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Problem Sheet 2

String Theory I

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1 Charge algebras for the classical string

In Hamiltonian mechanics, conserved charges are represented by functions on phase space that Poisson-commute with the Hamiltonian, and the action of the symmetry on an observable is implemented by the Poisson bracket. In this exercise, you will calculate the algebra of charges for spacetime Poincaré symmetry and worldsheet conformal symmetry of the closed string, written in oscillator coordinates on phase space, whose Poisson brackets take the form

$$\{\alpha^\mu_m, \alpha^\nu_n\}_{\text{P.B.}} = \{\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n\}_{\text{P.B.}} = im\eta^{\mu\nu}\delta_{m+n,0}, \quad \{p^\mu, x^\nu\}_{\text{P.B.}} = \eta^{\mu\nu}.$$

Question 1.1

Spacetime Poincaré symmetry charges for the closed string are written in terms of oscillator coordinates as follows:

$$P^\mu = p^\mu,$$
$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i\sum_{n=1}^{\infty}\frac{\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n}{n} - i\sum_{n=1}^{\infty}\frac{\tilde{\alpha}^\mu_{-n}\tilde{\alpha}^\nu_n - \tilde{\alpha}^\nu_{-n}\tilde{\alpha}^\mu_n}{n}.$$

Compute the Lie algebra of these charges.

Proof. Without ambiguity, the Poisson bracket shall be denoted by $\{-,-\}$. First we note that $p^\mu = \dfrac{2}{\ell}\alpha^\mu_0 = \dfrac{2}{\ell}\tilde{\alpha}^\mu_0$. So $\{p^\mu, \alpha^\nu_m\} = \{p^\mu, \tilde{\alpha}^\nu_m\} = 0$ for $m \neq 0$.

$$\{P^\mu, P^\nu\} = \{p^\mu, p^\nu\} = 0; \quad ✓$$
$$\begin{aligned}\left\{M^{\mu\nu}, P^\lambda\right\} &= \left\{x^\mu p^\nu, p^\lambda\right\} - \left\{x^\nu p^\mu, p^\lambda\right\} = -p^\nu\eta^{\mu\lambda} + p^\mu\eta^{\nu\lambda}\\ &= \eta^{\nu\lambda}P^\mu - \eta^{\mu\lambda}P^\nu. \quad ✓\end{aligned}$$

Before computing $\{M^{\mu\nu}, M^{\rho\sigma}\}$, we need

$$\begin{aligned}\{\alpha^\mu_{-n}\alpha^\nu_n, \alpha^\rho_{-m}\alpha^\sigma_m\} &= \alpha^\mu_{-n}\alpha^\sigma_m\{\alpha^\nu_n, \alpha^\rho_{-m}\} + \alpha^\nu_n\alpha^\sigma_{-m}\{\alpha^\mu_{-n}, \alpha^\sigma_m\}\\ &= \mathrm{i}m\delta_{m-n,0}\left(\alpha^\mu_{-n}\alpha^\sigma_m\eta^{\nu\rho} - \alpha^\rho_{-m}\alpha^\nu_n\eta^{\mu\sigma}\right).\end{aligned}$$

Therefore:

$$\begin{aligned}&\left\{\sum_{n=1}^{\infty}\frac{\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n}{n}, \sum_{m=1}^{\infty}\frac{\alpha^\rho_{-m}\alpha^\sigma_m - \alpha^\sigma_{-m}\alpha^\rho_m}{m}\right\}\\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}\left\{\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n, \alpha^\rho_{-m}\alpha^\sigma_m - \alpha^\sigma_{-m}\alpha^\rho_m\right\}\\ &= \mathrm{i}\sum_{n=1}^{\infty}\frac{1}{n}((\alpha^\mu_{-n}\alpha^\sigma_n\eta^{\nu\rho} - \alpha^\nu_n\alpha^\rho_{-n}\eta^{\mu\sigma}) + (\alpha^\nu_{-n}\alpha^\rho_n\eta^{\mu\sigma} - \alpha^\mu_n\alpha^\sigma_{-n}\eta^{\nu\rho})\\ &\quad - (\alpha^\nu_{-n}\alpha^\sigma_n\eta^{\mu\rho} - \alpha^\mu_n\alpha^\rho_{-n}\eta^{\nu\sigma}) - (\alpha^\mu_{-n}\alpha^\rho_n\eta^{\nu\sigma} - \alpha^\nu_n\alpha^\sigma_{-n}\eta^{\mu\rho}))\\ &= \mathrm{i}\eta^{\nu\rho}\sum_{n=1}^{\infty}\frac{\alpha^\mu_{-n}\alpha^\sigma_n - \alpha^\sigma_{-n}\alpha^\mu_n}{n} + \mathrm{i}\eta^{\mu\sigma}\sum_{n=1}^{\infty}\frac{\alpha^\nu_{-n}\alpha^\rho_n - \alpha^\rho_{-n}\alpha^\nu_n}{n}\\ &\quad - \mathrm{i}\eta^{\mu\rho}\sum_{n=1}^{\infty}\frac{\alpha^\nu_{-n}\alpha^\sigma_n - \alpha^\sigma_{-n}\alpha^\nu_n}{n} - \mathrm{i}\eta^{\nu\sigma}\sum_{n=1}^{\infty}\frac{\alpha^\mu_{-n}\alpha^\rho_n - \alpha^\rho_{-n}\alpha^\mu_n}{n}.\end{aligned}$$

