Summary of Differential Equations II

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1 Second-Order Linear Boundary Value Problems

1.1 Basic Concepts

A **second order linear ODE** is an equation of the form

$$\mathscr{L}y(x) = f(x) \tag{1.1}$$

where

$$\mathscr{L}y(x) := P_2(x)y''(x) + P_1(x)y'(x) + P_0(x)y(x) \tag{1.2}$$

is a **linear differential operator**.

It is **homogeneous** when $f = 0$; otherwise it is **inhomogeneous**. We shall use (H) and (N) to denote the two cases respectively.

1.2 Space of Solutions

Theorem 1.1. Space of Solutions

1. If y and $\tilde{y}$ satisfies (N), then $y - \tilde{y}$ satisfies (H).
2. The general solution of (N) is $y(x) = y_{PI}(x) + y_{CF}(x)$, where y_{PI} is any solution of (N), called the **particular solution**, and y_{CF} is the general solution of (H), called the **complementary function**.
3. The solutions of (H) forms a two-dimensional vector space:

$$y_{CF}(x) = c_1 y_1(x) + c_2 y_2(x) \tag{1.3}$$

where c_1, c_2 are arbitrary constants, and y_1, y_2 are any two **linearly independent** solutions to (H).

For two functions y_1, y_2, we define the **Wronskian** to be the determinant:

$$W[y_1, y_2] := \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x) \tag{1.4}$$

Theorem 1.2. Wronskian and Solutions

Two solutions y_1, y_2 of (H) are linearly dependent if and only if the Wronskian $W[y_1, y_2] = 0$.

Therefore any two solutions of (H) with non-vanishing Wronskian form a basis of the solution space of (H).

1.3 Solution Methods for Homogeneous Problem

If P_0, P_1 and P_2 are constants, then (H) admits solutions of the form $y(x) = \mathrm{e}^{mx}$, where m satisfies

$$P_2 m^2 + P_1 m + P_0 = 0 \tag{1.5}$$

known as the **auxiliary equation**. Let m_1, m_2 be the roots of the auxiliary euation. When $P_0, P_1, P_2 \in \mathbb{R}$, the general solution of (H) has the form

$$y(x) = \begin{cases} c_1 \mathrm{e}^{m_1 x} + c_2 \mathrm{e}^{m_2 x}, & m_1, m_2 \in \mathbb{R} \text{ and } m_1 \neq m_2; \\ (c_1 + c_2 x)\, \mathrm{e}^{mx}, & m_1 = m_2 = m \in \mathbb{R}; \\ \mathrm{e}^{ax} \left(c_1 \cos(bx) + c_2 \sin(bx)\right), & m_1 = a + b\mathrm{i},\ m_2 = a - b\mathrm{i},\ a, b \in \mathbb{R}. \end{cases} \tag{1.6}$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants.

A **Cauchy-Euler equation** takes the form

$$\alpha x^2 y''(x) + \beta x y'(x) + \gamma y(x) = 0 \tag{1.7}$$

where α, β, γ are constants. In this case (H) admits solutions of the form $y(x) = x^m$, where m satisfies

$$\alpha m(m-1) + \beta m + \gamma = 0 \tag{1.8}$$

Let m_1, m_2 be the roots of the equation (1.8). When $\alpha, \beta, \gamma \in \mathbb{R}$, the general solution of (H) has the form

$$y(x) = \begin{cases} c_1 x^{m_1} + c_2 x^{m_2}, & m_1, m_2 \in \mathbb{R} \text{ and } m_1 \neq m_2; \\ (c_1 + c_2 \ln x) x^m, & m_1 = m_2 = m \in \mathbb{R}; \\ x^a \left(c_1 \cos(b \ln x) + c_2 \sin(b \ln x)\right), & m_1 = a + b\mathrm{i},\ m_2 = a - b\mathrm{i},\ a, b \in \mathbb{R}. \end{cases} \tag{1.9}$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants.

Reduction of order: Suppose a solution $y_1(x)$ to (H) is known, then the solution to (H) is of the form

$$y(x) = v(x) y_1(x) \tag{1.10}$$

where $v(x)$ satisfies

$$v'(x) = \frac{\text{const}}{y_1(x)^2} \exp\left(-\int \frac{P_1(x)}{P_2(x)} \,\mathrm{d}x\right) \tag{1.11}$$

A further integration of (1.11) gives v and thus the general solution.

1.4 Variation of Parameters

Suppose that (H) is solved by $y(x) = c_1 y_1(x) + c_2 y_2(x)$ with linearly independent y_1, y_2. Then (N) is solved by

$$y(x) = c_1(x) y_1(x) + c_2(x) y_2(x) \tag{1.12}$$

where c_1', c_2' satisfy the linear equations

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f/P_2 \end{pmatrix} \tag{1.13}$$

After inversion and integration we obtain

$$c_1(x) = -\int^x \frac{f(\xi) y_2(\xi)}{P_2(\xi) W(\xi)} \,\mathrm{d}\xi \qquad c_2(x) = \int^x \frac{f(\xi) y_1(\xi)}{P_2(\xi) W(\xi)} \,\mathrm{d}\xi \tag{1.14}$$

1.5 Fitting Boundary Conditions

We consider (N) in the interval (a, b) with boundary data

$$y(a) = y(b) = 0 \tag{1.15}$$

As before, (N) is solved by

$$y(x) = c_1(x) y_1(x) + c_2(x) y_2(x) \tag{1.16}$$

We impose the boundary condition by setting

$$c_2(a) = c_1(b) = 0 \tag{1.17}$$

Then from equation (1.14) we obtain the solution to the BVP

$$y(x) = y_2(x)\int_a^x \frac{f(\xi)y_1(\xi)}{P_2(\xi)W(\xi)}\,\mathrm{d}\xi + y_1(x)\int_x^b \frac{f(\xi)y_2(\xi)}{P_2(\xi)W(\xi)}\,\mathrm{d}\xi := \int_a^b g(x,\xi)f(\xi)\,\mathrm{d}\xi \tag{1.18}$$

where

$$g(x,\xi) := \begin{cases} \dfrac{y_1(\xi)y_2(x)}{P_2(\xi)W(\xi)}, & a<\xi<x<b; \\ \dfrac{y_2(\xi)y_1(x)}{P_2(\xi)W(\xi)}, & a<x<\xi<b. \end{cases} \tag{1.19}$$

is called the **Green's function**.

1.6 Adjoint Problem

For $u, v \in L^2[a,b]$, we define the **inner product** of u and v to be

$$\langle u, v\rangle := \int_a^b u(x)\overline{v(x)}\,\mathrm{d}x \tag{1.20}$$

For a operator $\mathscr{L}$, the **adjoint operator** $\mathscr{L}^*$ is an operator with the largest domain $\mathrm{dom}(\mathscr{L}^*)$ satisfying

$$\langle \mathscr{L}u, v\rangle = \langle u, \mathscr{L}^* v\rangle \tag{1.21}$$

for all $u \in \mathrm{dom}(\mathscr{L})$ and $v \in \mathrm{dom}(\mathscr{L}^*)$. The uniqueness and existence of $\mathscr{L}^*$ follows from Riesz's Representation Theorem.

For a linear differential operator $\mathscr{L}$, we can obtain $\mathscr{L}^*$ using **integration by parts**. For $\mathscr{L}y = P_2y'' + P_1y' + P_0y$, the adjoint operator is given by

$$\mathscr{L}^* w = (P_2w)'' - (P_1w)' + P_0w = P_2w'' + (2P_2' - P_1)w' + (P_2'' - P_1' + P_0)w \tag{1.22}$$

Suppose that $\mathscr{L}$ is supplemented with **homogeneous boundary conditions** (BC):

$$\begin{aligned} \mathscr{B}_1 y &= \alpha_1 y(a) + \alpha_2 y'(a) + \beta_1 y(b) + \beta_2 y'(b) = 0, \\ \mathscr{B}_2 y &= \alpha_3 y(a) + \alpha_4 y'(a) + \beta_3 y(b) + \beta_4 y'(b) = 0. \end{aligned} \tag{BC}$$

Through integration by parts we obtain:

$$\langle \mathscr{L}y, w\rangle - \langle y, \mathscr{L}^* w\rangle = (K_1^* w)(\mathscr{B}_1 y) + (K_2^* w)(\mathscr{B}_2 y) + (K_1 y)(\mathscr{B}_1^* w) + (K_2 y)(\mathscr{B}_2^* w) \tag{1.23}$$

Then $\langle \mathscr{L}y, w\rangle = \langle y, \mathscr{L}^* w\rangle$ and (BC) enforce that $\mathscr{B}_1^* w = \mathscr{B}_2^* w = 0$, which are called the **adjoint boundary conditions** (BC*).

The problem ($\mathscr{L}$+BC*) is called

1. **fully self-abjoint**, if $\mathscr{L} = \mathscr{L}^*$ and BC=BC*;
2. **formally self-adjoint**, if $\mathscr{L} = \mathscr{L}^*$ but BC≠BC*.

Note that $\mathscr{L}$ is self-adjoint if and only if $P_1 = P_2'$. In this case, set $P_2 = -p$, $P_1 = -p'$ and $P_0 = q$, we can write $\mathscr{L}$ as

$$\mathscr{L}y = -(py')' + qy \tag{1.24}$$

which is called the **Sturm-Liouville operator**.

1.7 Fredholm Alternative

Consider the linear homogeneous and inhomogeneous ODEs

$$\mathscr{L}y = 0 \tag{H}$$

$$\mathscr{L}y = f \tag{N}$$

for $x \in (a,b)$, supplemented by linear homogeneous boundary conditions of the form

$$\begin{aligned} \mathscr{B}_1 y &= \alpha_1 y(a) + \alpha_2 y'(a) + \beta_1 y(b) + \beta_2 y'(b) = 0, \\ \mathscr{B}_2 y &= \alpha_3 y(a) + \alpha_4 y'(a) + \beta_3 y(b) + \beta_4 y'(b) = 0. \end{aligned} \tag{BC}$$

with $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ and $(\alpha_3, \alpha_4, \beta_3, \beta_4)$ linearly independent.

We also define the homogeneous adjoint equation

$$\mathscr{L}^* w = 0 \tag{H*}$$

and corresponding adjoint boundary conditions

$$\mathscr{B}_1^* w = \mathscr{B}_2^* w = 0 \tag{BC*}$$

Theorem 1.3. Fredholm Alternative for Homogeneous Linear ODE

Exactly one of the following possibilities occurs:

1. (H+BC) has only the zero solution. Then (N+BC) has a **unique solution**.
2. (H+BC) has non-trivial solutions, and so does (H^*+BC^*). In this case:
   (a) if $\langle f, w\rangle = 0$ for all w satisfying (H^*+BC^*), then (N+BC) has a **non-unique solution**;
   (b) otherwise, (N+BC) has **no solution**.

The condition $\langle f, w\rangle = 0$ is called the **solvability condition**.

Suppose that we have instead the **inhomogeneous boundary conditions**:

$$\begin{aligned} \mathscr{B}_1 y = \alpha_1 y(a) + \alpha_2 y'(a) + \beta_1 y(b) + \beta_2 y'(b) = \gamma_1, \\ \mathscr{B}_2 y = \alpha_3 y(a) + \alpha_4 y'(a) + \beta_3 y(b) + \beta_4 y'(b) = \gamma_2. \end{aligned} \tag{NBC}$$

for some constants γ_1, γ_2.

