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Problem Sheet 2

C2.6: Introduction to Schemes

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Question 1

i) Construct the scheme $\mathbb{P}^n_R$ bu gluing $(n+1)$ copies of $\mathbb{A}^n_R = \operatorname{Spec} R[y_1, ..., y_n]$, where for the i-th copy you use $y_1 = \dfrac{x_0}{x_i}, ..., y_n = \dfrac{x_n}{x_i}$ (*omit* $\dfrac{x_i}{x_i}$) (these generate a R-subalgebra of $S^{-1}R[x_0, ..., x_n]$ for S multiplicative set generated by $x_0, ..., x_n$.)

ii) Show that a homomorphism of rings $R \to S$ yields a natural map $\mathbb{P}^n_S \to \mathbb{P}^n_R$.

iii) Construct $\mathbb{P}^n_U$ for any open subscheme $U \subseteq \operatorname{Spec} R$. (*Compare lecture notes on* $\mathbb{A}^n_U$.)

iv) Construct $\mathbb{P}^n_X$: the projective n-space over any scheme X, and explain why a morphism $X \to Y$ induces a natural morphism $\mathbb{P}^n_X \to \mathbb{P}^n_Y$.

Proof. i) Let S be the multiplicative set in $R[x_0, ..., x_n]$ generated by $x_0, ..., x_n$. For $i \in \{0, ..., n\}$, we define $\varphi_i : R[y_1, ..., y_n] \to S^{-1}R[x_0, ..., x_n]$ by

$$y_1 \mapsto x_0/x_i, \ ..., \ y_i \mapsto x_{i-1}/x_i, \ y_{i+1} \mapsto x_{i+1}/x_i, \ ..., \ y_n \mapsto x_n/x_i.$$

Then let

$$R_i := \operatorname{im} \varphi_i = R\left[\frac{x_0}{x_i}, ..., \widehat{\frac{x_i}{x_i}}, ..., \frac{x_n}{x_i}\right], \qquad A_i := \operatorname{Spec} R_i$$

Equivalently, R_i is the zeroth grading of the localisation ring $R[x_0, ..., x_n]_{x_i}$. nice! Now let

$$R_{ij} := R_i\left[\frac{x_i}{x_j}\right] = (R_i)_{x_j/x_i} = \left(S_{ij}^{-1}R[x_0, ..., x_n]\right)_0$$

good comment

where S_{ij} is the multiplicative set generated by x_i, x_j. Let $A_{ij} := \operatorname{Spec} R_{ij}$. By symmetry we can see that $R_{ij} = R_{ji}$ and hence $A_{ij} = A_{ji}$. The localisation natural map $\psi : R_i \to R_{ij} = (R_i)_{x_j/x_i}$ induces an embedding $\operatorname{Spec} \psi : A_{ij} \hookrightarrow A_i$. And similarly there is an embedding $A_{ij} \hookrightarrow A_j$.

Now we define $\mathbb{P}^n_R$ as a topological space to be the push-out:

$$\mathbb{P}^n_R := A_0 \cup \cdots \cup A_n$$

where A_i and A_j are glued along A_{ij}. So in $\mathbb{P}^n_R$ we can say that $A_{ij} = A_i \cap A_j$.

As spectrum of rings, A_i carries the structure sheaf $\mathcal{O}_{\operatorname{Spec} R_i}$. We need the gluing of sheaves. Let ξ_{ij} be the composite isomorphism $\mathcal{O}_{A_i}|_{A_{ij}} \xrightarrow{\simeq} \mathcal{O}_{A_{ij}} \xrightarrow{\simeq} \mathcal{O}_{A_j}|_{A_{ij}}$ We need to check the compatibility conditions. It is clear that $\xi_{ii} = \text{id}$. Let $A_{ijk} := A_i \cap A_i \cap A_k = A_{ij} \cap A_{jk}$. It is easy to observe that $A_{ijk} = \operatorname{Spec} R_{ijk}$, where $R_{ijk} := \left(S_{ijk}^{-1}R[x_0, ..., x_n]\right)_0$, S_{ijk} is the multiplicative set generated by x_i, x_j, x_k. Then we simply have

$$\xi_{ik}|_{A_{ijk}} = \mathcal{O}_{A_k}|_{A_{ijk}} = \mathcal{O}_{A_{ijk}} = \xi_{jk} \circ \xi_{ij}|_{A_{ijk}}$$

Therefore there exists a unique sheaf $\mathcal{O}_{\mathbb{P}^n_R}$ on $\mathbb{P}^n_R$ such that $\mathcal{O}_{\mathbb{P}^n_R}|_{A_i} = \mathcal{O}_{A_i}$. This makes $\mathbb{P}^n_R$ a well-defined scheme.