Let $E^{\mu\nu} := \sum_{n=1}^{\infty} \frac{\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n}{n}$ and $\widetilde{E}^{\mu\nu} := \sum_{n=1}^{\infty} \frac{\widetilde{\alpha}^\mu_{-n}\widetilde{\alpha}^\nu_n - \widetilde{\alpha}^\nu_{-n}\widetilde{\alpha}^\mu_n}{n}$. Similarly for x^μ and p^μ,

$$\begin{aligned}
&\{x^\mu p^\nu - x^\nu p^\mu, x^\rho p^\sigma - x^\sigma p^\rho\} \\
&= (x^\mu p^\sigma \eta^{\nu\rho} - x^\rho p^\nu \eta^{\mu\sigma}) + (x^\nu p^\rho \eta^{\mu\sigma} - x^\sigma p^\mu \eta^{\nu\rho}) - (x^\nu p^\sigma \eta^{\mu\rho} - x^\rho p^\mu \eta^{\nu\sigma}) - (x^\mu p^\rho \eta^{\nu\sigma} - x^\sigma p^\nu \eta^{\mu\rho}) \\
&= \eta^{\nu\rho}(x^\mu p^\sigma - x^\sigma p^\mu) + \eta^{\mu\sigma}(x^\nu p^\rho - x^\rho p^\nu) - \eta^{\mu\rho}(x^\nu p^\sigma - x^\sigma p^\nu) - \eta^{\nu\sigma}(x^\mu p^\rho - x^\rho p^\mu).
\end{aligned}$$

Finally,

$$\begin{aligned}
\{M^{\mu\nu}, M^{\rho\sigma}\} &= \{x^\mu p^\nu - x^\nu p^\mu, x^\rho p^\sigma - x^\sigma p^\rho\} - \{E^{\mu\nu}, E^{\rho\sigma}\} - \left\{\widetilde{E}^{\mu\nu}, \widetilde{E}^{\rho\sigma}\right\} \\
&= \eta^{\nu\rho} M^{\mu\sigma} + \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\nu\sigma} M^{\mu\rho}.
\end{aligned}$$

The results coincide with the familiar Poincaré algebra. □

Question 1.2

Derive the expression for the conserved charges L_m and $\widetilde{L}_m$ used to impose the stress tensor constraints on the string phase space. Verify that their Lie algebra is the Witt algebra,

$$[L_m, L_n]_{\text{P.B.}} = i(m-n)L_{m+n}.$$

Show that the transformation of the oscillators under the action of these charges is given by

$$\begin{aligned}
\{L_m, \alpha^\mu_n\}_{\text{P.B.}} &= -in\alpha^\mu_{m+n}. \\
\left\{\widetilde{L}_m, \widetilde{\alpha}^\mu_n\right\}_{\text{P.B.}} &= -in\widetilde{\alpha}^\mu_{m+n}. \\
\{L_m, \widetilde{\alpha}^\mu_n\}_{\text{P.B.}} &= \left\{\widetilde{L}_m, \alpha^\mu_n\right\}_{\text{P.B.}} = 0.
\end{aligned}$$

Thus deduce the action of $\widetilde{L}_m$ and L_m on the space-time coordinate fields $X^\mu(\tau,\sigma)$. Confirm that your results agree with the action of the vector fields generating worldsheet conformal transformations, $V^\pm_m = -\frac{1}{2}e^{2im\sigma^\pm}\partial_\pm$.[a]

[a] Recall that the vector field $\xi^a(x)\partial_a$ generates a diffeomorphism $x^a \to \widetilde{x}^a = x^a + \xi^a(x)$.