Let $v(x)$ satisfies (NBC). Define $\tilde{y} := y - v$. Then $\tilde{y}(x)$ satisfies the problem

$$\mathscr{L}\tilde{y} = f - \mathscr{L}v =: \tilde{f} \tag{$\tilde{\text{H}}$}$$

The solvability condition becomes $\langle \tilde{f}, w\rangle = \langle f, w\rangle - \langle \mathscr{L}v, w\rangle = 0$. Substituting equation (1.23) we obtain the solvability condition:

$$\langle f, w\rangle = \gamma_1 K_1^* w + \gamma_2 K_2^* w \tag{1.25}$$

where K_1^* and K_2* are as in equation (1.23).

(*The solvability condition can also be obtained by taking the inner product of (N) with w.*)

Theorem 1.4. Fredholm Alternative for Inhomogeneous Linear ODE

Exactly one of the following possibilities occurs:

1. (H+BC) has only the zero solution. Then (N+NBC) has a **unique solution**.
2. (H+BC) has non-trivial solutions. In this case:
   (a) if $\langle f, w\rangle = \gamma_1 K_1^* w + \gamma_2 K_2^* w$ for all w satisfying (H^*+BC^*), then (N+NBC) has a **non-unique solution**;
   (b) otherwise, (N+NBC) has **no solution**.

2 Green's Function

2.1 Basic Properties

Consider the second-order linear BVP. Let the Green's function $g(x,\xi)$ given by equation (1.18) and (1.19).

Theorem 2.1. Properties of the Green's Function

1. $g(x,\xi)$ satisfies (H) for $x \neq \xi$:
$$\mathscr{L}_x g = P_2(x) g_{xx} + P_1(x) g_x + P_0(x) g = \delta(x-\xi) \tag{2.26}$$
   The subscript x emphasizes that the derivatives are with respect to x.
2. $g(x,\xi)$ satisfies (BC):
$$g(a,\xi) = g(b,\xi) = 0 \tag{2.27}$$
3. $g(x,\xi)$ is continuous at $x = \xi$:
$$\lim_{x\to\xi+} g(x,\xi) = \lim_{x\to\xi-} g(x,\xi) \tag{2.28}$$

4. $g_x(x,\xi)$ is discontinuous at $x = \xi$:

$$\lim_{x\to\xi+} g_x(x,\xi) - \lim_{x\to\xi-} g_x(x,\xi) = \frac{1}{P_2(\xi)} \tag{2.29}$$

2.2 Green's Function and General Linear BVP

In general, consider the linear differential operator of order n:

$$\mathscr{L} y(x) := \sum_{i=0}^{n} P_i(x) y^{(i)}(x) \tag{2.30}$$

assuming that $P_0(x), \ldots, P_n(x)$ are continuous and $P_n(x) \neq 0$. For a general n-th-order linear BVP in the interval (a,b), the inhomogeneous ODE

$$\mathscr{L} y(x) = f(x), \quad x \in (a,b) \tag{N}$$

is supplemented with n homogeneous linear boundary conditions which are linearly independent:

$$\mathscr{B}_i y\big|_{x=a,b} := \sum_{j=i}^{n} \left(\alpha_{i,j} y^{(j-1)}(a) + \beta_{i,j} y^{(j-1)}(b)\right) = 0 \tag{HBC}$$

(*If the boundary conditions are not inhomogeneous, then use the technique in Section 1.7.*)

The corresponding problem for the Green's function is

$$\mathscr{L}_x g(x,\xi) = \delta(x-\xi), \quad x,\xi \in (a,b) \tag{2.31}$$

with boundary conditions

$$\mathscr{B}_i g(x,\xi)\big|_{x=a,b} = 0 \tag{2.32}$$

By integrating equation (2.31), we deduce that the $(n-1)$**-th derivative** of $g(x,\xi)$ satifies the jump condition:

$$\frac{\partial^{n-1}}{\partial x^{n-1}} g(x,\xi)\bigg|_{x=\xi-}^{x=\xi+} = \frac{1}{P_n(\xi)} \tag{2.33}$$

and that all lower derivatives are continuous across $x = \xi$:

$$g(x,\xi)\big|_{x=\xi-}^{x=\xi+} = \frac{\partial}{\partial x} g(x,\xi)\bigg|_{x=\xi-}^{x=\xi+} = \cdots = \frac{\partial^{n-2}}{\partial x^{n-2}} g(x,\xi)\bigg|_{x=\xi-}^{x=\xi+} = 0 \tag{2.34}$$

After determining $g(x,\xi)$, the solution to (N)+(HBC) is given by

$$y(x) = \int_a^b g(x,\xi) f(\xi)\,\mathrm{d}\xi \tag{2.35}$$

2.3 Green's Function in terms of Adjoint

Theorem 2.2. Green's Function in terms of Adjoint

Suppose that $g(x,\xi)$ satisfies $\mathscr{L}_x g(x,\xi) = \delta(x-\xi)$ with homogeneous boundary conditions (HBC), then $g(\xi,x)$ satifies the adjoint equation $\mathscr{L}_x^* g(\xi,x) = \delta(x-\xi)$ and the adjoint boundary conditions (BC*).

In particular, if ($\mathscr{L}$+BC) is fully self-adjoint, then g is **symmetric**. That is, $g(x,\xi) = g(\xi,x)$.

Suppose that ($\mathscr{L}$+BC) has a non-trivial kernel. Then by Fredholm Alternative, $\mathscr{L}_x g(x,\xi) = \delta(x-\xi)$ implies the solvability condition:

$$0 = \langle g(x,\xi),\ \mathscr{L}^* w(x)\rangle_x = w(\xi) \tag{2.36}$$

which is never satisfied. In this situation **we cannot construct the Green's function**.

3 Eigenfunction Expansions

3.1 Constructing Eigenfunction Expansion for Inhomogeneous Problem

For a linear differential operator, we consider the **eigenvalue equation**:

$$\mathscr{L} y(x) = \lambda y(x) \tag{E}$$

The eigenvalues of $\mathscr{L}$ form a at most countable set $\{\lambda_i\}_{i=1,2,\ldots}$, and each of the eigenspace $\ker(\lambda_i \operatorname{id} - \mathscr{L})$ is finite-dimensional. So we have a corresponding set of non-trivial solutions $\{y_i(x)\}_{i=1,2,\ldots}$, known as **eigenfunctions**.

Theorem 3.1. Eigenvalues of the Adjoint Problem

The adjoint problem of (E):

$$\mathscr{L}^* w = \overline{\lambda} w \tag{E*}$$

has the same eigenvalues as the original problem.

Theorem 3.2. Orthogonal Eigenfunctions

The eigenfunction and adjoint eigenfunction corresponding to distict eigenvalues are orthogonal.

In other words, if $\mathscr{L} y_i = \lambda_i y_i$ and $\mathscr{L}^* w_i = \overline{\lambda_i} w_i$, then $\langle y_i, w_i \rangle = 0$.

Now we construct the solution of the inhomogeneous BVP $\mathscr{L} y = f$ (N) subject to linear homogeneous boundary conditions (HBC).

1. Solve the eigenvalue problem (E+HBC) to obtain the eigenvalue-eigenfunction pairs $\{(\lambda_j, y_j)\}_{j=1,2,\ldots}$;
2. Solve the adjoint eigenvalue problem (E*+HBC*) to obtain $\{(\lambda_j, w_j)\}_{j=1,2,\ldots}$;
3. Assume a solution to (N+HBC) of the form
$$y(x) = \sum_i c_i y_i(x) \tag{3.37}$$
4. By taking the inner product of (N) with w_k we obtain that the coefficients c_k satisfy that
$$\langle f, w_k \rangle = \lambda_k c_k \langle y_k, w_k \rangle \tag{3.38}$$

Note that if $\lambda_k = 0$ is an eigenvalue, then c_k is not determined by equation (3.38). In this case:

1. if $\langle f, w_k \rangle = 0$, then c_k is arbitrary, and (N+HBC) has non-unique solutions;
2. if $\langle f, w_k \rangle \neq 0$, then the solution does not exist.

The behavior is in line with Fredholm Alternative Theorem.

3.2 Inhomogeneous Boundary Conditions

We consider the inhomogeneous boundary conditions

$$\mathscr{B}_i y\big|_{x=a,b} = \gamma_i \tag{NBC}$$

We have two methods:

1. Use the technique in Section 1.7 and consider $\tilde{y} = y - v$;
2. Follow the process as for homogeneous boundary conditions. When taking the inner product of (N) with w_k and doing integration by parts, the boundary terms will generally present. So we will obtain a different form of equation (3.38).

3.3 Eigenfunction Expansion and Green's Function

Assuming that $\ker \mathscr{L}$ is trivial, we can write the solution as

$$y(x) = \sum_i \frac{\langle f, w_i \rangle}{\lambda_i \langle y_i, w_i \rangle} y_i(x) \tag{3.39}$$

Comparing the solution with

$$y(x) = \int_a^b g(x,\xi) f(\xi)\, \mathrm{d}\xi \tag{3.40}$$

We obtain the eigenfunction expansion of the Green's function:

$$g(x,\xi) = \sum_i \frac{w_i(\xi) y_i(x)}{\lambda_i \langle y_i, w_i \rangle} \tag{3.41}$$

3.4 Sturm-Liouville Theory

Sturm-Liouville theory concerns linear ODE of the form

$$\mathscr{L} y(x) = \lambda r(x) y(x) \tag{3.42}$$

where $\mathscr{L}$ is the **Sturm-Liouville operator**:

$$\mathscr{L} y(x) := -\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + q(x) y(x) \tag{3.43}$$

and $r(x) \geqslant 0$ is the **weighting function**.

Note that $\mathscr{L}$ is formally self-adjoint. It is fully self-adjoint if the boundary conditions take the separate form:

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0 \tag{3.44}$$

$$\beta_1 y(b) + \beta_2 y'(b) = 0 \tag{3.45}$$

The orthogonality condition for the eigenfunctions of $\mathscr{L}$ is

$$\langle y_j, y_k \rangle_r := \int_a^b y_j(x) \overline{y_k(x)} r(x)\, \mathrm{d}x = 0 \quad \text{for } j \neq k \tag{3.46}$$

Note that the integral with a weighting function does define an inner product provided $r > 0$ almost everywhere on $[a,b]$.

Theorem 3.3. Spectral Theorem for Sturm-Liouville Operators

Suppose that the functions p, q, r are real-valued. Then the eigenvalues of the Sturm-Liouville operator $\mathscr{L}$ on a finite domain $[a,b]$ are **real**, **countable**, and can be ordered as

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots \tag{3.47}$$

with $\lim_{i\to\infty} \lambda_i = +\infty$.

Theorem 3.4. Regular Sturm-Liouville Problem

Suppose that the Sturm-Liouville problem further satisfies that

1. $p(x), r(x) > 0$ for $x \in [a,b]$;
2. $q(x) \geqslant 0$ for $x \in [a,b]$;
3. $\alpha_1\alpha_2 \leqslant 0$ and $\beta_1\beta_2 \geqslant 0$ in the boundary conditions (3.44) and (3.45).

Then all eigenvalues $\lambda_k \geqslant 0$.