ii) The ring homomorphism $\varphi : R \to S$ induces $\widetilde{\varphi} : R[y_1, ..., y_n] \to S[y_1, ..., y_n]$ and hence a morphism of schemes $(\operatorname{Spec} \varphi, \varphi^{\#}) : \mathbb{A}^n_S \to \mathbb{A}^n_R$. From (i) we have constructed $\mathbb{P}^n_R$ as $\displaystyle\bigcup_{i=0}^{n} A_i$, where $A_i \cong \mathbb{A}^n_R$. Similarly $\mathbb{P}^n_S = \displaystyle\bigcup_{i=1}^{n} A'_i$, where $A'_i \cong \mathbb{A}^n_S$. Then we define f_i by $A'_i \xrightarrow{\simeq} \mathbb{A}^n_S \xrightarrow{\operatorname{Spec}\varphi} \mathbb{A}^n_R \xrightarrow{\simeq} A_i \hookrightarrow \mathbb{P}^n_R$. For $i \neq j$, it is clear that $f_i|_{A'_i \cap A'_j} = f_i|_{A'_{ij}} = f_j|_{A'_{ij}} = f_j|_{A'_i \cap A'_j}$. So by gluing lemma, we have a unique morphism $f : \mathbb{P}^n_S \to \mathbb{P}^n_R$ such that $f|_{A'_i} = f_i$.

iii) Since U is open in $\operatorname{Spec} R$, there exists $f_1, ..., f_m \in R$ such that $U = \bigcup_{i=1}^{m} D_{f_i}$. The localisation $R \to R_{f_i}$ induces an embedding of schemes $\mathbb{P}^n_{R_{f_i}} \hookrightarrow \mathbb{P}^n_R$ by (ii). So we can view each $\mathbb{P}^n_{R_{f_i}}$ as an open subscheme of $\mathbb{P}^n_R$. We define $\mathbb{P}^n_U := \bigcup_{i=1}^{m} \mathbb{P}^n_{R_{f_i}}$. This is also an open subscheme of $\mathbb{P}^n_R$.

iv) Let $\{X_1, ..., X_m\}$ be an affine open cover of X, so $X_i \cong \operatorname{Spec} R_i$ for some ring R_i. Since $X_i \cap X_j$ is open in X_i, by (iii) we have defined $\mathbb{P}^n_{X_i \cap X_j}$ as an open subscheme of $\mathbb{P}^n_{X_i}$. Now we can define $\mathbb{P}^n_X$ as the push-out

$$\mathbb{P}^n_{X_1} \cup \cdots \cup \mathbb{P}^n_{X_m}$$

where $\mathbb{P}^n_{X_i}$ and $\mathbb{P}^n_{X_j}$ are glued along $\mathbb{P}^n_{X_i \cap X_j}$.

Let $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of schemes. Let $\{Y_1, ..., Y_m\}$ be an affine open cover of Y. For each i, $X_i := f^{-1}(Y_i)$ is open in X. So there exists affine open subsets $X_{i1}, ..., X_{ik_i}$ such that $X_i = \bigcup_{j=1}^{k_i} X_{ij}$. Let $X_{ij} \cong \operatorname{Spec} R_{ij}$, and $Y_i \cong \operatorname{Spec} S_i$. Then $f|_{X_{ij}} : X_{ij} \to Y_i$ is $\operatorname{Spec} \psi_{ij} : \operatorname{Spec} R_{ij} \to \operatorname{Spec} S_i$ for some ring homomorphism $\psi_{ij} : S_i \to R_{ij}$. By (ii) ψ_{ij} induces $\widetilde{\psi}_{ij} : \mathbb{P}^n_{X_{ij}} \cong \mathbb{P}^n_{R_{ij}} \to \mathbb{P}^n_{S_i} \cong \mathbb{P}^n_{Y_i} \subseteq \mathbb{P}^n_Y$. Since $X = \bigcup_{i=1}^{m} \bigcup_{j=1}^{k_i} X_{ij}$, we can glue the morphisms $\widetilde{\psi}_{ij}$ to obtain the morphism $\mathbb{P}^n_X \to \mathbb{P}^n_Y$. Details of checking compatibility are omitted. □

Question 2

Let $(X, \mathcal{O}_X)$ be a scheme and $s \in \mathcal{O}_X(U)$. Show that $\{x \in U : s_x = 0 \in \mathcal{O}_{X,x}\}$ is open in U and need not be closed; and $\{x \in U : s(x) = 0 \in \kappa(x)\}$ is closed in U and need not be open.

[*Hint. Look at affine varieties.*]

Proof. Let $A := \{x \in U : s_x = 0 \in \mathcal{O}_{X,x}\}$. For $x \in A$, $s_x = 0$ implies that there exists some open $V \ni x$ such that $s = 0$ on V. Then $s_y = 0$ for all $y \in V$. Hence $V \subseteq A$, and A is open in U.

