Topology & Groups

Michaelmas 2016

Question Sheet 2

Questions with an asterisk $*$ beside them are optional.

1. Let $\alpha\colon S^n \to S^n$ be the antipodal map (defined by $\alpha(x) = -x$). Prove that α is homotopic to the identity if n is odd.

2. For any two maps $f, g\colon X \to S^n$ such that $f(x) \neq -g(x)$ for all $x \in X$, show that $f \simeq g$.

3. Let X be a contractible space and let Y be any space. Show that

   (i) X is path-connected;

   (ii) $X \times Y$ is homotopy equivalent to Y;

   (iii) any two maps from Y to X are homotopic;

   (iv) if Y is path-connected, any two maps from X to Y are homotopic.

The *wedge* $X \vee Y$ of two spaces X and Y, containing basepoints x and y, is the space obtained from the disjoint union of X and Y by identifying x and y. Often, the resulting space is independent of the choice of basepoints, in which case there is no need to specify them. (See Definition V.26.)

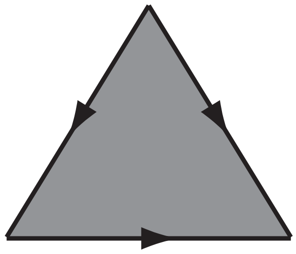
4. Prove that the following spaces are homotopy equivalent:

   (i) $S^1 \vee S^1$,

   (ii) the torus with one point removed,

   (iii) $\mathbb{R}^2$ minus two points.

$*$ 5. *(Harder)* For maps $f, g\colon S^{n-1} \to X$, let $X \cup_f D^n$ and $X \cup_g D^n$ be the spaces obtained by attaching n-cells to X along f and g respectively. Show that if f and g are homotopic maps $S^{n-1} \to X$, then $X \cup_f D^n$ and $X \cup_g D^n$ are homotopy equivalent. Deduce that the space (known as the 'dunce cap') obtained by identifying the three sides of a triangle, as shown overleaf, is contractible.



6. Let K and L be finite simplicial complexes. Prove that there are only countably many homotopy classes of maps $|K| \to |L|$.

7. Prove that any two maps $S^m \to S^n$, where $m < n$, are homotopic. [Hint: use the Simplicial Approximation Theorem.]

Topology & Groups 2 — Peize Liu

1. If n is odd, we have the natural embedding $S^n \subset \mathbb{C}^{\frac{n+1}{2}}$.
That is, $S^n = \{(z_1, \cdots, z_k) : |z_1|^2 + \cdots + |z_k|^2 = 1\}$ where $k = \frac{n+1}{2}$.
Let $H : S^n \times [0,1] \to S^n$ defined by:
$$H(z_1, \cdots, z_k; t) = e^{i\pi(t-1)}(z_1, \cdots, z_k).$$
~~Then we have : $H(z_1, \cdots, z_k$~~
This is well-defined because $\|e^{i\pi(t-1)}(z_1, \cdots, z_k)\|_2 = 1$ for all $z_1, \cdots, z_k \in \mathbb{C}$ and $t \in [0,1]$. Then $H(S^n \times [0,1]) \subset S^n$.
We have $H(z_1, \cdots, z_k; 0) = -(z_1, \cdots, z_k) = \alpha(z_1, \cdots, z_k)$ and $H(z_1, \cdots, z_k; 1) = (z_1, \cdots, z_k) = \mathrm{id}_{S^n}$.
Moreover, H is continuous on $S^n \times [0,1]$ as the exponential function is continuous.
Therefore H is a homotopy, α is homotopic to id_{S^n}.