Singular Sturm-Liouville Problem: Suppose that p vanishes at one of the endpoints. For example, $p(a) = 0$. Then the only boundary condition we can impose on a is that

$$y(x),\ y'(x),\ w(x),\ w'(x) \text{ are } \textbf{bounded} \text{ as } x \to a \tag{3.48}$$

If $p(a) = p(b) = 0$, then $[a,b]$ is called the **natural interval** for the problem.

For the inhomogeneous Sturm-Liouville problem $\mathscr{L} y = f$ with homogeneous boundary conditions, we have a simpler form of eigenfunction expansion as the Sturm-Liouville operator $\mathscr{L}$ is self-adjoint.

$$y(x) = \sum_i c_i y_i(x) \tag{3.49}$$

where

$$c_i = \frac{\langle f, y_i \rangle}{\lambda_i \langle y_i, y_i \rangle_r} \tag{3.50}$$

Finally, any second-order linear differential operator

$$\mathscr{L} y = P_2 y'' + P_1 y' + P_0 \tag{3.51}$$

can be converted into Sturm-Liouville form by multiplying a weighting function:

$$r \mathscr{L} y = -(p y')' + q y \tag{3.52}$$

where

$$r(x) = -\frac{1}{P_2(x)} \exp\left(\int \frac{P_1(x)}{P_2(x)} \mathrm{d}x \right); \qquad p(x) = -r(x) P_2(x); \qquad q(x) = r(x) P_0(x) \tag{3.53}$$

In addition, we have

$$\mathscr{L}^* (r y) = r \mathscr{L} y \tag{3.54}$$

where r is given in (3.53).

4 Power Series Solution of Linear ODEs

4.1 Ordinary and Singular Points

We concern n-th order homogeneous linear ODE of the form

$$\mathscr{L} y(x) = y^{(n)}(x) + P_{n-1} y^{(n-1)}(x) + \cdots + P_0 y(x) = 0 \tag{4.55}$$

We will seek a **power series expansion** of the solution in the nieghbourhood of some point $x = x_0$

The point x_0 is an **ordinary point** of the ODE (4.55), if all coefficients $P_j(x)$ are **analytic** in a neighbourhood of x_0. In this case,

1. all n linearly independent solutions of (4.55) are analytic in a nieghbourhood of x_0 and have the form

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \tag{4.56}$$

2. the radius of convergence of the series solution is at least the distance from x_0 to the nearest singularity of the coefficient functions $P_j(x)$.

The point x_0 is a **singular point** of the ODE (4.55), if at least one of the coefficient functions is not analytic near x_0. Then the general solution may not be analytic near x_0.

The point x_0 is a **regular singular point** of the ODE (4.55), if not all of the coefficient functions are analytic near x_0, but instead the modified coefficients

$$p_j(x) := P_j(x)(x - x_0)^{n-j} \tag{4.57}$$

are all analytic near x_0. Otherwise x_0 is an **irregular singular point**.

For the point at infinity, we change the variable $t = 1/x$, $\tilde{y}(t) = y(x)$. We say that $x = \infty$ is a ordinary/regular singular/irregular singular point of (4.55) for $y(x)$, if $t = 0$ is a ordinary/regular singular/irregular singular point of the ODE for $\tilde{y}(t)$.

4.2 Indicial Equation

We restrict attention to regular singular points of second-order ODEs. If $x = x_0$ is a regular singular point, then the ODE has the form

$$\mathscr{L} y(x) = y''(x) + \frac{p(x)}{x - x_0} y'(x) + \frac{q(x)}{(x - x_0)^2} y(x) = 0 \tag{4.58}$$

where $p(x)$, $q(x)$ are analytic near x_0, and can be expanded into power series:

$$p(x) = \sum_{k=0}^{\infty} p_k (x - x_0)^k; \qquad q(x) = \sum_{k=0}^{\infty} q_k (x - x_0)^k \tag{4.59}$$

We seek a solution in the form of **Frobenius series**

$$y(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^{\alpha+k} \tag{4.60}$$

with coefficients a_k to be determined. We shall set $a_0 \neq 0$ by choosing α appropriately.

We substitute (4.60) into (4.58) and equate the coefficients. At the lowest power $(x-x_0)^{\alpha-2}$, using the condition that $a_0 \neq 0$ we obtain that

$$F(\alpha) := \alpha(\alpha-1) + p_0\alpha + q_0 = 0 \tag{4.61}$$

which is called the **indicial equation**. The roots α_1, α_2 of $F(\alpha)$ are called **indicial exponents**. We order them such that $\operatorname{Re}\alpha_1 \geqslant \operatorname{Re}\alpha_2$.

4.3 First Series Solution

The coefficients of $(x-x_0)^{k+\alpha-2}$ satisfy

$$F(\alpha+k)a_k = -\sum_{j=0}^{k-1}((\alpha+j)p_{k-j} + q_{k-j})a_j \tag{4.62}$$

We take $\alpha = \alpha_1$. Then $F(\alpha+k) \neq 0$ for $k \in \mathbb{Z}_+$. By rearrangement we obtain the recurrence relation

$$a_k = -\frac{1}{F(\alpha_1+k)}\sum_{j=0}^{k-1}((\alpha_1+j)p_{k-j} + q_{k-j})a_j \tag{4.63}$$

By solving $a_1, a_2, a_3, \ldots$ successively we obtain the **first series solution**

$$y_1(x) = (x-x_0)^{\alpha_1}\sum_{k=0}^{\infty} a_k (x-x_0)^k \tag{4.64}$$

where $a_0 \neq 0$ can be chosen arbitrarily.

4.4 Second Series Solution: $\alpha_1 - \alpha_2 \notin \mathbb{Z}$

For the **second series solution**, if $\alpha_1 - \alpha_2 \notin \mathbb{Z}$, then $F(\alpha_2+k) \neq 0$ for $k \in \mathbb{Z}$. Similar to the first series solution, we obtain the Frobenius series

$$y_2(x) = (x-x_0)^{\alpha_2}\sum_{k=0}^{\infty} b_k (x-x_0)^k \tag{4.65}$$

with coefficients b_k satisfying the recurrence relations

$$b_k = -\frac{1}{F(\alpha_2+k)}\sum_{j=0}^{k-1}((\alpha_2+j)p_{k-j} + q_{k-j})b_j \tag{4.66}$$

and $b_0 \neq 0$ can be chosen arbitrarily.

4.5 Second Series Solution: $\alpha_1 = \alpha_2$

When $\alpha_1 = \alpha_2$, $F(\alpha)$ has a double root α_1.

Derivative Method: From the first series solution (with $a_0 = 1$)

$$y_1(x) = (x-x_0)^{\alpha_1} + \sum_{k=1}^{\infty} a_k (x-x_0)^{\alpha_1+k} \tag{4.67}$$

we consider the series

$$y(x;\alpha) = (x-x_0)^{\alpha} + \sum_{k=1}^{\infty} a_k(\alpha) (x-x_0)^{\alpha+k} \tag{4.68}$$

where α is a parameter. We can deduce that

$$y_2(x) = \left.\frac{\partial}{\partial\alpha} y(x;\alpha)\right|_{\alpha=\alpha_1} \tag{4.69}$$

is a solution to (4.58). The explicit form is

$$y_2(x) = y_1(x)\ln(x-x_0) + (x-x_0)^{\alpha_1}\sum_{k=0}^{\infty} b_k (x-x_0)^k \tag{4.70}$$

where $b_k = a_k'(\alpha_1)$.

In practice, it is easier to solve b_k by substituting (4.70) into (4.58) to obtain the recurrence relations for b_k.

4.6 Second Series Solution: $\alpha_1 - \alpha_2 \in \mathbb{Z}_+$

Suppose that $\alpha_1 - \alpha_2 = N > 0$. From the Frobenius method, the coefficient of $(x-x_0)^{\alpha_2+N-2}$ satisfies

$$F(\alpha_2 + N) b_N = -\sum_{j=0}^{N-1} ((\alpha_2 + j) p_{N-j} + q_{N-j}) b_j \tag{4.71}$$

1. If the RHS of equation (4.71) is non-zero, then the Frobenius method fails. To get a second solution, we again use the derivative method and try the solution of the form

$$y_2(x) = y_1(x) \ln(x - x_0) + (x - x_0)^{\alpha_2} \sum_{k=0}^{\infty} b_k (x - x_0)^k \tag{4.72}$$

   Note that the indicial exponent here is α_2 instead of α_1.

2. If the RHS of equation (4.71) is zero, then there is no contradiction, and b_N can be chosen arbitrarily. In this case we obtain the second series solution

$$y_2(x) = (x - x_0)^{\alpha_2} \sum_{k=0}^{\infty} b_k (x - x_0)^k \tag{4.73}$$

   where $b_0 \neq 0$ and b_N can be chosen arbitrarily. Since $\alpha_2 + N = \alpha_1$, changing b_N just corresponds to adding multiples of y_1.

   (*In practice, it is convenient to choose* $b_0 = 1$ *and* $b_N = 0$.)

Remark. In practice, if we obtain the closed form of the first solution using the Frobenius method, then the second solution can also be obtained by reduction of order introduced in Section 1.3.

5 Special Functions

5.1 Bessel Functions

Consider the **Helmholtz equation**:

$$\nabla^2 u + \lambda u = 0 \tag{5.74}$$

By separation of variables in the **cylindrical coordinates**, we obtain the ODE satisfied by the radial component:

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R_n}{\mathrm{d}r}\right) + \left(\lambda - \frac{n^2}{r^2}\right) R_n(r) = 0 \tag{5.75}$$

We eliminate λ by the rescaling $R(r) = y(x)$ with $x = r\sqrt{\lambda}$, resulting in

$$x^2 y''(x) + x y'(x) + (x^2 - n^2) y(x) = 0 \tag{5.76}$$

which is known as the **Bessel's equation** of order n.

Bessel's equation has a regular singular point at $x = 0$, with indicial equation

$$F(\alpha) = \alpha^2 - n^2 = 0 \tag{5.77}$$

The indicial exponents are $\alpha_1 = n$, $\alpha_2 = -n$. (When $n = 0$, the indicial equation has double roots $\alpha_1 = \alpha_2 = 0$.)

The first Frobenius series solution, with a specific normalisation of the leading coefficient in the expansion, defines the **Bessel functions of first kind**

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+n+1)} \left(\frac{x}{2}\right)^{2k} \tag{5.78}$$

We are only interested in the case when $n \in \mathbb{Z}$. Then the second solution can be given by the derivative method. A specifically chosen normalisation defines the **Bessel functions of second kind** or **Neumann functions**:

$$Y_n(x) = \frac{2}{\pi} \ln\left(\frac{x}{2}\right) J_n(x) - \frac{1}{\pi}\left(\frac{2}{x}\right)^n \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}\left(\frac{x^2}{4}\right)^k - \frac{1}{\pi}\left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{\psi(k+1) + \psi(n+k+1)}{k!(n+k)!}\left(-\frac{x^2}{4}\right)^k \tag{5.79}$$

where the **digamma function** $\psi(m) = -\gamma + \sum_{k=1}^{m-1} \frac{1}{k}$ for $m \in \mathbb{Z}_+$ and γ is the **Euler's constant**.