Consider $U = X = \operatorname{Spec} k[x, y]/\langle xy \rangle$, where k is an algebraically closed field. (*Geometrically this is the union of two lines* $\{x = 0\}$ *and* $\{y = 0\}$ *in the affine plane* $\mathbb{A}^2_k$.) Then $\mathcal{O}_X(U) \cong k[x, y]/\langle xy \rangle$. It is easy (either from algebra or geometry) to write down the prime spectrum:

$$\operatorname{Spec} k[x, y]/\langle xy \rangle = \{\{0\}, \langle x \rangle, \langle y \rangle\} \cup \{\langle x, y - a \rangle : a \in k\} \cup \{\langle x - a, y \rangle : a \in k\}$$

Let $s = x \in \mathcal{O}_X(U)$. We claim that A is not closed. Suppose that it is closed. Then $A = \mathbb{V}(\mathfrak{q})$ for some $\mathfrak{q} \in X$. Note that for $\mathfrak{p} \in X$,

$$s_\mathfrak{p} = 0 \in \mathcal{O}_{X,\mathfrak{p}} \iff x_\mathfrak{p} = 0 \in (k[x, y]/\langle xy \rangle)_\mathfrak{p} \iff \exists\, u \notin \mathfrak{p} : xu = 0$$

For $\mathfrak{p} = \{0\}$, $y \notin \mathfrak{p}$ and $xy = 0$. Therefore $x_{\{0\}} = 0$ and hence $\{0\} \in A$. Then $A = \overline{\{0\}} = X$.

On the other hand, if $xu = 0$, we write $u = \sum_{i=0}^{n} f_i(x) y^i$ and find that $f_0(x) = 0$, so y divides u. Hence $u \in \langle y \rangle$. It follows that $\langle y \rangle \notin A$. This is a contradiction. We have shown that A is not closed.

Let $B := \{x \in U : s(x) \neq 0 \in \kappa(x)\}$. If $s = 0$, then $B = \varnothing$ is trivially open in U. Suppose that $s \neq 0$. For $x \in B$, $s(x) \neq 0$ implies that $s_x \notin \mathfrak{m}_x$ where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$. Then f_x is a unit in $\mathcal{O}_{X,x}$ and there is some $g \in \mathcal{O}_X(U)$ with $f_x \cdot g_x = 1$. Then $f \cdot g = 1$ in some open $W \ni x$. For $y \in W$, $f_y \cdot g_y = 1 \in \mathcal{O}_{X,y}$. Then f_y is a unit in $\mathcal{O}_{X,y}$ and $f(y) \neq 0 \in \kappa(y)$. Hence $y \in B$, and B is open in U.

Consider $U = X = \operatorname{Spec} k[t]$, where k is an algebraically closed field. Then $\mathcal{O}_X(U) \cong k[t]$. Let $s = t \in k[t]$. Note that for $\mathfrak{p} \in X$, $\mathcal{O}_{X,\mathfrak{p}} \cong k[t]_{\mathfrak{p}}$ has the maximal ideal $\mathfrak{p} \cdot k[t]_{\mathfrak{p}}$. Then

$$s(\mathfrak{p}) = 0 \iff s_{\mathfrak{p}} \in \mathfrak{p} \cdot k[t]_{\mathfrak{p}} \iff s \in \mathfrak{p} \iff \langle t \rangle \subseteq \mathfrak{p} \iff \langle t \rangle = \mathfrak{p}$$

So $B = \{\langle t \rangle\}$. We claim that this is not an open set. Suppose that it is open. Then there is $\mathfrak{q} \in X$ such that $\mathbb{V}(\mathfrak{q}) = X \setminus \{\langle t \rangle\}$. Then $\langle x-1 \rangle \subseteq \mathfrak{q}$ and $\langle x+1 \rangle \subseteq \mathfrak{q}$ implies that $\mathfrak{q} = \langle 1 \rangle$. But $\mathfrak{q} \notin X$, contradiction. Hence B is not open. Geometrically, this is the point of origin in the affine line $\mathbb{A}^1_k$. ✓ □

Question 3

i) Let R_1, R_2 be rings. Use natural projections $R_1 \times R_2 \to R_i$ to show that $\operatorname{Spec} R_1 \coprod \operatorname{Spec} R_2 \cong \operatorname{Spec}(R_1 \times R_2)$. Show that $\operatorname{Spec}(R_1 \times R_2) = \{\mathfrak{p}_1 \times R_2, R_1 \times \mathfrak{p}_2 \colon \mathfrak{p}_1 \in \operatorname{Spec} R_1,\ \mathfrak{p}_2 \in \operatorname{Spec} R_2\}$.

ii) Let $(X, \mathcal{O}_X)$ be a scheme. U, V are disjoint affine open subsets of X. Show that $U \cup V$ is affine.

iii) Show that $(X, \mathcal{O}_X)$ is irreducible $\iff$ all affine open subsets of X are irreducible.

