\ z^{-1}xy\rangle \\ &\cong \langle x, y, z \mid x^2,\ y^2,\ x^{-1}zy^{-1}\rangle \\ &\cong \langle y, z \mid zy^{-1}zy^{-1},\ y^2\rangle \\ &\cong \langle y, z \mid z = yz^{-1}y,\ y^2\rangle \\ &\cong \langle y, z \mid yzy = z^{-1},\ y^2\rangle \end{aligned}$$

$yzy = z^{-1}$ inplies that $z^{-1}y^{-1}z^{-1}y^{-1}$ and $zyzy$ are relations in G. For a word $w \in F(\{y, z\})$, we insert the relation $z^{-1}y^{-1}z^{-1}y^{-1}$ into the middle of the substrings zy and zy^{-1}, and $zyzy$ into the middle of the substrings $z^{-1}y$ and $z^{-1}y^{-1}$. After each operation we invert the order of y^i and z^j in the substrings. Since w has finite length, eventually we will obtain that w equals y^lz^k in G. Since $y^2 = e$ in G, we may assume that $l \in \{0, 1\}$. In conclusion, every element in G can b written as z^k or yz^k.

Moreover, the expression is unique of each element, as we have proven in part (i) that $z^{k_1} \neq z^{k_2}$ for $k_1 \neq k_2$.

(iii) For $m, n \in \mathbb{Z}$, $|y(m) - y(n)| = |-m - (-n)| = |m - n|$. So $y \in D_\infty$. $|z(m) - z(n)| = |m + 1 - (n + 1)| = |m - n|$. So $z \in D_\infty$. Since

$$y \circ y(n) = y(-n) = n$$

and

$$y \circ z \circ y(n) = y \circ z(-n) = y(-n + 1) = n - 1 = z^{-1}(n)$$

y^2 and $yzy = z^{-1}$ are relations in D_∞. Therefore there exists a group epimorphism $\varphi : G \twoheadrightarrow D_\infty$ which sends y, z in G to y, z in D_∞. It remains to check the injectivity. For $z^k \in G$, $\varphi(z^k)(n) = n + k$. Therefore $z^k \in \ker\varphi$ implies that $k = 0$. For $yz^k \in G$, $\varphi(yz^k)(n) = -n + k$. Then $yz^k \notin \ker\varphi$ for all $k \in \mathbb{Z}$. We deduce that $\ker\varphi = \{e\}$. We conclude that $\varphi : G \to D_\infty$ is an isomorphism.

□

Question 6

(i) Let $n \geqslant 1$. Show that (12) and $(123\cdots n)$ generate S_n.

(ii) Show that $\mathbb{Q}$ is not finitely generated.

Proof. (i) We follow the convention of multiplication order which is consistent with map compositions. First, note that

$$\begin{aligned}(2\,3) &= (1\,2\,\cdots\,n)^{-1}(1\,2)(1\,2\,\cdots\,n)\\ (3\,4) &= (1\,2\,\cdots\,n)^{-1}(2\,3)(1\,2\,\cdots\,n)\\ &\cdots\\ (n\,1) &= (1\,2\,\cdots\,n)^{-1}(n-1\,n)(1\,2\,\cdots\,n)\end{aligned}$$

Hence $(1\,2)$ and $(1\,2\,\cdots\,n)$ generates $\{(1\,2),\ (2\,3),\ ...,\ (n-1\,n),\ (n\,1)\}$.

Second, note that

$$\begin{aligned}(1\,3) &= (2\,3)(1\,2)(2\,3)\\ (1\,4) &= (3\,4)(1\,2)(3\,4)\\ &\cdots\\ (1\,n) &= (n-1\,n)(1\,n-1)(n-1\,n)\end{aligned}$$

Hence $(1\,2)$ and $(1\,2\,\cdots\,n)$ generates $\{(1\,2),\ (1\,3),\ ...,\ (1\,n)\}$.

Third, note that $(i\,j) = (1\,i)(1\,j)(1\,i)$ for all $i, j \in \{1, ..., n\}$. Hence $(1\,2)$ and $(1\,2\,\cdots\,n)$ generates all transpositions in S_n.