✓ Good A+

2. There is a natural embedding $S^n \subset \mathbb{R}^{n+1}$ such that $S^n := \{x \in \mathbb{R}^{n+1} : \|x\|_2 = 1\}$.
We consider the homotopy given by projecting the straight-line homotopy onto S^n. Explicitly, let $H : X \times [0,1] \to S^n$ defined by:
$$H(x,t) = \frac{tf(x) + (1-t)g(x)}{\|tf(x) + (1-t)g(x)\|_2}$$
Since $f(x) \neq -g(x)$ for all $x \in X$, H is always well-defined and continuous on $X \times [0,1]$. Hence H is a homotopy:
$H(x,0) = g(x)/\|g(x)\| = g(x)$; $H(x,1) = f(x)/\|f(x)\| = f(x)$.
Hence $g \overset{H}{\simeq} f$ as required. ✓ A+

~~3.(i) Suppose X is not path-connected.~~

3.(i) Since X is contractible, $\exists\, x_0 \in X : id_X \simeq c_{x_0}$.
That is, there is a continuous mapping $H : X \times [0,1] \to X$
such that $H(x,0) = x$ ($\forall x \in X$) and $H(x,1) = x_0$.
Fix any $x \in X$, $H(x,\cdot) : [0,1] \to X$ is a (continuous) path
such that $H(x,0) = x$ and $H(x,1) = x_0$.
Hence x and x_0 are in the same path component.
X is path-connected. ✓

(ii) From (i) we have the homotopy $id_X \overset{H}{\simeq} c_{x_0}$, which can be
naturally extend to $id_{X\times Y} \overset{\widetilde{H}}{\simeq} c_{x_0} \times id_Y$ as follows:
$\widetilde{H}(x,y,t) = (H(x,t), y)$.
Now we define $\pi : X \times Y \to Y$ by $(x,y) \mapsto y$
and $\iota : Y \to X \times Y$ by $y \mapsto (x_0, y)$.
Then trivially $\pi \circ \iota = id_Y$ and $\iota \circ \pi \overset{\widetilde{H}}{\simeq} id_{X\times Y}$
Hence we have $X \times Y \simeq Y$. ✓ G.o.a.t

(iii) Suppose $f : Y \to X$ and $g : Y \to X$ are continuous. We claim
that $f \overset{K}{\simeq} g$ via $K : Y \times [0,1] \to X$ defined by:

$$K(y,t) = \begin{cases} H(f(y), 2t), & t \in [0, \frac{1}{2}] \\ H(g(y), 1-2t), & t \in (\frac{1}{2}, 1]. \end{cases}$$

Or simply: $f \overset{H\circ f}{\simeq} c_{x_0} \overset{H\circ g}{\simeq} g$.
K is continuous by the gluing lemma.
And $K(y,0) = f(y)$, $K(y,1) = g(y)$. ✓

(iv) Suppose $f : X \to Y$ and $g : X \to Y$ are continuous.
Let $\iota_f : \{x_0\} \to Y$ defined by $x_0 \mapsto f(x_0)$ and
$\iota_g : \{x_0\} \to Y$ defined by $x_0 \mapsto g(x_0)$.
Then $\iota_f \overset{\gamma}{\simeq} \iota_g$ via a path $\gamma : \{x_0\} \times [0,1] \to Y$, which exists

Campus

since Y is path-connected.
Moreover, we have $f \simeq l_f \circ c_{x_0}$ and $g \simeq l_g \circ c_{x_0}$.
By Lemma II.6 we have $f \simeq g$ as required.

4. We may assume that all $S^1 \vee S^1$ are homotopy equivalent regardless of the way of construction or embedding. And similar for (ii) and (iii).

(i) $\simeq$ (iii) : There is an embedding $\iota : S^1 \vee S^1 \to \mathbb{R}^2$ such that

~~$\iota(S^1 \vee S^1) = \{\vec{x} \in \mathbb{R}^2 : \|\vec{x} - 1\| = 1\} \cup \{\vec{x} \in \mathbb{R}^2 : \|\vec{x}+1\| = 1\}$~~.

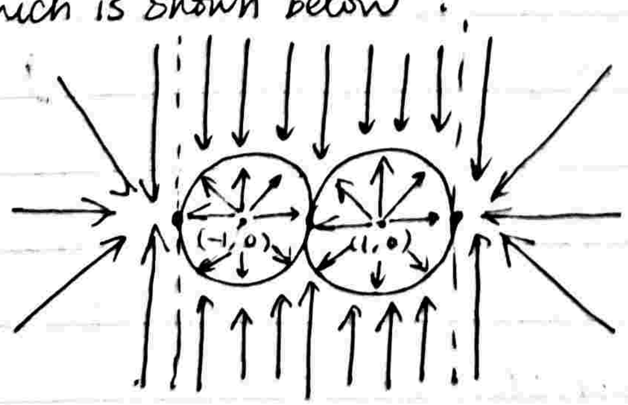
~~Let $r : \mathbb{R}^2 \setminus \{(1,0), (-1,$~~