[*Hint. For* $\Longleftarrow$, *consider* $\bigcup_i U_i = X = C_1 \cup C_2$. *Is* $U_i \cap U_j = \varnothing$ *possible?*]

iv) Suppose that $\mathcal{O}_X(U)$ is an integral domain for all affine $U \subseteq X$. Show that X is integral.

[*Hint. First show that* X *is irreducible. Then use Question 2.*]

v) Show that X is integral $\iff$ X is irreducible and reduced.

[*Hint. For (iv) and (v) use Sheet 1.*]

Finally deduce that $\operatorname{Spec} R$ is integral $\iff$ R is an integral domain.

Proof. i) Since $\operatorname{Spec} : \mathsf{CRing}^{\mathrm{op}} \to \mathsf{Aff}$ is an equivalence of catogory, it preserves products and coproducts. It is immediate that $\operatorname{Spec} R_1 \coprod \operatorname{Spec} R_2 \cong \operatorname{Spec}(R_1 \times R_2)$. ✓

Suppose that $\mathfrak{p} \in \operatorname{Spec}(R_1 \times R_2)$. Since $(0,0) = (1,0) \cdot (0,1) \in \mathfrak{p}$ and $\mathfrak{p}$ is prime, then either $(1,0) \in \mathfrak{p}$ or $(0,1) \in \mathfrak{p}$. If $(1,0) \in \mathfrak{p}$, then $R_1 \times \{0\} \subseteq \mathfrak{p}$. Note that $R_1 \times \{0\}$ is an ideal of $R_1 \times R_2$ and the quotient $(R_1 \times R_2)/(R_1 \times \{0\}) \cong R_2$. Then $\mathfrak{p}/(R_1 \times \{0\}) \cong \mathfrak{p}_2$ for some $\mathfrak{p}_2 \in \operatorname{Spec} R_2$. This shows that $\mathfrak{p} = R_1 \times \mathfrak{p}_2$. If $(0,1) \in \mathfrak{p}$, similarly we have $\mathfrak{p} = \mathfrak{p}_1 \times R_2$ for some $\mathfrak{p}_1 \in \operatorname{Spec} R_1$. Hence

$$\operatorname{Spec}(R_1 \times R_2) = \{\mathfrak{p}_1 \times R_2 \colon \mathfrak{p}_1 \in \operatorname{Spec} R_1\} \cup \{R_1 \times \mathfrak{p}_2 \colon \mathfrak{p}_2 \in \operatorname{Spec} R_2\}$$ ✓

ii) Suppose that $U \cong \operatorname{Spec} R$ and $V \cong \operatorname{Spec} S$ for some rings R, S. Since $U \cap V = \varnothing$, by (i) we have

iso of schemes by glueing

$$U \cup V = U \coprod V \cong \operatorname{Spec} R \coprod \operatorname{Spec} S \cong U \cup V \cong \operatorname{Spec}(R \times S)$$

Hence $U \cup V$ is affine. (*I don't like this* `\coprod` *symbol. But* `\sqcup` *is not a good choice either...*)

iii) "$\Longrightarrow$": Suppose that U is a reducible affine open subset of X. Then $U = U_1 \cup U_2$, where U_1, U_2 are non-empty proper subsets which are closed in U. By definition of subspace of topology and some set-theoretic massaging, U_1 and U_2 are also closed in X. Then $X = (X \cap U_1) \cup (X \cap U_2)$ is reducible. ✓

"$\Longleftarrow$": Suppose that all affine subsets of X are irreducible and X is reducible. If X is affine then the result is obvious. Otherwise, let $\{U_1, \dots, U_n\}$ be an affine open cover of X. And let $X = X_1 \cup X_2$, where X_1, X_2 are proper closed subsets. For each U_i, since $U_i = (X_1 \cap U_i) \cap (X_2 \cap U_i)$ and U_i is irreducible, either $U_i \cap X_1 = \varnothing$ or $U_i \cap X_2 = \varnothing$. Since both X_1, X_2 are non-empty, without loss of generality we assume that $U_1 \cap X_2 = \varnothing$ and $U_2 \cap X_1 = \varnothing$. So $U_1 \cap U_2 = \varnothing$. Now by (ii), $U_1 \cup U_2$ is affine. But ✗

I disagree that $U = C_1 \cup C_2 \Rightarrow C_1 = \phi$ or $C_2 = \phi$
It implies that $C_1 = U$ or $C_2 = U$

we know that $U_1 \cup U_2$ is disconnected, and hence is reducible. This contradicts our assumption. We conclude that X is irreducible.

iv) For an affine open subset $U \subseteq X$ such that $U \cong \operatorname{Spec} R$, $\mathcal{O}_X(U) \cong R$ is an integral domain. Hence U is irreducible by Question 3.(ii) of Sheet 1. Now by (iii) X is irreducible.