Proof. The conserved charges are given by

$$L_m = \frac{1}{2}\sum_{k\in\mathbb{Z}} \alpha_{m-k}\cdot\alpha_k, \qquad \widetilde{L}_m = \frac{1}{2}\sum_{k\in\mathbb{Z}} \widetilde{\alpha}_{m-k}\cdot\widetilde{\alpha}_k$$

The derivation is given in the lecture notes. You should have derived them here.

For the computation of the Poisson bracket, we refer to Question 2.2, but different from the quantum case, we have the simple formula

$$\{\alpha_{m-k}\cdot\alpha_k, \alpha_{n-\ell}\cdot\alpha_\ell\} = \mathrm{i}k\alpha_{m-k}\cdot\alpha_{n+k}\,(\delta_{k+\ell,0} + \delta_{k+n-\ell,0}) + \mathrm{i}(m-k)\alpha_{m+n-k}\cdot\alpha_k(\delta_{m-k+\ell,0} + \delta_{m+n-k-\ell,0}).$$

which gives $\{L_m, L_n\} = \mathrm{i}(m-n)L_{m+n}$ without the annoying term involving $\delta_{m+n,0}$.

The action of the Hamiltonian vector field X_{L_m} on the charges is given by

$$X_{L_m}(\alpha^\mu_n) = \{L_m, \alpha^\mu_n\} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \eta_{\nu\lambda}\left\{\alpha^\nu_{m-k}\alpha^\lambda_k, \alpha^\mu_n\right\}$$

$$= \frac{\mathrm{i}}{2} \sum_{k \in \mathbb{Z}} \eta_{\nu\lambda} \left(\alpha^{\nu}_{m-k} k \delta_{n+k,0} \eta^{\mu\lambda} + \alpha^{\lambda}_{k} (m-k) \delta_{m+n-k,0} \eta^{\mu\nu} \right)$$

$$= -\mathrm{i} n \alpha^{\mu}_{m+n}.$$

Since α_m commutes with $\widetilde{\alpha}_m$, we have $X_{L_m}(\widetilde{\alpha}^{\mu}_n) = 0$ trivially.

The spacetime coordinate fields are given by

$$X^{\mu}(\sigma^+, \sigma^-) = X^{\mu}_L(\sigma^+) + X^{\mu}_R(\sigma^-),$$

where

$$X^{\mu}_L(\sigma^+) = \frac{1}{2} x^{\mu} + \frac{1}{2} \ell^2 p^{\mu} \sigma^+ + \frac{\mathrm{i}}{2} \ell \sum_{n \neq 0} \frac{1}{n} \widetilde{\alpha}^{\mu}_n \, \mathrm{e}^{-2\mathrm{i} n \sigma^+}$$

$$X^{\mu}_R(\sigma^-) = \frac{1}{2} x^{\mu} + \frac{1}{2} \ell^2 p^{\mu} \sigma^- + \frac{\mathrm{i}}{2} \ell \sum_{n \neq 0} \frac{1}{n} \alpha^{\mu}_n \, \mathrm{e}^{-2\mathrm{i} n \sigma^-}$$

Therefore

$$X_{L_m}(X^{\mu}) = X_{L_m}(X^{\mu}_R) = \frac{\ell^2}{2} X_{L_m}(p^{\mu}) \sigma^+ + \frac{\mathrm{i}\ell}{2} \sum_{n \neq 0} \frac{1}{n} X_{L_m}(\alpha^{\mu}_n) \, \mathrm{e}^{-2\mathrm{i} n \sigma^-}$$

$$= \frac{\ell}{2} X_{L_m}(\alpha^{\mu}_0) \sigma^+ + \frac{\mathrm{i}\ell}{2} \sum_{n \neq 0} \frac{1}{n} X_{L_m}(\alpha^{\mu}_n) \, \mathrm{e}^{-2\mathrm{i} n \sigma^-}$$

$$= 0 + \frac{\ell}{2} \sum_{k \neq 0} \alpha^{\mu}_{m+n} \, \mathrm{e}^{-2\mathrm{i} n \sigma^-}$$

$$= \frac{\ell}{2} \sum_{n \neq m} \alpha^{\mu}_n \, \mathrm{e}^{-2\mathrm{i}(n-m)\sigma^-}$$