$\iota(S^1 \vee S^1) = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + y^2 = 1\} \cup \{(x,y) \in \mathbb{R}^2 : (x+1)^2 + y^2 = 1\}$

Let $r : \mathbb{R}^2 \setminus \{(1,0), (-1,0)\} \to S^1 \vee S^1$ defined by :

$$r(x,y) = \begin{cases} \dfrac{(x+1, y)}{\|(x+1,y)\|} + (-1,0), & (x+1)^2 + y^2 < 1 \\ \dfrac{(x-1,y)}{\|(x-1,y)\|} + (1,0), & (x-1)^2 + y^2 < 1 \\ (x, \sqrt{1-(x+1)^2}), & x \in [-2, 0) \wedge (x+1)^2 + y^2 > 1 \\ (x, \sqrt{1-(x-1)^2}), & x \in [0,2] \wedge (x-1)^2 + y^2 > 1 \\ \dfrac{(x+2, y)}{\|(x+2,y)\|} + (-2,0), & x < -2 \\ \dfrac{(x-2,y)}{\|(x-2,y)\|} + (2,0), & x > 2 \end{cases}$$

which is shown below :



Nice Picture

Intuitively r is continuous. Giving a formal proof here would be too horrible.

Then we have $r \circ \iota = id_{S^1 \vee S^1}$ and $\iota \circ r \overset{H}{\simeq} id_{\mathbb{R}^2 \setminus \{(1,0),(-1,0)\}}$ where $H : \mathbb{R}^2 \setminus \{(1,0),(-1,0)\} \times [0,1] \to \mathbb{R}^2 \setminus \{(1,0),(-1,0)\}$ is given by :

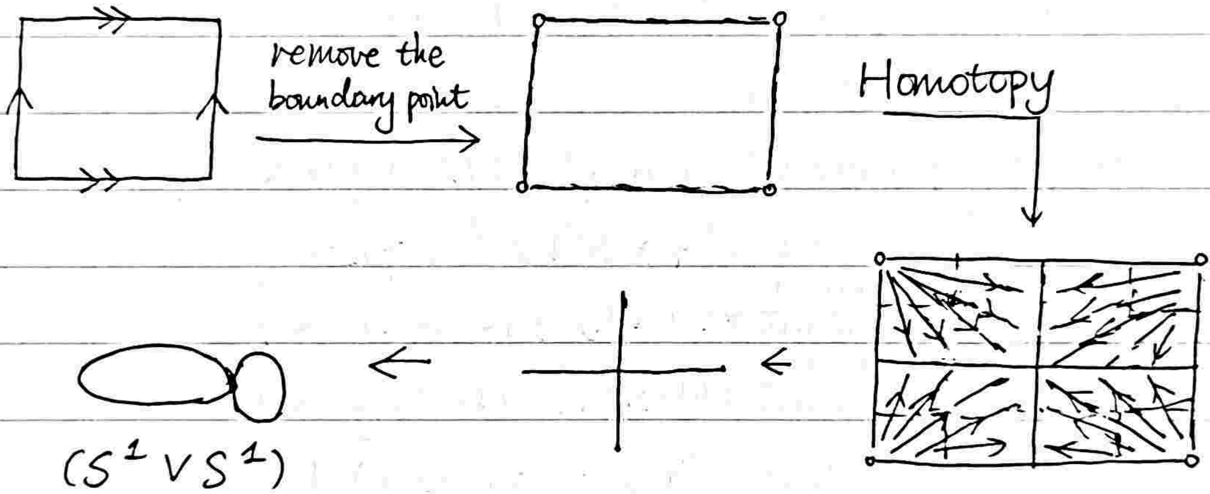
$H(x,y,t) = t(x,y) + (1-t)r(x,y)$.

$\Rightarrow H(x,y,0) = r(x,y)$, $H(x,y,1) = id_{\mathbb{R}^2 \setminus \{(1,0),(-1,0)\}}$.