Let V be an *arbitrary* open subset of X. By first half of (iii) V is also irreducible. Suppose that $f, g \in \mathcal{O}_X(V) \setminus \{0\}$ such that $fg = 0$. Let $V_f := \{x \in V : f(x) = 0\}$ and $V_g := \{x \in V : g(x) = 0\}$. Then $V_f \cup V_g = V$. By Question 2, V_f and V_g are closed in V. By irreducibility, we may assume that $V_f = V$. Let $\{V_1, \ldots, V_m\}$ be an affine open cover of V. For $V_i \cong \operatorname{Spec} R_i$, where R_i is an integral domain, let $f_i \in R_i$ be the restriction of f on V_i. Then $f_i(x) = f(x) = 0$ for all $x \in V_i$. Note that this implies that $f_i \in \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R_i$. Hence $f_i = 0$ because $\operatorname{Nil}(R_i) = \{0\}$. In particular $f_x = (f_i)_x = 0$ for all $x \in V_i$. This holds for each i, and by the covering, we have $f_x = 0$ for all $x \in V$. Hence $f = 0 \in \mathcal{O}_X(V)$. This is a contradiction. We deduce that $\mathcal{O}_X(V)$ is an integral domain. Therefore X is an integral scheme.

v) "$\Longrightarrow$": By definition integrality implies reducedness. Since X is integral, for any affine open subset $U \cong \operatorname{Spec} R$, R is an integral domain. Hence U is irreducible. By (iii) X is irreducible.

"$\Longleftarrow$": For any affine open subset $U \cong \operatorname{Spec} R$, U is irreducible and reduced. Hence $\mathcal{O}_X(U) \cong R$ is an integral domain. By (iv) X is integral.

We have $\operatorname{Spec} R$ is integral $\iff$ $\operatorname{Spec} R$ is irreducible and reduced $\iff$ R is an inetgral domain. $\square$

Question 4

Consider the scheme $Y = \operatorname{Spec} k[x,y]/(f)$ where $f = y^2 - x^2 - x^3$ and k is a field with $\operatorname{char} k \neq 2$.

i) Show that Y is an integral scheme.

ii) Draw a picture in $\mathbb{R}^2$ of the curve $f = 0$.

Now consider the functor of points $h_Y(X)$ for the following test schemes X:

iii) Let $X = \operatorname{Spec} k[\![x,y]\!]/(f)$. Using the "natural choice" of $\alpha \in h_Y(X)$, show that $\alpha^{-1}(Y)$ is reducible.

[*Hint. Newton binomial theorem:*

$$(1+x)^r = 1 + rx + \frac{r(r-1)}{1 \cdot 2}x^2 + \frac{r(r-1)(r-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots \in k[\![x,y]\!]$$

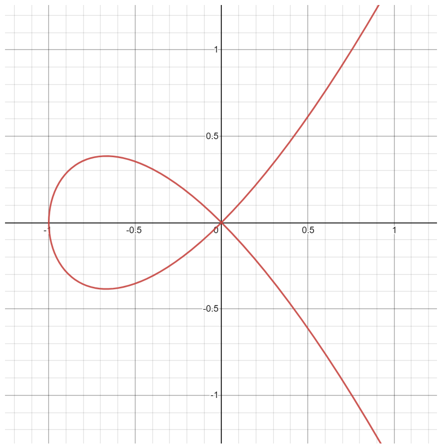
provided the fractions exist in k and $r \in \mathbb{Q}$. $\mathbb{Z} \ni \binom{2n}{n} = \frac{(2n)!}{n!n!}$ *and* $\binom{2n}{n}\frac{1}{n+1} = \binom{2n}{n+1}\frac{1}{n}$.]

iv) What would happen in (iii) for $X = \operatorname{Spec} \mathcal{O}_{Y,0}$? Comment in view of the picture in (ii).

Proof. i) By Question 3, it suffices to prove that $k[x,y]/\langle f \rangle$ is an integral domain. This is true if and only if $\langle f \rangle$ is a prime ideal, if and only if f is irreducible in $k[x,y]$. The remaining work is elementary school mathematics.

Suppose that $f = gh$ for non-constant $g, h \in k[x,y]$. By considering f, g, h as polynomials in y and comparing coefficients, we have $g(x,y) = y - \widetilde{g}(x)$ and $h(x,y) = y - \widetilde{h}(x)$ for some $\widetilde{g}, \widetilde{h} \in k[x]$. Then we have $\widetilde{g}(x) + \widetilde{h}(x) = 0$ and $\widetilde{g}(x)\widetilde{h}(x) = -x^2(x+1)$. The first equality suggests that $\deg \widetilde{g} = \deg \widetilde{h}$, so that $\deg(\widetilde{g}\widetilde{h})$ is even, contradicting the second equality. Hence f is irreducible.

ii) Sketch of $y^2 = x^2 + x^3$ in $\mathbb{R}^2$:



iii) Let $\iota\colon k[x,y]/\langle f\rangle \hookrightarrow k[\![x,y]\!]/\langle f\rangle$ be the canonical embedding. Then ι induces the morphism of schemes $\alpha = \operatorname{Spec}\iota\colon X \to Y$. We need to show that $\alpha^{-1}(Y) = X$ is reducible.