Finally, for any k-cycle in S_n, $(i_1\,\cdots\,i_k) = (i_1\,i_k)(i_1\,i_{k-1})\cdots(i_1\,i_2)$. All k-cycles are products of transpositions. In addition, all elements in S_n can be written uniquely as disjoint product of cycles. We conclude that $(1\,2)$ and $(1\,2\,\cdots\,n)$ generates S_n.

(ii) $\mathbb{Q}$ is Abelian group. In particular it is a $\mathbb{Z}$-module. Suppose that it is finitely generated. Notice that $2\mathbb{Z}$ is an ideal of $\mathbb{Z}$ and $(2\mathbb{Z})\mathbb{Q} = \mathbb{Q}$ since $2 \in \mathbb{Q}$ is a unit. Then by Nakayama's Lemma there exists $x \in 2\mathbb{Z}$ such that $(1 + x)\mathbb{Q} = 0$. But $x \neq -1$ implies that $(1 + x) \cdot 1 \neq 0$, which is a contradiction. Hence $\mathbb{Q}$ is not a finitely generated $\mathbb{Z}$-module.

□

Question 7

Write down all possible composition series of the following groups, verifying the Jordan-Hölder Theorem where appropriate.

$$\mathbb{Z}/12\mathbb{Z}, \qquad D_{10}, \qquad D_8, \qquad Q_8.$$

Proof. The composition series of $\mathbb{Z}/12\mathbb{Z}$:

$$\begin{aligned}&\{e\} \lhd \mathbb{Z}/2\mathbb{Z} \lhd \mathbb{Z}/4\mathbb{Z} \lhd \mathbb{Z}/12\mathbb{Z}\\ &\{e\} \lhd \mathbb{Z}/2\mathbb{Z} \lhd \mathbb{Z}/6\mathbb{Z} \lhd \mathbb{Z}/12\mathbb{Z}\\ &\{e\} \lhd \mathbb{Z}/3\mathbb{Z} \lhd \mathbb{Z}/6\mathbb{Z} \lhd \mathbb{Z}/12\mathbb{Z}\end{aligned}$$

The length of composition series of $\mathbb{Z}/12\mathbb{Z}$ is 3. The composition factors are C_2, C_2 and C_3.

Let $D_{10} = \langle \sigma, \tau \mid \sigma^5,\ \tau^2,\ \sigma\tau\sigma\tau \rangle$. The only non-trivial subgroup of D_{10} is $\langle \sigma \rangle$. Then the only composition series of D_{10} is

$$\{e\} \lhd \langle \sigma \rangle \lhd D_{10}$$

The composition factors are C_2 and C_5.

Let $D_8 = \langle \sigma, \tau \mid \sigma^4,\ \tau^2,\ \sigma\tau\sigma\tau \rangle$. Since D_8 is solvable, we know that every composition series of it has the form

$$\{e\} \lhd G_1 \lhd G_2 \lhd D_8,$$

where G_2 is a normal subgroup of D_8 of order 4, and G_1 is a subgroup of G_2 of order 2. If G_1 is cyclic, then $G_1 = \langle \sigma \rangle$. The only subgroup in G_1 is $\langle \sigma^2 \rangle$. If G_2 is not cyclic, then it contains onl elements of order 1 and 2. The only possibilitiies are $\langle \sigma^2, \tau \rangle$ and $\langle \sigma^2, \sigma\tau \rangle$, each of which is isomorphic to V_4. In summary, all composition series of D_8 are:

$$\begin{gathered}
\{e\} \lhd \langle \sigma^2 \rangle \lhd \langle \sigma \rangle \lhd D_8 \\
\{e\} \lhd \langle \sigma^2 \rangle \lhd \langle \sigma^2, \tau \rangle \lhd D_8 \\
\{e\} \lhd \langle \tau \rangle \lhd \langle \sigma^2, \tau \rangle \lhd D_8 \\
\{e\} \lhd \langle \sigma^2\tau \rangle \lhd \langle \sigma^2, \tau \rangle \lhd D_8 \\
\{e\} \lhd \langle \sigma^2 \rangle \lhd \langle \sigma^2, \sigma\tau \rangle \lhd D_8 \\
\{e\} \lhd \langle \sigma\tau \rangle \lhd \langle \sigma^2, \sigma\tau \rangle \lhd D_8 \\
\{e\} \lhd \langle \sigma^3\tau \rangle \lhd \langle \sigma^2, \sigma\tau \rangle \lhd D_8
\end{gathered}$$

All composition series of D_8 have length 3 and composition factors C_2, C_2, C_2.