Some properties of Bessel functions:

1. J_n and Y_n have infinite radius of convergence.
2. Each of J_n and Y_n has an infinite set of discrete zeros in $x > 0$. These zeros are labelled in order by $j_{n,1}, j_{n,2}, \ldots$ and $y_{n,1}, y_{n,2}, \ldots$.
3. Behaviour of Bessel functions as $x \to 0$:

   $J_n(0) = 0$ for $n > 0$ and $J_0(0) = 1$;

   $Y_n(x) = O(x^{-n})$ for $n > 0$ and $Y_0(x) = O(\ln x)$.
4. From the series expansion we can derive the following recurrence relations:

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x), \qquad J_{n+1}(x) = -2J_n'(x) + J_{n-1}(x) \tag{5.80}$$

   The same relations hold for Y_n.

Now we consider a physical problem of finding the normal modes of the oscillation on a circular drum. We impose the boundary conditions on the ODE (5.75):

$$R_n(r) \text{ is bounded as } r \to 0; \qquad R_n(a) = 0. \tag{5.81}$$

which leads to the eigenvalues

$$\lambda = \frac{j_{n,m}^2}{a^2}, \quad n \in \mathbb{N},\ m \in \mathbb{Z}_+ \tag{5.82}$$

with corresponding eigenfunctions

$$R_{n,m}(r) = J_n\left(\frac{j_{n,m}}{a} r\right) \tag{5.83}$$

The **normal frequencies** are $\omega_{n,m} = j_{n,m}\dfrac{c}{a}$.

By rescaling r we now assume that $a = 1$. The equation (5.75) can be put into Sturm-Liouville form by multiplying r:

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R_n}{\mathrm{d}r}\right) + \frac{n^2}{r} R_n(r) = \lambda r R_n(r) \tag{5.84}$$

with eigenvalues and corresponding eigenfunctions

$$\lambda_{n,m} = j_{n,m}^2, \qquad R_{n,m}(r) = J_n(j_{n,m} r) \tag{5.85}$$

from which we can derive the orthogonality relations

$$\int_0^1 J_n(j_{n,\ell} r) J_n(j_{n,m} r)\, r\, \mathrm{d}r = 0 \quad \text{for } \ell \neq m \tag{5.86}$$

and

$$\int_0^1 J_n(j_{n,m} r)^2\, r\, \mathrm{d}r = \frac{1}{2}\big(J_n'(j_{n,m})\big)^2 \tag{5.87}$$

5.2 Legendre Functions

By separation of variables of the Helmholtz equation in the **spherical coordinates**, we obtain the ODE satisfied by the azimuthal component:

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \left(\lambda - \frac{m^2}{\sin^2\theta}\right)\Theta(\theta) = 0 \tag{5.88}$$

Change of variable $\Theta(\theta) = y(x)$ with $x = \cos\theta$. We obtain the **associated Legendre equation**

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \left(\lambda - \frac{m^2}{1-x^2}\right)y(x) = 0 \tag{5.89}$$

The solutions are called the **associated Legendre functions**. When $m = 0$, we have the **Legendre equation**

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \lambda y(x) = 0 \tag{5.90}$$

The solutions are called the **Legendre functions**.

The associated Legendre equation has regular singular points at $x = \pm 1$ and $x = \infty$. The indicial exponents for $x = \pm 1$ are $m/2$ and $-m/2$. $[-1,1]$ is a natural interval for the associated Legendre equation and we can pose it as a singular Sturm-Liouville problem:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x}\right) + \frac{m^2}{1-x^2}y(x) = \lambda y(x) \quad \text{for } -1 < x < 1; \qquad y(x) \text{ is bounded as } x \to \pm 1 \tag{5.91}$$

The eigenvalues are given by $\lambda = \ell(\ell+1)$ with integer $\ell \geqslant m \geqslant 0$, corresponding to eigenfunctions $P_\ell^m(x)$. The orthogonality relations are

$$\int_{-1}^{1} P_k^m(x) P_\ell^m(x) = 0 \quad \text{for } k \neq \ell \tag{5.92}$$

For $m = 0$ and $\ell \geqslant 0$, the Legendre functions $P_\ell(x) := P_\ell^0(x)$ are **polynomials of degree** ℓ. The **Legendre polynomials** are given explicitly by the **Rodrigues' formula**:

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}(x^2-1)^\ell \tag{5.93}$$

More explicitly,

$$P_\ell(x) = \sum_{n=0}^{\ell} \frac{1}{(n!)^2}\frac{(\ell+n)!}{(\ell-n)!}\left(\frac{x-1}{2}\right)^n \tag{5.94}$$

The orthogonality relation of $P_\ell(x)$

$$\int_{-1}^{1} P_\ell(x)^2 \,\mathrm{d}x = \frac{2}{2\ell+1} \tag{5.95}$$

The recurrence relations satified by $P_\ell(x)$:

$$(2\ell+1)xP_\ell(x) = (\ell+1)P_{\ell+1}(x) + \ell P_{\ell-1}(x); \tag{5.96}$$

$$P'_{\ell+1}(x) = xP'_\ell(x) + (\ell+1)P_\ell(x) \tag{5.97}$$

The associate Legendre functions and the Legendre polynomials are related by

$$P_\ell^m(x) = (-1)^m \left(1-x^2\right)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} P_\ell(x) \tag{5.98}$$

Then $P_\ell^m(x)$ is a polynomial if and only if m is even. Combining (5.98) with the Rodrigues' formula we obtain

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!}\left(1-x^2\right)^{m/2} \frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}}\left(x^2-1\right)^\ell \tag{5.99}$$

for which we obtain

$$\int_{-1}^{1} P_\ell^m(x)^2 \,\mathrm{d}x = \frac{2(\ell+m)!}{(2\ell+1)(\ell-m)!} \tag{5.100}$$

The recurrence relations satified by $P_\ell^m(x)$:

$$(2\ell+1)xP_\ell^m(x) = (\ell-m+1)P_{\ell+1}^m(x) + (\ell+m)P_{\ell-1}^m(x) \tag{5.101}$$

A second, linearly independent solution of the Legendre equation with $m = 0$ is given by the **Legendre function of second kind**, denoted by $Q_\ell(x)$, which are unbounded as $x \to \pm 1$. For $\ell = 0$,

$$Q_0(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \tag{5.102}$$

Similar to the first kind, the **associated Legendre functions of second kind** are given by

$$Q_\ell^m(x) = (-1)^m \left(1-x^2\right)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} Q_\ell(x) \tag{5.103}$$

5.3 Orthogonal Polynomials

There are other families of orthogonal polynomials, which are solutions to some second-order linear ODEs, satisfying the orthogonality relations

$$\langle p_m, p_n \rangle_r := \int_a^b p_m(x)p_n(x)r(x)\,\mathrm{d}x = 0 \quad \text{for } m \neq n \tag{5.104}$$

for some **weighting function** $r(x)$.

These orthogonal polynomials can also be obtained by applying Gram-Schmidt orthogonalisation process to the set $\{1, x, x^2, \ldots\}$ with respect to certain inner product on the interval $[a,b]$, and also by considering the **generalised Rodrigues' formula**:

$$p_n(x) = \frac{1}{k_n r(x)}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(r(x)X^n\right), \quad \text{where } X = \begin{cases} (b-x)(a-x), & |a|,|b| < \infty \\ x-a, & |a| < \infty, b = \infty \\ 1, & -a = b = \infty \end{cases} \tag{5.105}$$

where k_n is some constant. The following table provides a full classification of orthogonal polynomials:

Orthogonal Polynomials		2nd-order Linear ODE	$[a,b]$	X	$r(x)$	α,β
Jacobi	$P_n^{(\alpha,\beta)}$	$(1-x^2)y''(x)+(a+bx)y'(x)+\lambda y(x)=0$	$[-1,1]$	x^2-1	$(1-x)^{\alpha}(x+1)^{\beta}$	$\alpha,\beta>-1$
Gegenbauer	$P_n^{(\alpha,\alpha)}$				$(1-x)^{\alpha}(x+1)^{\alpha}$	$\alpha,\beta=-1$
Chebyshev	$T_n^{(\pm)}$				$(1-x)^{\pm 1/2}(x+1)^{\pm 1/2}$	$\alpha,\beta=\pm 1/2$
Legendre	P_n				1	$\alpha,\beta=0$
Associated Laguerre	L_n^{α}	$xy''(x)+(\alpha+1-x)y'(x)+\lambda y(x)=0$	$[0,+\infty)$	x	$e^{-x}x^{\alpha}$	$\alpha>-1$
Laguerre	L_n				e^{-x}	$\alpha=0$
Hermite	H_n	$y''-2xy'(x)+\lambda y(x)=0$	$(-\infty,+\infty)$	1	e^{-x^2}	

6 Asymptotic Analysis

6.1 Asymptotic Expansion

We say that $f(x)=O(g(x))$ or ***f* is of order *g*** as $x\to x_0$, if

$$\exists\delta>0\quad\exists A>0\quad\forall x\in(x_0-\delta,x_0+\delta)\quad|f(x)|<A|g(x)| \tag{6.106}$$

We say that $f(x)\sim g(x)$ or ***f* is asymptotic to *g*** as $x\to x_0$, if

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=1 \tag{6.107}$$

We say that $f(x)=o(g(x))$ or $f(x)\ll(g(x))$ as $x\to x_0$, if

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=0 \tag{6.108}$$

We are particularly interested in the behaviour of $f(\varepsilon)$ as $\varepsilon\to 0$.

A sequence of functions $\{\varphi_k(\varepsilon)\}_{k\in\mathbb{N}}$ is an **asymptotic sequence** as $\varepsilon\to 0$, if $\varphi_{k+1}(\varepsilon)=o(\varphi_k(\varepsilon))$ for each $k\in\mathbb{N}$ as $\varepsilon\to 0$.

A function $f(\varepsilon)$ has **asymptotic expansion** of the form

$$f(\varepsilon)\sim\sum_k a_k\varphi_k(\varepsilon)\quad\text{as }\varepsilon\to 0 \tag{6.109}$$

if

1. the **gauge functions** φ_k form an asymptotic sequence;
2. $f(\varepsilon)-\sum_{k=0}^{N}a_k\varphi_k(\varepsilon)=o(\varphi_N(\varepsilon))$ for $N\in\mathbb{N}$ as $\varepsilon\to 0$.

Some properties of asymptotic expansion:

1. Given a sequence of gauge functions $\{\varphi_k\}$, the coefficients a_k are unique;
2. The function defines the expansion but not *vice versa*.

Remark. We have $e^{1/\varepsilon}=o(\varepsilon^k)$ as $\varepsilon\to 0$ for all $k>0$. $e^{1/\varepsilon}$ is said to be **exponentially small** or **transcendentally small**.

6.2 Approximate Roots of Algebraic Equations

In this section we consider algebraic equations of the form $f(x;\varepsilon)=0$. We are interested in the asymptotic expansion of the solutions in the limit $\varepsilon\to 0$. The general method can be summarised as follows:

1. Scale the variable to get a **dominant balance**, so that at least two of the terms balance and are much larger than the remaining terms in the equation.

2. Plug in an asymptotic expansion $x = \sum_k x_k \varphi_k(\varepsilon)$, whose form is usually clear from the equation.

   For example, if $f(x,\varepsilon)$ is a rational function in x and ε, we normally will try

$$x = x_0 + x_1\varepsilon + x_2\varepsilon^2 + x_3\varepsilon^3 + \cdots \tag{6.110}$$

3. By equating the coefficients of each gauge function $\varphi_k(\varepsilon)$, obtain the coefficients $x_0, x_1, x_2, \ldots$ in the expansion.
4. Repeat for any other possible dominant balances in the equation to obtain approximations for other roots.