The phrasing of the question is confusing as the reducibility of $\alpha^{-1}(Y) = X$ *has nothing to do with* α. true

By Question 3 of Sheet 1, it suffices to show that the nilradical N of $k[\![x,y]\!]/\langle f\rangle$ is not prime. First we claim that there exists $\eta(x) \in k[\![x]\!]$ such that $\eta(x)^2 = x+1$. To prove this, we observe that

$$\frac{1}{n!}\prod_{m=0}^{n-1}\left(\frac{1}{2}-m\right) = \frac{(-1)^{n-1}}{n!2^n}\prod_{m=1}^{n-1}(2m-1) = \frac{(-1)^{n-1}}{n!2^n}\frac{(2n)!}{n!2^n}\frac{1}{2n-1} = \frac{(-1)^{n-1}}{2^{2n}}\binom{2n}{n}\frac{1}{2n-1}$$

is an element of k, because $\operatorname{char} k \neq 2$ and $\binom{2n}{n}\frac{1}{2n-1} \in \mathbb{Z}$. Then by the Newton's binomial theorem,

$$\eta(x) = 1 + \sum_{n=1}^{\infty}\frac{1}{n!}\prod_{m=0}^{n-1}\left(\frac{1}{2}-m\right)x^m \in k[\![x]\!]$$

satisfies that $\eta(x)^2 = 1+x$. Then

$$f(x) = y^2 - x^2(x+1) = y^2 - x^2\eta(x)^2 = (y + x\eta(x))(y - x\eta(x)) \in k[\![x,y]\!]$$

Second, we need the following lemma (*Atiyah & MacDonald Exercise 1.5.(i)*): for any ring R, $p \in R[\![x]\!]$ is a unit if and only if the constant term $a_0 \in R$ of p is a unit in R. The proof is straightforward and we omit it. In particular, if a_0 is irreducible in R, then p is irreducible in $R[\![x]\!]$.

Third, we need the fact that $k[\![x,y]\!]$ is a unique factorisation domain. The proof is very lengthy and we omit it.

With these results in hand, we can prove our claim that N is not prime. Since the constant term of $\eta(x)$ is 1, $\eta(x)$ is a unit in $k[\![x]\!]$. We know that x is irreducible in $k[\![x]\!]$ ($\langle x\rangle$ is the unique maximal ideal). Then $x\eta(x)$ is irreducible in $k[\![x]\!]$. Note that $y \pm x\eta(x) \in k[\![x]\!][\![y]\!]$ has constant term $x\eta(x) \in k[\![x]\!]$. So $y \pm x\eta(x)$ are irreducible in $k[\![x]\!][\![y]\!] = k[\![x,y]\!]$. Since $k[\![x,y]\!]$ is a UFD, the ideals $\langle y \pm x\eta(x)\rangle$ are prime. In the quotient ring $k[\![x,y]\!]/\langle f\rangle$, since $\langle 0\rangle$ is not prime, $\langle y \pm x\eta(x)\rangle$ are minimal prime.

Finally, we claim that $\langle y + x\eta(x)\rangle \neq \langle y - x\eta(x)\rangle$. Suppose not. Then there exists a unit $u \in k[\![x,y]\!]$ such that $y + x\eta(x) = u(y - x\eta(x))$. Then $(u-1)y = (u+1)x\eta(x)$. So $u - 1 = u + 1 = 0$. Since $2 \neq 0$ in k, this is impossible. We deduce that $\langle y + x\eta(x)\rangle$ and $\langle y - x\eta(x)\rangle$ are distinct in $k[\![x,y]\!]$. Therefore they are distinct minimal prime ideals of $k[\![x,y]\!]/\langle f\rangle$. We conclude that N is not prime, and $\alpha^{-1}(Y) = X$ is reducible.

iv) $0 \in Y$ *is ambiguous. It could be the zero ideal* $\langle 0\rangle \in Y$, *or could be the maximal ideal corresponding to the point* $(0,0)$ *in* $\mathbb{A}^2_k$, *which is* $\langle x,y\rangle$.