For the vector field $V^-_m = -\frac{1}{2} \mathrm{e}^{2\mathrm{i} m \sigma^-} \partial_-$, we find that

$$V^-_m(X^{\mu}) = V^-_m(X^{\mu}_R) = -\frac{1}{2} \mathrm{e}^{2\mathrm{i} m \sigma^-} \left(\frac{\ell}{2} p^{\mu} + \frac{\mathrm{i}\ell}{2} \sum_{n \neq 0} \frac{1}{n} \alpha^{\mu}_n (-2\mathrm{i} n) \, \mathrm{e}^{-2\mathrm{i} n \sigma^-} \right)$$

$$= -\frac{1}{2} \mathrm{e}^{2\mathrm{i} m \sigma^-} \left(\ell \sum_{n \in \mathbb{Z}} \alpha^{\mu}_n \, \mathrm{e}^{-2\mathrm{i} n \sigma^-} \right)$$

$$= -\frac{\ell}{2} \sum_{n \in \mathbb{Z}} \alpha^{\mu}_n \, \mathrm{e}^{-2\mathrm{i}(n-m)\sigma^-}$$

There are some discrepancies but supposely we should find that $X_{L_m}(X^{\mu}) = V^-_m(X^{\mu})$ and similarly $X_{\widetilde{L}_m}(X^{\mu}) = V^+_m(X^{\mu})$. □

2 Charge algebras for the quantum string

Upon quantization, Poisson brackets for the oscillators are promoted to commutation relations for corresponding creation and annihilation (and zero-mode) operators on the string Fock space,

$$[\alpha^{\mu}_m, \alpha^{\nu}_n] = [\widetilde{\alpha}^{\mu}_m, \widetilde{\alpha}^{\nu}_n] = m \eta^{\mu\nu} \delta_{m+n,0}, \quad [p^{\mu}, x^{\nu}] = -i \eta^{\mu\nu},$$

where $\alpha^\mu_{-m} = (\alpha^\mu_m)^\dagger$ and similarly for the $\widetilde{\alpha}$'s. In this exercise you will study the Fock space operators corresponding to the conserved charges encountered in our analysis of the classical string.

Question 2.1

Write the generators of spacetime Poincaré symmetry in terms of oscillator creation and annihilation operators. By direct computation or otherwise, show that the commutation relations for these charges are precisely those of the Poincaré algebra computed in problem 1(a).

Proof. The generators of the Poincaré group are the usual ones:

$$P^\mu = p^\mu,$$
$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i\sum_{n=1}^{\infty} \frac{\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n}{n} - i\sum_{n=1}^{\infty} \frac{\widetilde{\alpha}^\mu_{-n}\widetilde{\alpha}^\nu_n - \widetilde{\alpha}^\nu_{-n}\widetilde{\alpha}^\mu_n}{n},$$

but now all operators are defined on some Hilbert space. To show that we have the same Poincaré algebra relation (with an extra factor of i, due to the difference in convention of the Lie brackets), there are two ways:

- We can go through the calculation of Question 1.(a) again with caution that we have kept track of the ordering of the operators correctly. This is indeed the case.
- Alternatively, we may argue that the Poisson brackets must be preserved in the canonical quantisation scheme. More explicitly, canonical quantisation is a map Q from real-valued C^∞-functions on classical phase space T^*M to the set of self-adjoint operators on a Hilbert space $\mathcal{H}$, such that $Q(1) = \mathrm{id}_{\mathcal{H}}$ and $Q(\{f,g\}) = i\hbar[Q(f), Q(g)] + \mathcal{O}(\hbar^2)$[1]. The existence of such quantisation map Q is assumed in physics. Mathematically, we can construct Q through the schemes of either geometric quantisation or deformation quantisation.

  In this method, however, we may never know if we have the correct form of the operators P^μ and $M^{\mu\nu}$, as they depend on the quantisation Q we construct. The choice of quantisation is indeed a problem as already shown in the lectures. □

Question 2.2

In the quantum theory, we define worldsheet conformal generators by

$$L_m = \frac{1}{2}\sum_{k=-\infty}^{\infty} \alpha_{m-k}\cdot\alpha_k$$
$$L_0 = \frac{1}{2}\alpha_0\cdot\alpha_0 + \sum_{k=1}^{\infty}\alpha_{-k}\cdot\alpha_k$$

Argue that the commutation relations of these operators must take the form

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n,0}$$

where $A(m)$ is a c-number (i.e., just a number) depending on m. Now determine the form of $A(m)$, either by brute force, or by arguing as follows:

i) First argue the $A(-m) = -A(m)$.