Let $Q_8 = \{\pm 1, \pm \boldsymbol{i},\ \pm \boldsymbol{j}, \pm \boldsymbol{k}\}$. Since Q_8 is solvable, we know that every composition series of it has the form

$$\{1\} \lhd G_1 \lhd G_2 \lhd Q_8,$$

where G_2 is a normal subgroup of D_8 of order 4, and G_1 is a subgroup of G_2 of order 2. The normal subgroups of order 4 are $\langle \pm \boldsymbol{i} \rangle$, $\langle \pm \boldsymbol{j} \rangle$ and $\langle \pm \boldsymbol{k} \rangle$, each of which is cyclic. Therefore all composition series of Q_8 are:

$$\begin{gathered}
\{1\} \lhd \langle -1 \rangle \lhd \langle \boldsymbol{i} \rangle \lhd Q_8 \\
\{1\} \lhd \langle -1 \rangle \lhd \langle \boldsymbol{j} \rangle \lhd Q_8 \\
\{1\} \lhd \langle -1 \rangle \lhd \langle \boldsymbol{k} \rangle \lhd Q_8
\end{gathered}$$

All composition series of Q_8 have length 3 and composition factors C_2, C_2, C_2. □

Question 8

Let

$$\{e\} \lhd G_1 \lhd G \qquad \text{and} \qquad \{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_r = G$$

be two composition series for a group G. Why is $r \geqslant 2$? Why is $r = 2$ if $H_{r-1} = G_1$? Show that if $H_{r-1} \neq G_1$ then $G_1 \cap H_{r-1} = \{e\}$. Show that $G_1 H_{r-1}$ is normal in G and that $G/G_1 = H_{r-1}$. Deduce that $r = 2$.

Proof. If $r = 0$, then $G = \{e\}$, which is impossible. If $r = 1$, then $\{e\} \lhd G$ is a composition series. In particular, G is simple. Then $G_1 = \{e\}$ or G, which is impossible. Hence $r \geqslant 2$.

If $H_{r-1} = G_1$, then H_{r-1} is simple. $\{e\} \lhd H_{r-1} \lhd G$ is a composition series. We deduce that $r = 2$.

If $H_{r-1} \neq G_1$, then let $K = G_1 \cap H_{r-1}$. Since G and H_{r-1} are normal subgroups of G, K is normal in G. In particuler K is normal in G_1. But G_1 is simple implies that $K = \{e\}$.

For $a_1, a_2 \in G_1$ and $b_1, b_2 \in H_{r-1}$, $a_1b_1, a_2b_2 \in G_1H_{r-1}$. $a_1b_1(a_2b_2)^{-1} = a_1b_1b_2^{-1}a_2^{-1}$. Since $H_{r-1} \lhd G$, $b_1b_2^{-1} \in H_{r-1}$ and there exists $b_3 \in H_{r-1}$ such that $b_1b_2^{-1}a_2^{-1} = a_2b_3$. Hence $a_1b_1(a_2b_2)^{-1} = a_1a_2b_3 \in G_1H_{r-1}$. By subgroup test, $G_1H_{r-1} \leqslant G$. For $g \in G$, $a \in G_1$ and $b \in H_{r-1}$, $g^{-1}(ab)g = (g^{-1}ag)(g^{-1}bg) \in G_1H_{r-1}$. Hence $G_1H_{r-1} \lhd G$.

We have $G_1 \triangleleft G_1 H_{r-1} \triangleleft G$. If $G_1 H_{r-1} = G_1$, then $H_{r-1} \subseteq G$, contradicting that $G_1 \cap H_{r-1} = \{e\}$ and $H_{r-1} \neq \{e\}$. Therefore $G_1 H_{r-1} = G$. By second isomorphism theorem,

$$\frac{G}{G_1} = \frac{G_1 H_{r-1}}{G_1} \cong \frac{H_{r-1}}{G_1 \cap H_{r-1}} \cong H_{r-1}.$$

But we know that G/G_1 is simple. Hence H_{r-1} is simple. $\{e\} \triangleleft H_{r-1} \triangleleft G$ is a composition series. We conclude that $r = 2$. $\square$