Summary of Algebraic Curves

Peize Liu
St Peter's College, University of Oxford

July 30, 2021

Contents

Examinable Syllabus

B3.3 Algebraic Curves

- Projective spaces, homogeneous coordinates, projective transformations.
- Algebraic curves in the complex projective plane. Irreducibility, singular and nonsingular points, tangent lines.
- Bezout's Theorem (the proof will not be examined). Points of inflection, and normal form of a nonsingular cubic.
- Nonsingular algebraic curves as Riemann surfaces. Meromorphic functions, divisors, linear equivalence. Differentials and canonical divisors. The group law on a nonsingular cubic.
- The Riemann-Roch Theorem (the proof will not be examined). The geometric genus. Applications.

1 Projective Geometry

This section is a brief review on *ASO. Projective Geometry*. All vector spaces are assumed to be finite-dimensional. $\mathbb{F}$ denotes a field, which is usually $\mathbb{R}$ or $\mathbb{C}$, unless claimed otherwise.

1.1 Projective Spaces and Transformations

Definition 1.1. Projective Spaces

Let V be a vector space. The projective space $\mathbb{P}(V)$ of V is the set of 1-dimensional subspaces of V.

Let $V = \mathbb{F}^{n+1}$ be a finite-dimensional vector space. The projective space $\mathbb{P}(\mathbb{F}^{n+1})$ is usually denoted by $\mathbb{FP}^n$.

Remark. From *A5. Topology* and *B3.5. Topology and Groups*, we know that topologically the real projective space $\mathbb{RP}^n$ is the quotient of S^n by identifying the antipodal points:

$$\mathbb{RP}^n = S^n/\{x \sim -x\}$$

$\mathbb{RP}^n$ is a **connected**, **compact** and **Hausdorff** space, with the fundamental group

$$\pi_1(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}, & n = 1; \\ \mathbb{Z}/2\mathbb{Z}, & n \geqslant 2. \end{cases}$$

Definition 1.2. Homogeneous Coordinates

Let V be a $(n+1)$-dimensional vector space. We fix a basis $\{v_0, ..., v_n\}$ of V. For each $[v] \in \mathbb{P}(V)$, v is called a **representative vector** of $[v]$. We expand v with respect to the basis:

$$v = \sum_{i=0}^{n} x_i v_i$$

The coordinates $[v] = [x_0 : \cdots : x_n]$ is called the homogeneous coordinates. They are unique for a point in $\mathbb{P}(V)$ up ton scaling by a factor:

$$[x_0 : \cdots : x_n] = [\lambda x_0 : \cdots : \lambda x_n], \qquad (\lambda \neq 0)$$

Assume that $\operatorname{char} \mathbb{F} = 0$. Let $U_i := \{[x_0 : \cdots : x_n] \in \mathbb{FP}^n : \ x_i \neq 0\}$ be an open set in $\mathbb{FP}^n$. We note that any point in U_0 has a unique representation $[y_0 : \cdots : 1 : \cdots : y_n]$, which gives a homeomorphism $U_i \cong \mathbb{F}^n$. In addition we note that $\mathbb{FP}^n \setminus U_i \cong \mathbb{FP}^{n-1}$. So we have the decomposition:

$$\mathbb{FP}^n \cong \mathbb{FP}^{n-1} \cup \mathbb{F}^n$$

Remark. When $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, we note that $\mathbb{FP}^n$ has an open cover $\bigcup_{i=0}^n U_i$. We can utilise this to construct $\mathbb{FP}^n$ as an n-dimensional (real/complex) smooth manifold.

Example 1.3. Riemann Sphere

Topologically, $\mathbb{CP}^1 \cong S^2 \cong \mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. This is known as the **Riemann sphere**. See *A2. Complex Analysis* and *B3.2. Geometry of Surfaces* for detail.

Definition 1.4. Projective Linear Subspaces

Let $\mathbb{P}(V)$ be a projective space of V. A projective linear subspace $\mathbb{P}(U)$ is the set of 1-dimensional subspaces of the vector subspace $U \leqslant V$. The dimension of $\mathbb{P}(U)$ is defined to be $\dim U - 1$.

In particular, if $\dim \mathbb{P}(U) = 1$, then $\mathbb{P}(U)$ is called a **projective line**.