We illistrate the process in the following example:

Example 6.1

Solve approximately the quadratic equation

$$\varepsilon x^2 + x - 1 = 0 \tag{6.111}$$

in the limit as $\varepsilon \to 0$.

Solution. First we try to balance the terms x and -1, which implies that $x = \mathrm{O}(1)$. We seek the asymptotic expansion

$$x = x_0 + x_1\varepsilon + x_2\varepsilon^2 + x_3\varepsilon^3 + \mathrm{O}(\varepsilon^4) \tag{6.112}$$

By equating the terms $\mathrm{O}(1)$, $\mathrm{O}(\varepsilon)$, $\mathrm{O}(\varepsilon^2)$ and $\mathrm{O}(\varepsilon^3)$, we can solve x_0, x_1, x_2 and x_3 successively. The approximation of this root is

$$x^+ \sim 1 - \varepsilon + 2\varepsilon^2 - 5\varepsilon^3 + \mathrm{O}(\varepsilon^4) \tag{6.113}$$

Second, we try to balance the terms εx^2 and 1, which suggests that $x = \mathrm{O}(\varepsilon^{-1/2})$. But now x is much larger than εx^2 and 1. So we will have a problem.

Third, we try to balance the terms εx^2 and x, which suggests that $x = \mathrm{O}(\varepsilon^{-1})$. We use the scaling $x = \varepsilon^{-1} y$. Then the equation becomes

$$y^2 + y - \varepsilon = 0 \tag{6.114}$$

We seek the asymptotic expansion

$$y = y_0 + y_1\varepsilon + y_2\varepsilon^2 + y_3\varepsilon^3 + \mathrm{O}(\varepsilon^4) \tag{6.115}$$

By equating the terms we solve y_0, y_1, y_2 and y_3 successively. The approximation for x is

$$x^- \sim -\varepsilon^{-1} - 1 + \varepsilon - 2\varepsilon^2 + \mathrm{O}(\varepsilon^3) \tag{6.116}$$

Finally, we observe that the asymptotic expansions of $x^\pm$ agree with the asymptotic expansion of the exact solution

$$x^\pm = \frac{-1 \pm \sqrt{1+4\varepsilon}}{2\varepsilon} \tag{6.117}$$

near $\varepsilon = 0$. □

The example above is a **singular perturbation** problem because setting $\varepsilon = 0$ reduces the degree of the problem.

6.3 Regular Perturbations in ODEs

We consider ODEs of the form $f(y, y', \ldots, y^{(n)}; x; \varepsilon) = 0$. We are interested in the asymptotic expansion of the solutions $y(x;\varepsilon)$ in the limit $\varepsilon \to 0$. We will not summary the general method but instead present an example to illistrate the process.

Example 6.2. Small Oscillation of a Pendulum

Solve approximately the initial value problem

$$y''(x) + \frac{\sin(\varepsilon y(x))}{\varepsilon} = 0 \qquad y(0) = 0,\ y'(0) = 1 \tag{6.118}$$

Solution. From the Taylor exapnsion of $\sin(\varepsilon y(x))$ we observe that the problem only contains *even* powers of ε. So we seek a solution of the form

$$y(x;\varepsilon) = y_0(x) + \varepsilon^2 y_2(x) + O(\varepsilon^4) \tag{6.119}$$

By equating the terms $O(1)$ and $O(\varepsilon^2)$ we obtain the equations satisfied by y_0 and y_2:

$$O(1): \qquad y_0'' + y_0 = 0 \qquad y_0(0) = 0,\ y_0'(0) = 1 \tag{6.120}$$

$$O(\varepsilon^2): \qquad y_2'' + y_2 = \frac{y_0^3}{6} \qquad y_2(0) = 0,\ y_2'(0) = 1 \tag{6.121}$$

From which we obtain the asymptotic expansion of the solution:

$$y(x;\varepsilon) \sim \sin x + \varepsilon^2\left(\frac{3}{64}\sin x + \frac{1}{192}\sin 3x - \frac{1}{16}x\cos x\right) + O(\varepsilon^4) \tag{6.122}$$

□

Note that the expansion is valid when $x = O(1)$. When $x = O(\varepsilon^{-2})$, the expansion becomes **non-uniform** as the second term becomes the same order as the leading term. In this problem, the non-uniformity arises from the **secular term** proportional to $x\cos x$. In general, we expect to find a secular term in the solution whenever the right hand-side of (6.121) contains a term in the complementary function.

6.4 Boundary Layers in IVP

For a ODE like Example 6.2, the regular asymptotic expansion

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots \tag{6.123}$$

may only be valid for a limited range of values of x. In addition, it is not clear how to determine the solution uniquely if a boundary condition is imposed in a region where the asymptotic expansion is not valid, which called a **boundary layer**.

For a problem involving boundary layers, we use the method of **matched asymptotic expansion**. We construct two different asymptotic expansions for the solution, one in the **outer region**, which is the **outer expansion**, and the other in the boundary layer, which is the **inner expansion**. Then we join them up as a **composite expansion** by **asymptotic matching**:

Composite Expansion = Inner Expansion + Outer Expansion − Common Limit

We illustrate the process by considering a singular perturbation problem:

Example 6.3

Solve approximately the initial value problem for $x > 0$:

$$\varepsilon y'(x) + y(x) = e^{-x} \qquad y(0) = 0 \tag{6.124}$$

Solution. First we consider $x = O(1)$. We try $y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots$. The leading-order term y_0:

$$y_0(x) = e^{-x} \tag{6.125}$$

cannot satisfy the condition $y_0(0) = 0$. We recognise that there is a boundary layer near $x = 0$. So $y_0(x) = e^{-x}$ is the outer expansion.

To obtain the inner expansion we have to rescale x to get a dominant balance for the equation. We set $x = \delta X$ and $y(x) = Y(X)$ where $\delta \ll 1$ is to be determined. The problem becomes

$$\frac{\varepsilon}{\delta}Y'(X) + Y(X) = e^{-\delta X}, \qquad Y(0) = 0 \tag{6.126}$$

We can balance all three terms by setting $\delta = \varepsilon$. This corresponds to $x = O(\varepsilon)$. The equation in the inner region becomes

$$Y'(X) + Y(X) = e^{-\varepsilon X} \sim 1 - \varepsilon X + \cdots \tag{6.127}$$

We try $Y(x) = Y_0(x) + \varepsilon Y_1(x) + \varepsilon^2 Y_2(x) + \cdots$. The leading order satifies

$$Y_0'(X) + Y_0(X) = 1 \qquad Y_0(0) = 0 \tag{6.128}$$

which is solve by $Y_0(X) = 1 - e^{-X}$. THis gives the inner expansion.

The leading-order matching principle is demonstrated by

$$\lim_{x\to 0} y_0(x) = \lim_{X\to\infty} Y_0(X) = 1 \tag{6.129}$$

Hence the composite expansion is given by

$$y_{\text{comp}}(x) = y_0(x) + Y_0(X) - 1 = \mathrm{e}^{-x} - \mathrm{e}^{-x/\varepsilon} \tag{6.130}$$

□

6.5 Boundary Layers in BVP

For a general second-order linear ODE

$$\varepsilon y''(x) + P_1(x)y'(x) + P_2(x)y(x) = R(x), \qquad x \in (a,b) \tag{6.131}$$

with boundary conditions given at $x = a$ and $x = b$. Assume that P_0, P_1 and R are smooth and bounded, and that $P_1(x) \neq 0$ and **does not change sign** in the interval $x \in [a,b]$. Then the boundary layer is at

1. the **left-hand** boundary $x = a$, it $P_1(x) > 0$;
2. the **right-hand** boundary $x = b$, it $P_1(x) < 0$.

(assuming that $\varepsilon \to 0^+$.)

Once we locate the boundary layer, we can apply the process described in Section 6.4 to obtain the inner and outer expansion. Usually the inner expansion will contain an integrating constant, which should be determined by the matching principle.

For more complicated situations, the boundary layers may

1. appear at both ends of the domain, or
2. appear in the intermediate region, or
3. appear nested, with one boundary layer inside another one.

The difficulties arise in these situations can be

1. It may not be clear where to look for boundary layers;
2. The boundary layers may require the rescaling of the dependent variable y as weel as the independent variable x;
3. We may have to solve the full ODE with no simplification.

6.6 Poincaré-Lindstedt Method

For problems of **slowly varying oscillations** (including the example 6.2), we seek periodic solution and want to suppress the secular terms. We can use the **Poincaré-Lindstedt method**, which is a simplified version of the more general **method of multiple scales**.

Example 6.4

Solve the initial value problem for $x > 0$

$$y''(x) + (1+\varepsilon)y(x) = 0, \qquad y(0) = 1,\ y'(0) = 0 \tag{6.132}$$

Solution. Make the substitution $X = \omega x$ where the frequency $\omega(\varepsilon)$ is not known in advanced. The problem becomes

$$\omega^2 Y''(X) + (1+\varepsilon)Y(X) = 0, \qquad Y(0) = 1,\ Y'(0) = 0 \tag{6.133}$$

We expand both y and ω in powers of ε:

$$Y(X) \sim Y_0(X) + \varepsilon Y_1(X) + \mathrm{O}(\varepsilon^2) \qquad \omega \sim 1 + \varepsilon\omega_1 + \mathrm{O}(\varepsilon^2) \tag{6.134}$$

At O(1), we obtain that $Y_0(X) = \cos X$. At $\mathrm{O}(\varepsilon)$, we find that $Y_1(X)$ satisfies

$$Y_1''(X) + Y_1(X) = -2\omega_1 Y_0''(X) - Y_0(X) = (2\omega_1 - 1)\cos X \qquad Y_1(0) = Y_1'(0) = 0 \tag{6.135}$$

We must eliminate the resonant term $(2\omega_1 - 1)\cos X$, so we set $\omega_1 = 1/2$. Thus the oscillation frequency is given by the asymptotic expansion

$$\omega = 1 + \frac{1}{2}\varepsilon + \mathrm{O}(\varepsilon^2) \tag{6.136}$$

which agrees with the expansion of the exact solution

$$y(x) = \cos\left(\sqrt{1+\varepsilon}\,x\right) \tag{6.137}$$

□

6.7 WKB Approximation

When the small parameter multiplies the highest derivative in an ODE, it does not always lead to the formation of boundary layers, but is also possible for the solution to exhibit **rapid oscillations** instead. We can use the **WKB approximation**. This method is particularly useful for analysing the Schrödinger equation in the semi-classical limit.