- For the firsr case, $\mathcal{O}_{Y,\langle 0\rangle} = (k[x,y]/\langle f\rangle)_{\langle 0\rangle} = \operatorname{Frac}(k[x,y]/\langle f\rangle)$. So $\operatorname{Spec}\mathcal{O}_{Y,\langle 0\rangle}$ is the prime spectrum of a field, and is a singleton as a set. The natural choice of $\alpha \in h_Y(X)$ is $\alpha = \operatorname{Spec}\varphi$, where $\varphi : k[x,y]/\langle f\rangle \to \mathcal{O}_{Y,\langle 0\rangle}$ is the embedding into the field of fractions. X is trivially irreducible.
- For the second case, $\mathcal{O}_{Y,\langle 0\rangle} = (k[x,y]/\langle f\rangle)_{\langle x,y\rangle}$ is the localisation of an integral domain, which is again an integral domain. Then by Question 3, $X = \operatorname{Spec}\mathcal{O}_{Y,\langle 0\rangle}$ is an integral scheme, and hence is irreducible.

that's what it's referring to

This suggests that the ring of formal power series $k[\![x,y]\!]/\langle f\rangle$ better reflects the local geometric property of the affine variety at $(0,0)$ than the stalk $\mathcal{O}_{Y,0}$, as it captures the fact that the variety is locally reducible as shown on the graph. $\square$

this is the 'completion' of $\frac{k[x,y]}{(f)}$ at $\langle x,y\rangle$ so that's why we also want to think about this point.

Question 5

An element e is called idempotent if $e^2 = e$. In integral domains, only $0, 1$ are idempotents; e is an idempotent $\implies$ $1-e$ is an idempotent.

Let $X = \operatorname{Spec} R$.

i) Show that $X = D_e \sqcup D_{1-e}$ for all idempotents $e \in R$.

[*Hint. What is $e = e(\mathfrak{p}) \in \kappa(\mathfrak{p})$?*]

Example. In 3.(i), $\operatorname{Spec} R_1 \times R_2 = D_{(1,0)} \sqcup D_{(0,1)} = \operatorname{Spec} R_1 \sqcup \operatorname{Spec} R_2$.

ii) Show that $D_f \cap D_g = \varnothing \iff fg$ is nilpotent.

Example: $D_e \cap D_{1-e} = \varnothing$ in (i).

iii) Show that $U \subseteq X$ is open and closed $\iff$ there exists a unique idempotent $e \in R$ with $U = D_e$.

[*Hint. $U = \mathbb{V}(I)$, $V = X \setminus U = \mathbb{V}(J)$. Show that $(IJ)^N = 0$ for some N, hence $1 = 1^{2N} \in I^N + J^N$.*]

iv) Show that $\{$connected component of $\mathfrak{p} \in X\} = \mathbb{V}(\langle$idempotents $e \in R$ with $e(\mathfrak{p}) = 0 \in \kappa(\mathfrak{p})\rangle)$.

[*Use the fact from topology: If a topological space X is compact, and that it has a basis of compact open subsets, the intersection of any two of which is compact, then the connected component of $x \in X$ is $\bigcap\{$clopen $U \ni x\}$.*]

Finally deduce that $\operatorname{Spec} R$ is connected $\iff$ $0, 1$ are the only idempotents of R.

Proof. i) To show that $X = D_e \sqcup D_{1-e}$, it suffices to show that $D_e \cap D_{1-e} = \varnothing$ and $D_e^c \cap D_{1-e}^c = \varnothing$. Note that $D_e = \{\mathfrak{p} \in X : e \notin \mathfrak{p}\}$ and $D_{1-e} = \{\mathfrak{p} \in X : 1-e \notin \mathfrak{p}\}$.

Suppose that $\mathfrak{p} \in X$ such that $\mathfrak{p} \in D_e \cap D_{1-e}$. Then $e, 1-e \notin \mathfrak{p}$. But then $0 = e - e^2 = e(1-e) \notin \mathfrak{p}$ since $\mathfrak{p}$ is prime. This is impossible. So $D_e \cap D_{1-e} = \varnothing$.

Suppose that $\mathfrak{p} \in X$ such that $\mathfrak{p} \in D_e^c \cap D_{1-e}^c$. Then $e, 1-e \in \mathfrak{p}$. But then $1 = e + (1-e) \in \mathfrak{p}$. This is impossible. So $D_e^c \cap D_{1-e}^c = \varnothing$.

ii) We have

$$\begin{aligned} D_f \cap D_g = \varnothing &\iff \neg\exists\, \mathfrak{p} \in X\colon f \notin \mathfrak{p} \wedge g \notin \mathfrak{p} \\ &\iff \forall\, \mathfrak{p} \in X\colon f \in \mathfrak{p} \vee g \in \mathfrak{p} \\ &\iff \forall\, \mathfrak{p} \in X\colon fg \in \mathfrak{p} \\ &\iff fg \in \operatorname{Nil}(R) \\ &\iff fg \text{ is nilpotent.} \end{aligned}$$

iii) Since U is clopen, $U = \mathbb{V}(\mathfrak{p})$ and $U^c = \mathbb{V}(\mathfrak{q})$ for some $\mathfrak{p}, \mathfrak{q} \in X$. $\operatorname{Spec} R = \mathbb{V}(\mathfrak{p}) \sqcup \mathbb{V}(\mathfrak{q})$ implies that $\mathfrak{p} + \mathfrak{q} = R$ and $\sqrt{\mathfrak{pq}} = \sqrt{\{0\}}$. The first one implies that there exist $f \in \mathfrak{p}$, $g \in \mathfrak{q}$ such that $f + g = 1$. Then $fg \in \mathfrak{pq} \subseteq \operatorname{Nil}(R)$ is nilpotent. There exists $N \in \mathbb{N}$ such that $(fg)^N = 0$. On the other hand, we have $1 = (f+g)^{2N} \in \langle f \rangle^N + \langle g \rangle^N$. So there exist $a, b \in R$ such that $1 = af^N + bg^N$. Now let $e = bg^N$ and $1 - e = af^N$. Then $e(1-e) = 0$ and hence e is idempotent.