Hence we have $S^1 \vee S^1 \simeq \mathbb{R}^2 \setminus \{(1,0),(-1,0)\}$.

(i) $\simeq$ (ii) : An intuitive illistruation would be as follows :

A torus is obtained by identifying the sides of a square :



Explicitly, ~~suppose the torus $T \subset \mathbb{R}^3$ is given by~~ :

~~$T = \{(x,y,z) \in \mathbb{R}^3 : (\sqrt{x^2+y^2} - R)^2 + z^2 = r^2\}$~~

suppose the torus $T \subset \mathbb{R}^4$ is given by :

$T = \{(\cos\theta_1, \sin\theta_1, \cos\theta_2, \sin\theta_2) \in \mathbb{R}^4 : \theta_1, \theta_2 \in [0, 2\pi]\}$.

We have the following homeomorphisms

$\sigma_1 : [0,1]^2 \to T_1$, $(t_1,t_2) \mapsto (\cos\pi t_1, \sin\pi t_1, \cos\pi t_2, \sin\pi t_2)$

$\sigma_2 : [0,1]^2 \to T_2$, $(t_1,t_2) \mapsto (\cos\pi(t_1-1), \sin\pi(t_1-1), \cos\pi t_2, \sin\pi t_2)$

$\sigma_3 : [0,1]^2 \to T_3$, $(t_1,t_2) \mapsto (\cos\pi t_1, \sin\pi t_1, \cos\pi(t_2-1), \sin\pi(t_2-1))$

$\sigma_4 : [0,1]^2 \to T_4$, $(t_1,t_2) \mapsto (\cos\pi(t_1-1), \sin\pi(t_1-1),$
$\cos\pi(t_2-1), \sin\pi(t_2-1))$ ...

We can see that $T = T_1 \cup T_2 \cup T_3 \cup T_4$.

If we remove $p := (1,0,1,0) \in T$, we can see that

$\sigma_1^{-1}(p) = (0,0)$, $\sigma_2^{-1}(p) = (1,0)$, $\sigma_3^{-1}(p) = (0,1)$, $\sigma_4^{-1}(p) = (1,1)$

$[0,1]^2 \setminus \{(0,0)\}$, $[0,1]^2 \setminus \{(1,0)\}$, $[0,1]^2 \setminus \{(0,1)\}$, $[0,1]^2 \setminus \{(1,1)\}$

have the following homotopy retract:

$$r_1(x,y) = \begin{cases} (1, \frac{y}{x}), & y \le x \\ (\frac{x}{y}, 1), & x > y \end{cases} \qquad r_2(x,y) = \begin{cases} (0, \frac{y}{1-x}), & y \le 1-x \\ (1-\frac{1-x}{y}, 1), & y > 1-x \end{cases}$$

$$r_3(x,y) = \begin{cases} (\frac{x}{1-y}, 0), & x \le 1-y \\ (1, 1-\frac{1-y}{x}), & x > 1-y \end{cases} \qquad r_4(x,y) = \begin{cases} (0, 1-\frac{1-y}{1-x}), & y \geq x \\ (1-\frac{1-x}{1-y}, 0), & y < x \end{cases}$$

Let $r : T \setminus \{p\} \to T$ be defined as

$r(x) = \sigma_i \circ r_i \circ \sigma_i^{-1}(x)$ for $x \in T_i$, $i = 1,2,3,4$.

It is obvious that r is a homotopy retract.

Moreover, $r(T\setminus\{p\}) = \{(-1,0,\cos\theta_2, \sin\theta_2) \in \mathbb{R}^4 : \theta_2 \in [0,2\pi]\}$
$\cup \{(\cos\theta_1, \sin\theta_1, -1, 0) \in \mathbb{R}^4 : \theta_1 \in [0,2\pi]\}$
$\cong S^1 \vee S^1$.

Hence we have $T \setminus \{p\} \simeq S^1 \vee S^1$ as required.

(ii) $\simeq$ (iii) : This is immediate by the translation of homotopy equivalence.