We concern ODEs in the form

$$\varepsilon^2 y''(x) + Q(x) y = 0 \tag{6.138}$$

with $Q(x) \neq 0$ in the domain of interest. We assume a WKB asymptotic expansion in the form

$$y(x) = A(x;\varepsilon)\,\mathrm{e}^{\mathrm{i}u(x)/\varepsilon} \tag{6.139}$$

where both the **phase** $u(x)$ and the **amplitude** $A(x;\varepsilon)$ are to be determined. The equation is transformed to

$$\left(Q(x) - u'(x)^2\right) A(x) + \mathrm{i}\varepsilon \left(2u'(x)A'(x) + u''(x)A(x)\right) + \varepsilon^2 A''(x) = 0 \tag{6.140}$$

At leading order O(1), we get the **eikonal equation** $u'(x)^2 = Q(x)$ so the phase is given by

$$u(x) = \pm \int^x \sqrt{Q(t)}\,\mathrm{d}t \tag{6.141}$$

We consider a regular asymptotic expansion of $A(x;\varepsilon) \sim A_0(x) + \varepsilon A_1(x) + \mathrm{O}(\varepsilon^2)$. The leading-order amplitude satisfies

$$\frac{2A_0'(x)}{A_0(x)} + \frac{u''(x)}{u'(x)} = 0 \tag{6.142}$$

which solves into

$$A_0(x) = \mathrm{const} \cdot Q(x)^{-1/4} \tag{6.143}$$

Therefore at leading order, the approximate solution is given by

$$y(x) \sim Q(x)^{-1/4} \left(C_1 \exp\left(\frac{\mathrm{i}}{\varepsilon} \int^x \sqrt{Q(t)}\,\mathrm{d}t \right) + C_2 \exp\left(-\frac{\mathrm{i}}{\varepsilon} \int^x \sqrt{Q(t)}\,\mathrm{d}t \right) \right) \qquad \text{as } \varepsilon \to 0 \tag{6.144}$$

At the point $x = x_e$ where $Q(x_e) = 0$, we observe that $A(x) \to \infty$ as $x \to x_e$. Such points where the WKB approximation breaks down are called **classical turning points**. At the turning points we must consider inner expansions and match them with the outer WKB expansion.

7 Dirac δ-Function and Distributions

7.1 Test Functions and Distributions

$f : \mathbb{R} \to \mathbb{R}$ is said to be **locally (Lebesgue) integrable**, if it is (Lebesgue) integrable on any compact subset of $\mathbb{R}$.

A map $\phi : \mathbb{R} \to \mathbb{R}$ is a **test function**, if it is smooth with compact support. A simple example of test function is

$$\phi(x) = \begin{cases} \exp\left(\dfrac{C}{(x-a)(x-b)} \right), & x \in (a,b); \\ 0, & \text{otherwise.} \end{cases} \tag{7.145}$$

The set of test functions forms a vector space $\mathscr{D}$.

Convergence in $\mathscr{D}$: We say that a sequence of test functions $\{\phi_n\}_{n\in\mathbb{N}}$ converges to ϕ in $\mathscr{D}$, if:

1. there exists $R > 0$ such that $\phi(x) = \phi_n(x) = 0$ for all $|x| > R$ and $n \in \mathbb{N}$;
2. $\phi_n^{(k)}$ converges uniformly to $\phi^{(k)}$ for all derivatives $k \in \mathbb{N}$.

We denote it by $\phi_n \xrightarrow{\mathscr{D}} \phi$.

A **distribution** or **generalised function** F is a continuous linear functional from $\mathscr{D}$ to $\mathbb{R}$. The continuity of F is in the sense that

$$\phi_n \xrightarrow{\mathscr{D}} \phi \quad \Longrightarrow \quad F(\phi_n) \to F(\phi) \tag{7.146}$$

The set of distributions forms a vector space $\mathscr{D}'$, which is a subspace of the dual space of $\mathscr{D}$.

It is common to use the "inner product" $\langle F, \phi \rangle$ to denote $F(\phi)$.

Theorem 7.1

Suppose that $f:\mathbb{R}\to\mathbb{R}$ is a continuous function such that

$$\int_{-\infty}^{+\infty} f(x)\phi(x)\,\mathrm{d}x = 0 \tag{7.147}$$

for all $\phi\in\mathscr{D}$. Then $f = 0$.

Given a locally integrable function $f:\mathbb{R}\to\mathbb{R}$, we can define a distribution F_f by

$$F_f:\ \phi\mapsto\int_{-\infty}^{+\infty} f(x)\phi(x)\,\mathrm{d}x \tag{7.148}$$

Such distributions are called **regular distributions**. Note that the above theorem shows that the map $f\mapsto F_f$ restricted to continuous functions is injective.

The distributions that are not regular are called **singular distributions**.

Convergence in distribution: We say that a sequence of distributions $\{F_n\}_{n\in\mathbb{N}}$ converges to F_n in $\mathscr{D}'$, if

$$\lim_{n\to\infty}\langle F_n,\phi\rangle = \langle F,\phi\rangle \tag{7.149}$$

for all $\phi\in\mathscr{D}$. We denote it by $F_n\xrightarrow{\mathscr{D}'}F$.

For $F\in\mathscr{D}'$ and $a\in\mathbb{R}$, we define the **translation of F through a** to be the distribution given by

$$F(x-a):\ \phi(x)\mapsto\langle F(x),\phi(x+a)\rangle \tag{7.150}$$

For regular distribution F_f, this corresponds to a change of variable in the integral:

$$\langle f(x-a),\phi(x)\rangle = \int_{-\infty}^{+\infty} f(x-a)\phi(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} f(x)\phi(x+a)\,\mathrm{d}x = \langle f(x),\phi(x+a)\rangle \tag{7.151}$$

For $F\in\mathscr{D}'$ and $f\in\mathrm{C}^{\infty}(\mathbb{R})$, we define the distribution fF to be

$$fF:\ \phi\mapsto\langle F,f\phi\rangle \tag{7.152}$$

For $F\in\mathscr{D}'$, we define the **distributional derivative** of F to be the distribution:

$$F':\ \phi\mapsto-\langle F,\phi'\rangle \tag{7.153}$$

For regular distribution F_f, this corresponds to integration by parts:

$$\langle f',\phi\rangle = \int_{-\infty}^{+\infty} f'(x)\phi(x)\,\mathrm{d}x = f(x)\phi(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f(x)\phi'(x)\,\mathrm{d}x = -\langle f,\phi'\rangle \tag{7.154}$$

As test functions are infinitely differentiable, the distributions are all "infinitely differentiable".

The product rule $(fF)' = f'F + fF'$ holds for distributional derivatives.

All distributions G have **antiderivative** F such that $G = F'$: Fix $\phi_1\in\mathscr{D}$ such that $\displaystyle\int_{-\infty}^{+\infty}\phi_1(x)\,\mathrm{d}x = 1$. Given a test function ϕ we can write

$$\phi = K\phi_1 + \phi_0 \tag{7.155}$$

where $K = \displaystyle\int_{-\infty}^{+\infty}\phi(x)\,\mathrm{d}x$ and $\phi_0\in\mathscr{D}$ such that $\displaystyle\int_{-\infty}^{+\infty}\phi_0(x)\,\mathrm{d}x = 0$. Define

$$\psi(x) = \int_{-\infty}^{x}\phi_0(t)\,\mathrm{d}t \tag{7.156}$$

Then $\psi\in\mathscr{D}$ and $\psi' = \phi_0$. We then can define F by

$$F:\phi\mapsto-\langle G,\psi\rangle \tag{7.157}$$

7.2 δ-Function

Informally, **δ-function** $\delta:\mathbb{R}\to\mathbb{R}$ is defined to be a function with the properties

$$\delta(x)=0 \quad \text{for } x\neq 0; \qquad \int_{-\infty}^{+\infty}\delta(x)\,\mathrm{d}x=1 \tag{7.158}$$

We know that such function does not exist by Lebesgue integration theory. However, in modern math we define the δ-function to be the singular distribution:

$$\delta:\phi\mapsto\phi(0) \tag{7.159}$$

The relation is often represented (incorrectly) by an integral:

$$\int_{-\infty}^{+\infty}\phi(x)\delta(x)\,\mathrm{d}x=\phi(0) \tag{7.160}$$

The translation of δ through $a\in\mathbb{R}$ is the distribution

$$\delta(x-a):\ \phi\mapsto\phi(a) \tag{7.161}$$

which is represented by the integral:

$$\int_{-\infty}^{+\infty}\phi(x)\delta(x-a)\,\mathrm{d}x=\phi(a) \tag{7.162}$$

The **Heaviside function** H is defined to be the distribution:

$$H:\ \phi\mapsto\int_0^{\infty}\phi(x)\,\mathrm{d}x \tag{7.163}$$

which is a regular distribution of the real functions $\chi_{[0,\infty)}$ or $\chi_{(0,\infty)}$.

It is easy to show that δ is the distributional derivative of H. On the other hand, the derivative of δ is the distribution:

$$\delta':\ \phi\mapsto-\phi'(0) \tag{7.164}$$

This can also be found by the ordinary definition of derivative:

$$\delta'(x)=\lim_{h\to 0}\frac{\delta(x+h)-\delta(x)}{h} \tag{7.165}$$

δ-function can be approximated by a sequence of functions, which can be generated by the following theorem

Theorem 7.2

Suppose that $f:\mathbb{R}\to\mathbb{R}$ is integrable with $\int_{-\infty}^{+\infty}f(x)\,\mathrm{d}x=1$. For $\varepsilon>0$ we define $f_\varepsilon(x):=\varepsilon^{-1}f(x/\varepsilon)$. Then we have $F_{f_\varepsilon}\xrightarrow{\mathscr{D}'}\delta$ as $\varepsilon\to 0$.

Some common examples are

$$\frac{1}{\sqrt{2\pi\varepsilon}}\exp\left(-\frac{x^2}{2\varepsilon^2}\right)\xrightarrow{\mathscr{D}'}\delta \quad \text{as } \varepsilon\to 0 \qquad \frac{1}{2\varepsilon}\chi_{(-\varepsilon,\varepsilon)}\xrightarrow{\mathscr{D}'}\delta \quad \text{as } \varepsilon\to 0 \tag{7.166}$$

Let $g\in C^1(\mathbb{R})$. We can define the define the distribution $\delta(g(x))$ to be the distributional limit of $F_{f_\varepsilon\circ g}$ as $\varepsilon\to 0$, where f_ε is defined in Theorem 7.2.

Theorem 7.3. Composition of Delta Function with a Function

Suppose that $g\in C^1(\mathbb{R})$ and that $\mathbb{R}$ can be expressed as an at most countable union of disjoint intervals I_α such that each $g|_{I_\alpha}$ is invertible. In addition we assume that all roots of g are simple. Then we have

$$\delta(g(x))=\sum_{a\in g^{-1}(\{0\})}\frac{1}{|g'(a)|}\delta(x-a) \tag{7.167}$$

As a direct corollary, we have:

$$\delta(ax)=\frac{1}{|a|}\delta(x), \qquad \delta(x^2-a^2)=\frac{1}{2|a|}\left(\delta(x+a)+\delta(x-a)\right) \tag{7.168}$$

8 Laplace Transform

8.1 Basic Properties

For $f:(0,\infty)\to\mathbb{C}$ and $p\in\mathbb{C}$ such that $f(x)\,\mathrm{e}^{-px}\in L^1(0,\infty)$, we define the **Laplace transform** $\overline{f}(p)$ of $f(x)$ to be

$$\overline{f}(p) := \int_0^\infty f(x)\,\mathrm{e}^{-px}\,\mathrm{d}x \tag{8.169}$$

$\overline{f}$ is also denoted by $\mathscr{L}[f]$, where $\mathscr{L}$ is an integral operator that represents Laplace transform.

Theorem 8.1. Existence of Laplace Transform

Suppose that $f:[0,\infty)$ is measurable and there exist constants $M, x_0, c\in\mathbb{R}$ such that $|f(x)|\leqslant M\mathrm{e}^{cx}$ a.e. for $x\geqslant x_0$. Then $\overline{f}(p)$ exists for all $\mathrm{Re}(p)>c$.