[1] The explicit dependence on $\hbar$ is necessary as it is proven that Q can never be an isomorphism of Lie algebras.

ii) Now use the Jacobi identity for the commutator algebra to show that for $k+m+n=0$, one has

$$(n-m)A(k)+(k-n)A(m)+(m-k)A(n)=0$$

iii) Therefore deduce that in general the c-number term takes the form

$$A(m)=c_1 m+c_3 m^3$$

where c_1 and c_3 are constant c-numbers.

iv) By evaluating the expectation value of $[L_m, L_{-m}]$ in the oscillator ground state $|0;0\rangle$ for $m=1$ and $m=2$, determine the values of c_1 and c_3.

Proof. We begin with considering the following commutator:

$$\begin{aligned}
&[\alpha_{m-k}\cdot\alpha_k, \alpha_{n-\ell}\cdot\alpha_\ell] \\
&\quad=\eta_{\mu\nu}\eta_{\rho\sigma}[\alpha^\mu_{m-k}\alpha^\nu_k, \alpha^\rho_{n-\ell}\alpha^\sigma_\ell] \\
&\quad=\eta_{\mu\nu}\eta_{\rho\sigma}\left(\alpha^\mu_{m-k}\alpha^\rho_{n-\ell}[\alpha^\nu_k,\alpha^\sigma_\ell]+\alpha^\mu_{m-k}\alpha^\sigma_\ell[\alpha^\nu_k,\alpha^\rho_{n-\ell}]+\alpha^\rho_{n-\ell}\alpha^\nu_k[\alpha^\mu_{m-k},\alpha^\sigma_\ell]+\alpha^\nu_k\alpha^\sigma_\ell[\alpha^\mu_{m-k},\alpha^\rho_{n-\ell}]\right) \\
&\quad=\eta_{\mu\nu}\eta_{\rho\sigma}\left(k\eta^{\nu\sigma}\delta_{k+\ell,0}\alpha^\mu_{m-k}\alpha^\rho_{n-\ell}+k\eta^{\nu\rho}\delta_{k+n-\ell,0}\alpha^\mu_{m-k}\alpha^\sigma_\ell\right. \\
&\qquad\qquad\left.+(m-k)\eta^{\mu\sigma}\delta_{m-k+\ell,0}\alpha^\rho_{n-\ell}\alpha^\nu_k+(m-k)\eta^{\nu\rho}\delta_{m+n-k-\ell,0}\alpha^\nu_k\alpha^\sigma_\ell\right) \\
&\quad=k\alpha_{m-k}\cdot\alpha_{n+k}\left(\delta_{k+\ell,0}+\delta_{k+n-\ell,0}\right)+(m-k)\alpha_{m+n-k}\cdot\alpha_k\delta_{m-k+\ell,0}+(m-k)\alpha_k\cdot\alpha_{m+n-k}\delta_{m+n-k-\ell,0}
\end{aligned}$$

Then the commutator for L_m and L_n is given by

$$\begin{aligned}
[L_m,L_n]&=\frac14\sum_{k\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}}[\alpha_{m-k}\cdot\alpha_k,\alpha_{n-\ell}\cdot\alpha_\ell] \\
&=\frac14\sum_{k\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}}k\alpha_{m-k}\cdot\alpha_{n+k}\left(\delta_{k+\ell,0}+\delta_{k+n-\ell,0}\right)+\frac14\sum_{k\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}}(m-k)\alpha_{m+n-k}\cdot\alpha_k\delta_{m-k+\ell,0} \\
&\quad+\frac14\sum_{k\in\mathbb{Z}}\sum_{\ell\in\mathbb{Z}}(m-k)\alpha_k\cdot\alpha_{m+n-k}\delta_{m+n-k-\ell,0} \\
&=\frac12\sum_{k\in\mathbb{Z}}k\alpha_{m-k}\cdot\alpha_{n+k}+\frac14\sum_{k\in\mathbb{Z}}(m-k)(\alpha_{m+n-k}\cdot\alpha_k+\alpha_k\cdot\alpha_{m+n-k}) \\
&=\frac12\sum_{k\in\mathbb{Z}}k\alpha_{m-k}\cdot\alpha_{n+k}+\frac14\sum_{k\in\mathbb{Z}}(m-k)(2\alpha_{m+n-k}\cdot\alpha_k+\eta_{\mu\nu}[\alpha^\mu_k,\alpha^\nu_{m+n-k}]) \\
&=\frac12\sum_{k\in\mathbb{Z}}(k-n)\alpha_{m+n-k}\cdot\alpha_k+\frac12\sum_{k\in\mathbb{Z}}(m-k)\alpha_{m+n-k}\cdot\alpha_k+\frac14\sum_{k\in\mathbb{Z}}k(m-k)\eta_{\mu\nu}\eta^{\mu\nu}\delta_{m+n,0} \\
&=\frac12(m-n)\sum_{k\in\mathbb{Z}}\alpha_{m+n-k}\cdot\alpha_k+\frac{D}{4}\delta_{m+n,0}\sum_{k\in\mathbb{Z}}k(m-k)
\end{aligned}$$