For $\mathfrak{a} \in X$, we have

$$\mathfrak{a} \in \mathbb{V}(\mathfrak{p}) \implies \mathfrak{p} \subseteq \mathfrak{a} \implies f \in \mathfrak{a} \implies 1 - e = af^n \in \mathfrak{a} \implies e \notin \mathfrak{a} \implies \mathfrak{a} \in D_e$$

Hence $U \subseteq D_e$. Similarly $U^c \subseteq D_{1-e}$. But $X = U \sqcup U^c = D_e \sqcup D_{1-e}$. We must have $U = D_e$.

It remains to show that e is unique. Suppose that there is an idempotent e' such that $D_e = D_{e'}$. Then $D_e \cap D_{1-e} = D_e \cap D_{1-e'} = D_{e(1-e')} = \varnothing$ and similarly $D_{e'(1-e)} = \varnothing$. It follows that $e(1-e')$ and $e'(1-e)$ are nilpotent. But they are also idempotents. Hence $e(1-e') = e'(1-e) = 0$. Finally, $e - e' = e(1-e') - e'(1-e) = 0$. The idempotent e is unique.

For the converse direction, $X = D_e \sqcup D_{1-e}$ implies that D_e is clopen.

iv) It is straightforward to check that X satisfies the condition in the given fact. So the connected component Y of $\mathfrak{p} \in X$ is the intersection of clopen sets containing x. By (iii), we have

$$Y = \bigcap \{D_e \colon \mathfrak{p} \in D_e\} = \bigcap \{D_e \colon e \notin \mathfrak{p}\} = \bigcap \{D_{1-e} \colon e \in \mathfrak{p}\} = \bigcap \{\mathbb{V}(\langle e \rangle) \colon e \in \mathfrak{p}\} = \mathbb{V}(\langle e : e \in \mathfrak{p} \rangle)$$

If $\operatorname{Spec} R$ is disconnected, then $\operatorname{Spec} R = U \sqcup V$ for some non-empty clopen $U, V \subseteq \operatorname{Spec} R$. Then $U = D_e$ some idempotents $e \in R$. Since $U \neq \varnothing$ or $\operatorname{Spec} R$, $e \neq 0$ or 1.

Conversely, suppose that $e \in R \setminus \{0, 1\}$ is an idempotent. Then $\operatorname{Spec} R = D_e \sqcup D_{1-e}$ is disconnected. $\square$

Question 6

*A **family of schemes** is a morphism $f : X \to B$ of schemes. Think of this as the collection of schemes $X_b = f^{-1}(b) = \operatorname{Spec}(K(b) \times_B X)$ (fibre product: on affines this is the tensor product of algebras). A family of closed subschemes of Y over B is a closed subscheme*

$$\begin{tikzcd} X \arrow[r, phantom, "\subseteq"] \arrow[dr, dashed] & Y \times B \arrow[d, "\text{project}"] \\ & B \end{tikzcd}$$

i) Let $B = \operatorname{Spec} k[t] = \mathbb{A}^1_k$. $B^* = D_0 = \operatorname{Spec} k[t, t^{-1}] = \mathbb{A}^1_k \setminus \{0\}$, and $X^* = \mathbb{V}(x^2 - t^2) \subseteq \mathbb{A}^1_{B^*} = \operatorname{Spec} k[t, t^{-1}, x]$. Calculate the closure X of $X^* \subseteq \mathbb{A}^1_B = \operatorname{Spec} k[t, x]$ and the fibre X_0. (*Think of X_0 as the "limit" of X_b as $b \to 0$.*)

ii) Repeat (i) for $X^* = \mathbb{V}(xy - t) \subseteq \mathbb{A}^2_{B^*} = \operatorname{Spec} k[t, t^{-1}, x, y]$. What pictures over $k = \mathbb{R}$ and $k = \mathbb{C}$ does this correspond to? (*Only consider closed points for the picture.*)

iii) For the family $X = \operatorname{Spec} \mathbb{Z}[x, y]/(x^2 - y^2 - 5) \to B = \operatorname{Spec} \mathbb{Z}$ (*induced by obvious map on rings*), what are the fibres $X_{(0)}, X_{(2)}, X_{(3)}, X_{(5)}$?

Show that this is a flat family (*the notes will help*). What happens if you replace $x^2 - y^2 - 5$ by $2x^2 - 2y^2 - 6$?