Theorem 8.2. Watson's Lemma

Suppose that $f:[0,\infty]\to\mathbb{C}$ is continuous and the Laplace transform $\overline{f}(p)$ of $f(x)$ exist for some $p_0\in\mathbb{C}$. Then the asymptotic behaviour of $\overline{f}$ as $\mathrm{Re}\,p\to\infty$ is given by

$$\overline{f}(p)\sim\frac{f(0)}{p}+\mathrm{O}(p^{-2}) \tag{8.170}$$

Some basic properties of Laplace transform are summarised as follows: (we assume that the transformed functions exist and that f is sufficiently smooth and well-behaved as $x\to 0$)

1. $\mathscr{L}$ is linear: $\mathscr{L}[af(x)+bg(x)] = a\mathscr{L}[f(x)]+b\mathscr{L}[g(x)]$ for $a,b\in\mathbb{C}$;
2. $\mathscr{L}[f(x)\,\mathrm{e}^{-ax}] = \overline{f}(p+a)$ for $a\in\mathbb{C}$;
3. $\mathscr{L}[f'(x)] = p\overline{f}(p)-f(0)$;
4. $\mathscr{L}[f''(x)] = p^2\overline{f}(p)-pf(0)-f'(0)$;
5. $\mathscr{L}[f^{(n)}(x)] = p^n\overline{f}(p)-\sum\limits_{k=1}^{n}p^{n-k}f^{(k-1)}(0)$ for $n\in\mathbb{N}$;
6. $\mathscr{L}[xf(x)] = -\dfrac{\mathrm{d}\overline{f}}{\mathrm{d}p}$;
7. $\mathscr{L}[x^nf(x)] = (-1)^n\dfrac{\mathrm{d}^n\overline{f}}{\mathrm{d}p^n}$ for $n\in\mathbb{N}$;
8. $\mathscr{L}\big[f(x-a)H(x-a)\big] = \mathrm{e}^{-ap}\overline{f}(p)$, where $H(x)$ is the Heaviside function and $a>0$.

8.2 Summary of Laplace Transforms of Common Functions

1. $\mathscr{L}[1] = 1/p$ for $\mathrm{Re}\,p>0$;
2. $\mathscr{L}[x] = 1/p^2$ for $\mathrm{Re}\,p>0$;
3. $\mathscr{L}[x^a] = \Gamma(a+1)/p^{a+1}$ for $\mathrm{Re}\,p>0$, where $a>-1$;
4. $\mathscr{L}[\mathrm{e}^{ax}] = 1/(p-a)$ for $\mathrm{Re}\,p>\mathrm{Re}\,a$, where $a\in\mathbb{C}$;
5. $\mathscr{L}[\cos ax] = p/(p^2+a^2)$ for $\mathrm{Re}\,p>|\mathrm{Im}\,\omega|$, where $a\in\mathbb{C}$;
6. $\mathscr{L}[\sin ax] = a/(p^2+a^2)$ for $\mathrm{Re}\,p>|\mathrm{Im}\,\omega|$, where $a\in\mathbb{C}$;
7. $\mathscr{L}[\delta(x-a)] = \mathrm{e}^{-ap}$, where $a>0$;
8. $\mathscr{L}[J_0(x)] = (1+p^2)^{-1/2}$, where $J_0(x)$ is the Bessel function of first kind of order zero.

These formulae are particularly useful for finding the **inverse Laplace transform** of common functions.

8.3 Convolution and Inversion

For functions f, g whose Laplace transform exists for $\operatorname{Re} p > c$, we define the **convolution** $f * g$ by

$$(f * g)(x) = \int_0^x f(t) g(x-t)\,\mathrm{d}t \tag{8.171}$$

Theorem 8.3. Convolution Theorem for Laplace Transform

The Laplace transform of the convolution of f and g is the product of the Laplace transform of f and g:

$$\mathscr{L}[f * g] = \mathscr{L}[f] \cdot \mathscr{L}[g] \tag{8.172}$$

Theorem 8.4. Injectivity of Laplace Transform

Suppose that $f : [0, \infty) \to \mathbb{C}$ is continuous and bounded by $M\mathrm{e}^{cx}$. If $\overline{f}(p) = 0$ for $\operatorname{Re}(p) > c$, then $f = 0$.

Theorem 8.5. Inversion Theorem for Laplace Transform

Suppose that $f : (0, \infty) \to \mathbb{C}$ is piecewise smooth and $\overline{f}(p)$ exists for $\operatorname{Re}(p) > c$. Then for $x > 0$,

$$f(x) = \frac{1}{2\pi\mathrm{i}} \int_{\sigma - \mathrm{i}\infty}^{\sigma + \mathrm{i}\infty} \overline{f}(p)\,\mathrm{e}^{px}\,\mathrm{d}p \tag{8.173}$$

for some $\sigma > c$.

From the inversion theorem we can deduce the **term-by-term Laplace inversion**: Let $f : [0, \infty) \to \mathbb{C}$ be a function that is analytic near $x = 0$. Assume that

$$\overline{f}(p) = \sum_{n=0}^{\infty} \frac{a_n}{p^{n+1}} \qquad \text{for } \operatorname{Re}(p) > c \geqslant 0 \tag{8.174}$$

Then

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \tag{8.175}$$

The common techniques for Laplace inversion are summarised below:

1. Use the properties and common results in Section 8.1 and 8.2, and the convolution theorem.
2. Directly compute the contour integral in the inversion theorem;
3. Expand the function into Laurent series and use the term-by-term Laplace inversion.

For a linear ODE for $y(x)$ with $y(0)$ specified, we can apply the Laplace transform to the equation. If the coefficients of $y(x)$ are constants, then the transformed problem is an algebraic equation of $\overline{y}(p)$; if the coefficients of $y(x)$ are polynomials in x, then he transformed problem is an ODE of $\overline{y}(p)$. We can solve $\overline{y}(p)$ and apply the inverse Laplace transform to find the solution $y(x)$.

9 Fourier Transform

9.1 Basic Properties, Examples, Convolution and Inversion

Let $f \in L^1(\mathbb{R})$. We define the **Fourier transform** $\hat{f}(s)$ of $f(x)$ to be

$$\hat{f}(s) = \int_{-\infty}^{+\infty} f(x)\,\mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}x \tag{9.176}$$

$\hat{f}$ is also denoted by $\mathscr{F}[f]$, where $\mathscr{F}$ is an integral operator that represents Fourier transform.

Remark. There are many other definitions of Fourier transform of f in various texts, like

$$\int_{-\infty}^{+\infty} f(x)\,\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}x, \text{ or } \qquad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\,\mathrm{e}^{-\mathrm{i}sx}\,\mathrm{d}x, \text{ or } \qquad \int_{-\infty}^{+\infty} f(x)\,\mathrm{e}^{-2\pi\mathrm{i}sx}\,\mathrm{d}x \tag{9.177}$$

Theorem 9.1. Riemann-Lebesgue Lemma

Suppose that $f \in L^1(\mathbb{R})$. Then $\hat{f}(s) \to 0$ as $|\mathrm{Re}(s)| \to \infty$.

Some basic properties of Fourier transform are summarised as follows: (assume all functions in $L^1(\mathbb{R})$, and sufficiently smooth if necessary)

1. $\mathscr{F}$ is linear: $\mathscr{F}[af(x)+bg(x)] = a\mathscr{F}[f(x)] + b\mathscr{F}[g(x)]$ for $a, b \in \mathbb{C}$;
2. $\mathscr{F}[f(x/a)] = a\hat{f}(sa)$ for $a > 0$;
3. $\mathscr{F}\left[\mathrm{e}^{-\mathrm{i}sa} f(x)\right] = \hat{f}(s+a)$ for $a \in \mathbb{R}$;
4. $\mathscr{F}\left[f^n(x)\right] = (\mathrm{i}s)^n \hat{f}(s)$;
5. $\mathscr{F}\left[x^n f(x)\right] = \mathrm{i}^n \dfrac{\mathrm{d}^n \hat{f}}{\mathrm{d}s^n}$.

Summary of Fourier transforms of some common functions:

1. Indicator function: $\mathscr{F}\left[\chi_{[-1,1]}\right] = \dfrac{2\sin s}{s}$;
2. δ-function: $\mathscr{F}[\delta(x-a)] = \mathrm{e}^{-\mathrm{i}sa}$, where $a \in \mathbb{R}$;
3. Exponential function: $\mathscr{F}\left[\mathrm{e}^{-a|x|}\right] = \dfrac{2a}{a^2+s^2}$, where $a > 0$;
4. Gaussian function: $\mathscr{F}\left[\mathrm{e}^{-a^2x^2}\right] = \dfrac{\sqrt{\pi}}{a}\exp\left(-\dfrac{s^2}{4a^2}\right)$, where $a > 0$;
5. Constant function: $\mathscr{F}[1] = 2\pi\delta(s)$.

Note that the $\mathscr{F}[1] = 2\pi\delta(s)$ should only be understood in the sense of distributions, in which we have to work on a larger class of test function. This is beyond the scope of this course.

Theorem 9.2. Convolution Theorem for Fourier Transform

For $f, g \in L^1(\mathbb{R})$, we define the **convolution** $f \star g$ (*note the differnce in the limits!*) to be

$$(f \star g)(x) = \int_{-\infty}^{+\infty} f(t)g(x-t)\,\mathrm{d}t \tag{9.178}$$

Then $f \star g \in L^1(\mathbb{R})$. In particular, the Fourier transform of the convolution of f and g is the product of the Fourier transform of f and g:

$$\mathscr{F}\left[f \star g\right] = \mathscr{F}\left[f\right] \cdot \mathscr{F}\left[g\right] \tag{9.179}$$

Theorem 9.3. Inversion Theorem for Fourier Transform

Suppose that $f : \mathbb{R} \to \mathbb{C}$ is continuous and $f(x), \hat{f}(s) \in L^1(\mathbb{R})$. Then

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(s)\,\mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}s \tag{9.180}$$

9.2 Fourier Transform in $L^2(\mathbb{R})$

Theorem 9.4. Parseval–Plancherel identity

For $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we have

$$\langle \hat{f}, \hat{g} \rangle := \int_{-\infty}^{+\infty} \hat{f}(s)\overline{\hat{g}(s)}\,\mathrm{d}s = 2\pi \int_{-\infty}^{+\infty} f(x)\overline{g(x)}\,\mathrm{d}x = 2\pi\,\langle f, g \rangle \tag{9.181}$$

As a corollary, we have $\left\|\mathscr{F}[f]\right\|_2 = \sqrt{2\pi}\left\|f\right\|_2$ for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. A slightly modified definition of the Fourier Transform can be extended to a **unitary operator** on the inner product space $L^2(\mathbb{R})$.

9.3 Application to PDEs

We consider the **Dirichlet problem on the upper half plane**:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad x \in \mathbb{R},\ y > 0 \tag{9.182}$$

with the boundary conditions

$$u(x,0) = f(x) \text{ for some } f \in L^1(\mathbb{R}) \qquad u(x,y) \text{ is bounded as } x^2 + y^2 \to \infty \tag{9.183}$$

We apply Fourier transform to the x-variable. The equation transforms into

$$\hat{u}_{yy}(s,y) = s^2 \hat{u}(s,y) \tag{9.184}$$

with the transformed boundary condition

$$\hat{u}(s,0) = \hat{f}(s) \tag{9.185}$$

Equation (9.184) is a ODE. The solution is

$$\hat{u}(s,y) = f(s)\,\mathrm{e}^{-y|s|} \tag{9.186}$$

Using the convolution theorem and the known inverse of $\mathrm{e}^{-y|s|}$, we can write the solution as

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{f(s)}{y^2 + (x-s)^2}\,\mathrm{d}s \tag{9.187}$$

which is known as the **Poisson's solution**.