A+

6. By the simplicial approximation theorem, for any continuous mapping $f : |K| \to |L|$, there exists a (barycentric) subdivision K' of K and a simplicial map $g : K' \to L$ such that $|g| \simeq f$.

The theorem in fact gives an injection from the set of all continuous mappings $|K| \to |L|$ to the set of all simplicial mappings $K' \to L$. The latter set is countable, because K has countably many barycentric subdivisions $K^{(r)}$ $(r \in \mathbb{N})$, and for each $r \in \mathbb{N}$, there are finitely many simplicial mappings $K^{(r)} \to L$ (which send vertices to vertices).

✓ Great A+

7. ~~We~~ Suppose $f: S^m \to S^n$ is continuous. We claim that there exists a non-surjective ~~function~~ continuous mapping $h: S^m \to S^n$ such that $f \simeq h$.

We know that there exists homeomorphisms: $\sigma: \Delta^m \to S^m$ and $\rho: \Delta^n \to S^n$.

Consider $\tilde{f} := \rho^{-1} \circ f \circ \sigma : \Delta^m \to \Delta^n$, which is continuous.

By the simplicial approximation theorem, there exists a subdivision $(\Delta^m)'$ of Δ^m and a simplicial mapping $g: (\Delta^m)' \to \Delta^n$ such that $\tilde{f} \simeq |g|$.

$|g|$ is not surjective because the inside of Δ^n is not in the image of $|g|$, as we have $n > m$.

$\Rightarrow f \simeq \rho \circ |g| \circ \sigma^{-1}$, which is not surjective.

(Without loss of generality we now consider f itself non-surjective.)

Suppose $p \in S^n \setminus \mathrm{Im} f$. Then $f: S^m \to S^n \setminus \{p\}$.

But $S^n \setminus \{p\} \cong \mathbb{R}^n$, which is contractible.

By Q3.(iii), f is null-homotopic.

Hence any ~~two~~ ~~ma~~ continuous mapping $S^m \to S^n$ is null-homotopic, and the result follows. ✓ A+ ▨

5. We first claim that there exists a homotopy retract

$r : D^n \times [0,1] \to (D^n \times \{0\}) \cup (S^{n-1} \times [0,1])$.

There exists a natural embedding $D^n \times [0,1] \subset \mathbb{R}^{n+1}$ such that $D^n \times [0,1] := \{(x_1, \cdots x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n} x_i^2 = 1 ;\ x_{n+1} \in [0,1]\}$

Let $p = (0, \cdots, 0, 2) \in \mathbb{R}^{n+1}$. For each $q \in D^n \times [0,1]$, we define $r(q)$ to be the intersection of the line through p and q and the "wall" $(D^n \times \{0\}) \cup (S^{n-1} \times [0,1])$. Intuitively this gives a homotopy ~~restro~~ retract.

Suppose $f : S^{n-1} \to X$ and $g : S^{n-1} \to X$ are homotopic via the homotopy $H : S^{n-1} \times [0,1] \to X$. This induces an attachment $X \sqcup_H (D^n \times [0,1])$, which contains $X \cup_f D^n$ and $X \cup_g D^n$ as its subspaces.

Since $D^n \times [0,1] \simeq (D^n \times \{0\}) \cup (S^{n-1} \times [0,1])$,

we have $X \sqcup_H (D^n \times [0,1]) \simeq X \sqcup_H (D^n \times \{0\} \cup S^{n-1} \times [0,1])$

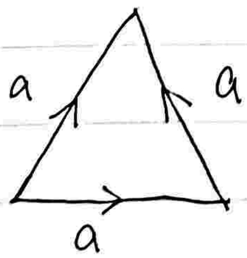
$$\simeq X \sqcup_H (D^n \times \{0\}) = X \cup_f D^n$$

Similarly since $D^n \times [0,1] \simeq (D^n \times \{1\}) \cup (S^{n-1} \times [0,1])$

we have $X \sqcup_H (D^n \times [0,1] \simeq X \sqcup_g D^n$. ✓ Nice

Hence $X \sqcup_f D^n \simeq X \sqcup_g D^n$ as required.

The dunce cap is constructed by the following side identification:



(It so should be $S^1 \cup_f D^2$ but I fail to construct $f : S^1 \to S^1$ ...)