Let $\mathbb{P}(U_1)$ and $\mathbb{P}(U_2)$ be two projective subspaces of $\mathbb{P}(V)$. We define the **projective span** of $\mathbb{P}(U_1)$ and $\mathbb{P}(U_2)$ to be

$$\langle \mathbb{P}(U_1), \mathbb{P}(U_2) \rangle := \mathbb{P}(U_1 + U_2)$$

Proposition 1.5. Projective Points and Lines

Let $\mathbb{P}(V)$ be a projective space.

- Through two distinct points in $\mathbb{P}(V)$ there passes a unique projective line;
- Two distinct projective lines in $\mathbb{P}(V)$ intersect in a unique point.

Proof. Trivial by dimension counting. □

Proposition 1.6. Projective Dimension of Intersection

Let $\mathbb{P}(V)$ be a projective space. Let L_1, L_2 be two projective subspaces. Then

$$\dim(L_1 \cap L_2) = \dim L_1 + \dim L_2 - \dim \langle L_1, L_2 \rangle$$

We adopt the convention that $\dim \varnothing = -1$.

Proof. This is the projective version of the vector space dimension formula for intersection:

$$\dim(U_1 \cap U_2) = \dim U_1 + \dim U_2 - \dim(U_1 + U_2)$$ □

Definition 1.7. Projective Transformations

Let $T : V \to W$ be an *invertible* linear map between vector spaces. Then it induces a map $\tau : \mathbb{P}(V) \to \mathbb{P}(W)$ via the following diagram:

$$\begin{array}{ccc} \mathbb{P}(V) & \xrightarrow{\tau} & \mathbb{P}(W) \\ \mathbb{P}\uparrow & & \uparrow\mathbb{P} \\ V & \xrightarrow{T} & W \end{array}$$

That is, $\tau([v]) = [Tv]$. τ is called a projective transformation.

The group of projective transformations in $\mathbb{P}(V)$ is called the **projective general linear group** of V and is denoted by $\mathrm{PGL}(V)$. If $V = \mathbb{F}^n$, then the projective general linear group is denoted by $\mathrm{PGL}(n, \mathbb{F})$.

Example 1.8. Möbius Group

The group of projective transformations on the Riemann sphere is $\mathrm{PGL}(2, \mathbb{C})$, which coincides with the group of **Möbius transformations** Möb:

$$\left\{ T(z) = \frac{az+b}{cz+d} : \ ad - bc \neq 0 \right\}$$

Definition 1.9. General Position

Suppose that $\dim V = n + 1$. The $n + 2$ points $X_0, ..., X_{n+1}$ in $\mathbb{P}(V)$ is said to be in general position, if any subset of $(n + 1)$ of the points is represented by linearly independent representative vectors.

Theorem 1.10. General Position Theorem

Let $\{X_0, ..., X_{n+1}\}$ and $\{Y_0, ..., Y_{n+1}\}$ be two sets of points in the n-dimensional projective space $\mathbb{P}(V)$, which are in general position repsectively. Then there exists a unique projective transformation $\tau : \mathbb{P}(V) \to \mathbb{P}(V)$ such that $\tau(X_i) = Y_i$ for each $i \in \{0, ..., n+1\}$.

Proof. Let $X_i = [v_i]$ for $i = 0, \ldots, n+1$, that is, $v_i \in V$ are representative vectors for X_i. The general position hypothesis implies that $v_0, \ldots, v_n$ form a basis for the vector space V. Then for the last point X_{n+1}, we have

$$v_{n+1} = \sum_{i=0}^{n} \lambda_i v_i$$

for some scalars λ_i. Now, all λ_i are nonzero, again using the general position hypothesis: if one were to be zero, then we would get a dependency relation between v_{n+1} and n of the other v_i. So we may in fact replace v_i by $\lambda_i v_i$ and take

$$v_{n+1} = \sum_{i=0}^{n} v_i$$

as representative vector for our last point. Again using the general position hypothesis, this representation of v_{n+1} is unique.

Similarly we can take $Y_i = [w_i]$ for $i = 0, \ldots, n+1$, with $w_{n+1} = \sum_{i=0}^{n} w_i$ where $w_0, \ldots, w_n$ is another basis of V. Now there exists an invertible linear transformation T of V with $T(v_i) = w_i$ for $i = 0, \ldots, n$. Linearity and the formulae for v_{n+1}, w_{n+1} imply that $T(v_{n+1}) = w_{n+1}$ also, as required.