10 Calculus of Variations

10.1 Euler-Lagrange Equation

We consider the general problem of finding the function $y(x)$ which gives a stationary value to the functional

$$I[y] = \int_a^b F(x, y, y')\,\mathrm{d}x \tag{10.188}$$

For this problem, we consider a subset of test functions, called **bump functions** $\eta(x)$ on $[a,b]$, which satisfy:

1. $\eta(x) = 0$ unless $x \in [a,b]$;
2. $0 < \eta(x) \leqslant 1$ for $x \in (a,b)$.

Equation (7.145) with $C = 1$ is an example of a bump function.

The following theorem is a generalisation of Theorem 7.1:

Theorem 10.1

Suppose that $f : [a,b] \to \mathbb{R}$ is continuous and c_1, c_2 are constants such that

$$c_1\eta(a) + c_2\eta(b) + \int_a^b f(x)\eta(x)\,\mathrm{d}x = 0 \tag{10.189}$$

for all bump function $\eta(x)$ on $[a,b]$. Then $f = 0$.

Let $y : [a,b] \to \mathbb{R}$ be a minimiser of the functional $I[y]$ (either subject to the constraints $y(a) = c_1$ and $y(b) = c_2$, or with no constraints at all) and η is a bump function on $[a,b]$. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} I[y + \alpha\eta]\bigg|_{\alpha=0} = 0 \tag{10.190}$$

from which we can derive the **Euler-Lagrange equation** for $y(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} = 0 \tag{10.191}$$

If either $y(a)$ or $y(b)$ (or both) are not fixed, then $y(x)$ is subject to the **natural boundary conditions** $\left.\frac{\partial F}{\partial y'}\right|_{x=a} = 0$ or $\left.\frac{\partial F}{\partial y'}\right|_{x=b} = 0$ (or both) respectively.

Suppose that $\frac{\partial F}{\partial y} = 0$. Then y is called an **ignorable coordinate** of F, in which case we have

$$\frac{\partial F}{\partial y'} = \text{const} \tag{10.192}$$

Suppose that $\frac{\partial F}{\partial x} = 0$. Then we have the **Beltrami's identity**:

$$\mathscr{H} := y' \frac{\partial F}{\partial y'} - F = \text{const} \tag{10.193}$$

10.2 Application to Mechanics

We generalise the problem to several dependent variables $q_1(t), \dots, q_n(t)$. We consider the stationary values of the functional

$$S[\boldsymbol{q}] := \int_a^b \mathscr{L}(t; q_1, \dots, q_n; \dot{q}_1, \dots, \dot{q}_n)\, \mathrm{d}t \tag{10.194}$$

In mechanics, $q_1, \dots, q_n$ are **generalised coordinates** which are functions of time t, $S[\boldsymbol{q}]$ is called the **action**, and $\mathscr{L}(t; q_1, \dots, q_n; \dot{q}_1, \dots, \dot{q}_n)$ is called the **Lagrangian**. We have the famous **Principle of Least Action**:

Principle 10.2. Principle of Least Action

Suppose that a mechanical system is subject to **holonomic**, **workless** constraints, and all forces are **conservative**. Then the motion of the system $(q_1(t), \dots, q_n(t))$ extremise the action

$$S[\boldsymbol{q}] = \int_a^b \mathscr{L}(t; q_1, \dots, q_n; \dot{q}_1, \dots, \dot{q}_n)\, \mathrm{d}t \tag{10.195}$$

The principle of least action serves as the fundamental postulate of physics. In classical mechanics, the Lagrangian $\mathscr{L} = T - V$, where T and V are the kinetic and potential energy of the mechanical system.

The extremisers $q_1(t), \dots, q_n(t)$ satisfy the **Euler-Lagrange equations**:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{q}_i} - \frac{\partial \mathscr{L}}{\partial q_i} = 0, \quad i \in \{1, \dots, n\} \tag{10.196}$$

with either the fixed boundary conditions $q_i(a) = c_i$, $q_i(b) = d_i$, or the natural boundary conditions $\left.\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right|_{t=a} = \left.\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right|_{t=b} = 0$.

We define the **generalised momenta** $p_i := \frac{\partial \mathscr{L}}{\partial \dot{q}_i}$. The ignorable coordinates implies the conservation of generalised momenta:

$$\frac{\partial \mathscr{L}}{\partial q_i} = 0 \quad \Longrightarrow \quad p_i = \text{const}, \quad i \in \{1, \dots, n\} \tag{10.197}$$

We define the **Hamiltonian**

$$\mathscr{H} := \sum_{i=1}^{n} p_i \dot{q}_i - \mathscr{L} = \sum_{i=1}^{n} \dot{q}_i \frac{\partial \mathscr{L}}{\partial \dot{q}_i} - \mathscr{L} \tag{10.198}$$

The Beltrami's identity becomes

$$\frac{\partial \mathscr{L}}{\partial t} = 0 \quad \Longrightarrow \quad \mathscr{H} = \text{const} \tag{10.199}$$

In other words, the time-independent Lagrangian implies the conservation of Hamiltonian.

The above theorems hint a more general connection between continuous symmetries and conserved quantities:

Theorem 10.3. Noether's Theorem

We say that the function $\boldsymbol{\rho}(t; \boldsymbol{q}; \dot{\boldsymbol{q}})$ generates a **infinitesimal symmetry** of the Lagrangian $\mathscr{L}$, if there exists a (sufficiently smooth) function $f(t; \boldsymbol{q}; \dot{\boldsymbol{q}})$ such that

$$\left.\frac{\partial}{\partial \alpha}\mathscr{L}(t; \boldsymbol{q} + \alpha\boldsymbol{\rho}; \dot{\boldsymbol{q}} + \alpha\dot{\boldsymbol{\rho}})\right|_{\alpha=0} = \frac{\mathrm{d}}{\mathrm{d}t} f(t; \boldsymbol{q}(t); \dot{\boldsymbol{q}}(t)) \tag{10.200}$$

for all (sufficiently smooth) $\boldsymbol{q}(t)$.

Then $\boldsymbol{\rho}$ induces the **conserved quantity**:

$$F = \sum_{i=1}^{n} \rho_i \frac{\partial \mathscr{L}}{\partial \dot{q}_i} - f = \text{const} \tag{10.201}$$

10.3 Generalisations

We generalise the problem to several independent variables $y = y(x_1, ..., x_n)$. We consider the stationary values of the functional

$$I[y] = \int_R F(x_1, ..., x_n; y; y_1, ..., y_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n \tag{10.202}$$

where $R \subseteq \mathbb{R}^n$ is a region and $y_i := \dfrac{\partial y}{\partial x_i}$ for $i \in \{1, ..., n\}$.

The minimisers of $I[y]$ with fixed boundary conditions satisfy the Euler-Lagrange equation:

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial F}{\partial y_i} - \frac{\partial F}{\partial y} = 0 \tag{10.203}$$

We can also generalise the problem to include higher derivatives. We consider the stationary values of the functional

$$I[y] = \int_a^b F(x; y, y', ..., y^{(n)})\, \mathrm{d}x \tag{10.204}$$

The minimisers of $I[y]$ satisfy the Euler-Lagrange equation:

$$\sum_{k=0}^{n} (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k} \frac{\partial F}{\partial y^{(k)}} = 0 \tag{10.205}$$

If $n = 2$, the natural boundary conditions for this problem are

$$\frac{\partial F}{\partial y'} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial F}{\partial y''} = 0 \qquad\qquad \frac{\partial F}{\partial y''} = 0 \tag{10.206}$$

when evaluted at $x = a$ and $x = b$.

10.4 Integral Constraints

We consider the problem of finding the stationary value of the functional

$$I[y] = \int_a^b F(x, y, y')\, \mathrm{d}x \tag{10.207}$$

subject to an integral constraint

$$J[y] = \int_a^b G(x, y, y')\, \mathrm{d}x = C \tag{10.208}$$

We can introduce a **Lagrange multiplier** λ so that the minimisers $y(x)$ satisfy the Euler-Lagrange equation for $F - \lambda G$:

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial}{\partial y'} (F - \lambda G) \right) - \frac{\partial}{\partial y} (F - \lambda G) = 0 \tag{10.209}$$

Note that in this case the natural boundary conditions will be

$$\frac{\partial}{\partial y'} (F - \lambda G) = 0 \qquad \text{at } x = a \text{ and } x = b \tag{10.210}$$

10.5 Application to Sturm-Liouville Theory

We notice that the Sturm-Liouville equation

$$-(p(x) y'(x))' + q(x) y(x) = \lambda r(x) y(x) \qquad x \in [a, b] \tag{10.211}$$

is the Euler-Lagrange equation for the variational problem of finding stationary values of

$$I[y] = \int_a^b \left(p(x)y'(x)^2 + q(x)y(x)^2\right) \mathrm{d}x \tag{10.212}$$

subject to

$$J[y] = \int_a^b r(x)y(x)^2 \,\mathrm{d}x = \text{const} \tag{10.213}$$

The corresponding fixed or natural boundary conditions are given respectively by

$$y = 0 \qquad\qquad py' = 0 \tag{10.214}$$

evaluted at $x = a$ and $x = b$.

In particular, an eigenvalue λ_n and its corresponding eigenfunction y_n satisfy that

$$\lambda_n = \frac{I[y_n]}{J[y_n]} = \frac{\displaystyle\int_a^b \left(p(x)y'(x)^2 + q(x)y(x)^2\right) \mathrm{d}x}{\displaystyle\int_a^b r(x)y(x)^2 \,\mathrm{d}x} \tag{10.215}$$

from which we may recover some standard results in the Sturm-Liouville theory, for example, the orthogonality relation:

$$\int_a^b r(x)y_n(x)y_m(x) = 0 \qquad \text{for } m \neq n \tag{10.216}$$

in the eigenfunction expansion

$$y(x) = \sum_i c_i y_i(x) \tag{10.217}$$

An interesting application is the **Rayleigh-Ritz approximation** or **variational principle**. Suppose that we have a Sturm-Liouville problem

$$\mathscr{L}y(x) := -(p(x)y'(x))' + q(x)y(x) = \lambda r(x)y(x) \tag{10.218}$$

and we would like to obtain an **upper bound** its **smallest eigenvalue** λ_0. We have

$$\lambda_0 = \frac{\langle \mathscr{L}y_0, y_0\rangle_r}{\langle y_0, y_0\rangle_r} \leqslant \frac{\langle \mathscr{L}y, y\rangle_r}{\langle y, y\rangle_r} = \frac{\displaystyle\int_a^b \left(p(x)y'(x)^2 + q(x)y(x)^2\right) \mathrm{d}x}{\displaystyle\int_a^b r(x)y(x)^2 \,\mathrm{d}x} \tag{10.219}$$

for all (sufficiently smooth) $y(x)$ that satisfy the same boundary conditions.

The method ia particularly useful in estimating the upper bound of the ground state energy for a given quantum system.

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