For $m+n\neq 0$, the second term vanishes and we have $[L_m,L_n]=(m-n)L_{m+n}$. For $m+n=0$, $L_{m+n}=L_0$ is special, in the sense that [2]

$$L_0=\frac12\alpha_0\cdot\alpha_0+\frac12\sum_{k=1}^{\infty}(2\alpha_{-k}\cdot\alpha_k-\eta_{\mu\nu}[\alpha^\mu_k,\alpha^\nu_{-k}])=\frac12\sum_{k\in\mathbb{Z}}\alpha_{-k}\cdot\alpha_k-\frac{D}{2}\sum_{k=1}^{\infty}k=\frac12\sum_{k\in\mathbb{Z}}\alpha_{-k}\cdot\alpha_k+\frac{D}{24}$$

We still have an unsolved "infinity", namely $\sum_{k\in\mathbb{Z}}k(m-k)$. There is no way to preceed from here.

[2] The black magic $\sum_{n=1}^{\infty}n=-1/12$ appears finally...

Alternatively, we take a step back and look at

$$[L_m, L_{-m}] = \frac{1}{2}\sum_{k\in\mathbb{Z}} k\alpha_{m-k}\cdot\alpha_{-m+k} + \frac{1}{4}\sum_{k\in\mathbb{Z}}(m-k)(\alpha_{-k}\cdot\alpha_k + \alpha_k\cdot\alpha_{-k})$$

By symmetry, the sum over $\mathbb{Z}$ can be written as sum over $\mathbb{N}$:

$$\begin{aligned}
[L_m, L_{-m}] &= \frac{1}{2}\sum_{k\in\mathbb{Z}}(m+k)\alpha_{-k}\cdot\alpha_k + \frac{m}{2}\alpha_0\cdot\alpha_0 + \frac{1}{2}\sum_{k=1}^{\infty} m(\alpha_{-k}\cdot\alpha_k + \alpha_k\cdot\alpha_{-k}) \\
&= m\alpha_0\cdot\alpha_0 + \frac{1}{2}\sum_{k=1}^{\infty}(m+k)\alpha_{-k}\cdot\alpha_k + \frac{1}{2}\sum_{k=1}^{\infty}(m-k)\alpha_k\cdot\alpha_{-k} + \frac{1}{2}\sum_{k=1}^{\infty} m(\alpha_{-k}\cdot\alpha_k + \alpha_k\cdot\alpha_{-k}) \\
&= m\left(\alpha_0\cdot\alpha_0 + \sum_{k=1}^{\infty}(\alpha_{-k}\cdot\alpha_k + \alpha_k\cdot\alpha_{-k})\right) + \sum_{k=1}^{\infty} k(\alpha_{-k}\cdot\alpha_k - \alpha_k\cdot\alpha_{-k}) \\
&= 2m\left(\frac{1}{2}\alpha_0\cdot\alpha_0 + \sum_{k=1}^{\infty}\alpha_{-k}\cdot\alpha_k\right) - mD\sum_{k=1}^{\infty} k - D\sum_{k=1}^{\infty} k^2
\end{aligned}$$

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Unfortunately we still have the annoying infinities... □

3 Open string at level two

Question 3.1

Derive the conditions for a state of the form

$$|\psi\rangle = (L_{-2} + \gamma L_{-1}L_{-1})\,|0;p\rangle$$

to be spurious and physical.

Question 3.2

Explain why this is the only form for an additional state (beyond those of the form $L_{-1}\,|\chi_1\rangle$) that one must examine at level two when looking for physical spurious states.

Question 3.3

Construct the reduced Hilbert space at level two for the open string with normal ordering constant $a = 1$. Give a D-dimensional spacetime interpretation of your results.