If S is another linear transformation inducing a projective transformation with the required property, then $Sv_i = \mu_i w_i$ for $i = 0, \ldots, n+1$, where μ_i are nonzero scalars. Now

$$\mu_{n+1} w_{n+1} = S v_{n+1} = \sum_{i=0}^{n} S v_i = \sum_{i=0}^{n} \mu_i w_i$$

so $w_{n+1} = \sum_{i=0}^{n} (\mu_i / \mu_{n+1}) w_i$ and by uniqueness of this representation we see all the μ_i are equal. Hence $S = \mu T$ and they induce the same projective map. ☐

Corollary 1.11. Coordinate Version of General Position Theorem

Let $\{X_0, ..., X_{n+1}\}$ be a set of points in the n-dimensional projective space $\mathbb{P}(V)$, which is in general position. Then there exists a basis of V in which the points are represented by the homogeneous coordinates:

$$X_0 = [1 : 0 : \cdots : 0], \quad X_1 = [0 : 1 : \cdots : 0], \quad \cdots, \quad X_n = [0 : 0 : \cdots : 1], \quad X_{n+1} = [1 : 1 : \cdots : 1]$$

The General Position Theorem is useful in proving the following two classical theorems in projective geometry.

Theorem 1.12. Desargues' Theorem

Let P, A, A', B, B', C, C' be seven distinct points in a projective space such that the lines AA', BB' and CC' are distinct and concurrent at P. Then the points of intersection $AB \cap A'B'$, $BC \cap B'C'$, $CA \cap C'A'$ are collinear.

Proof. As in the proof of the General Position Theorem above, we can choose representative vectors p, a, a', b, b', c, c' for our points such that

$$p = a + a' = b + b' = c + c'$$

Now these equations imply $a - b = b' - a'$, so $a - b$ is a representative vector for $AB \cap A'B'$. Similarly $b - c$ and $c - a$ are representative vectors for $BC \cap B'C'$ and $CA \cap C'A'$ respectively. But $(a - b) + (b - c) + (c - a) = 0$, so these three representative vectors are linearly dependent, hence the points they represent are collinear. ☐

Theorem 1.13. Pappus' Theorem

Let A, B, C and A', B', C' be two pairs of collinear triples of distinct points in a projective plane. Then the three points $BC' \cap B'C$, $CA' \cap C'A$, $AB' \cap A'B$ are collinear.

Proof. Without loss of generality, we can assume that A, B, C', B' are in general position. If not, then two of the three required points coincide, so the conclusion is trivial. By the General Position Theorem, we can then assume that

$$A = [1:0:0], \quad B = [0:1:0], \quad C' = [0:0:1], \quad B' = [1:1:1]$$

The line AB is defined by the 2 -dimensional subspace $\{(x_0, x_1, x_2) \in \mathbb{F}^3 : x_2 = 0\}$, so the point C, which lies on this line, is of the form $C = [1:c:0]$ and $c \neq 0$ since $A \neq C$. Similarly the line $B'C'$ is $x_0 = x_1$, so $A' = [1:1:a]$ with $a \neq 1$.

The line BC' is defined by $x_0 = 0$ and $B'C$ is defined by the span of $(1,1,1)$ and $(1,c,0)$, so the point $BC' \cap B'C$ is represented by the linear combination of $(1,1,1)$ and $(1,c,0)$ for which $x_0 = 0$, i.e.

$$(1,1,1) - (1,c,0) = (0, 1-c, 1)$$

The line $C'A$ is given by $x_1 = 0$, so similarly $CA' \cap C'A$ is represented by

$$(1,c,0) - c(1,1,a) = (1-c, 0, -ca)$$

Finally AB' is given by $x_1 = x_2$, so $AB' \cap A'B$ is

$$(1,1,a) + (a-1)(0,1,0) = (1,a,a)$$

But then

$$(c-1)(1,a,a) + (1-c,0,-ca) + a(0,1-c,1) = 0$$

Thus the three vectors span a 2-dimensional subspace and so the three points lie on a projective line. □

1.2 Plane Curves

From now on we mainly work in $\mathbb{CP}^2$ and occasionally in $\mathbb{RP}^2$. The benefit of working in complex spaces is explained by the following theorem in commutative algebra:

Theorem 1.14. Hilbert's Nullstellensatz

Let $\mathbb{F}$ be an algebraically closed field. Let I be an ideal in $\mathbb{F}[x_1, ..., x_n]$. We define the **algebraic set**

$$\mathcal{V}(I) := \{x \in \mathbb{F}^n : \ \forall f \in I \ (f(x) = 0)\}$$

Then $\mathcal{V}(I) = \mathcal{V}(\sqrt{I})$, where

$$\sqrt{I} := \{f \in \mathbb{F}[x_1, ..., x_n] : \ \exists m \in \mathbb{N} \ (f^m \in I)\}$$

is called the **radical** of I.

Proof. See *B2.2. Commutative Algebra.* □

Corollary 1.15. Polynomial with Repeated Factors

Let $P, Q \in \mathbb{C}[x_1, ..., x_n]$. Then $\mathcal{V}(P) = \mathcal{V}(Q)$ if and only if there exists $m, n \in \mathbb{Z}_+$ such that $P \mid Q^n$ and $Q \mid P^m$; or equivalently, if and only if P and Q have the same irreducible factors (possibly with different multiplicities).

Corollary 1.16

A complex polynomial with no repeated factors is uniquely determined (up to scaling by a constant) by its set of zeros.

Definition 1.17. Homogeneous Polynomials, Plane Projective Curves

A polynomial $P(x, y, z) \in \mathbb{F}[x, y, z]$ is called homogeneous of degree d, if

$$P(\lambda x, \lambda y, \lambda z) = \lambda^d P(x, y, z)$$

for $\lambda \in \mathbb{F}$. It is clear that $\{P(x, y, z) = 0\}$ is a well-defined subset of $\mathbb{FP}^2$.

If $P(x, y, z)$ is homogeneous of degree $d > 0$ and has *no repeated factors*, then $P(x, y, z) = 0$ defines a plane projective curve C in $\mathbb{FP}^2$. The degree of C is d.

Remark. More rigourously, we define the plane projective curve to be an *equivalent class* of polynomials, with $P \sim Q$ if and only if P and Q have the same repeated factors (so that their sets of zeros are the same in the complex projective plane).

Remark. By definition, a plane projective curve is a closed subset of $\mathbb{FP}^2$, and hence is compact and Hausdorff. For $\mathbb{F} = \mathbb{C}$, such plane curves are in fact **Riemann surfaces**.

Definition 1.18. Irreducibility, Components

A projective plane curve C defined by $P(x, y, z) = 0$ is said to be irreducible, if P is irreducible.

An irreducible curve D defined by $Q(x, y, z) = 0$ is said to be a component of C, if Q divides P.

Definition 1.19. Singular Points

The point $[a : b : c] \in \mathbb{FP}^2$ is called a singular point of C, if

$$\frac{\partial P}{\partial x}(a, b, c) = \frac{\partial P}{\partial y}(a, b, c) = \frac{\partial P}{\partial z}(a, b, c) = 0$$

C is said to be non-singular if it has no singular points.

Remark. Since P is homogeneous of degree d, differentiating with respect to λ we obtain the **Euler's relation**:

$$x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} + z\frac{\partial P}{\partial z} = dP$$

So the vanishing of the partial derivatives actually implies the vanishing of P and hence $[a : b : c] \in C$.

A polynomial $p(x, y) \in \mathbb{F}[x, y]$ defines a affine plane curve C in $\mathbb{F}^2$ by $p(x, y) = 0$.

- For a projective curve C defined by $P(x, y, z) = 0$, we can identify $\mathbb{F}^2$ with U_2, and hence $C \cap U_2$ is an affine curve defined by $P(x, y, 1) = 0$;
- Conversely, for an affine curve defined $p(x, y) = 0$, it extends to a projective curve defined by $P(x, y, z) = z^d p\left(\frac{x}{z}, \frac{y}{z}\right)$ for sufficiently large $d > 0$.

Lemma 1.20

Suppose that $P(x, y) \in \mathbb{C}[x, y]$ is homogeneous of degree d. Then P can be factorised into a product of linear polynomials:

$$P(x, y) = \prod_{i=1}^{d} (\alpha_i x + \beta_i y)$$

Proof. Suppose that $P(x, y)$ is a homogeneous polynomial of degree n. Then there exists $a_0, ..., a_n \in \mathbb{C}$ such that

$$P(x, y) = \sum_{i=0}^{n} a_i x^i y^{n-i}$$

Let m be the largest integer such that $a_m \neq 0$. Let $Q(x) = \sum_{i=0}^{n} a_i x^i$. By the fundamental theorem of algebra, Q factorises into linear factors: $Q(x) = a_m \prod_{i=1}^{m}(x - \lambda_i)$. For $y \neq 0$, we have

$$P(x,y) = y^n \sum_{i=0}^{m} a_i \left(\frac{x}{y}\right)^i = y^n Q(x/y) = a_m y^n \prod_{i=1}^{m}\left(\frac{x}{y} - \lambda_i\right) = a_m y^{n-m} \prod_{i=1}^{m}(x - \lambda_i y)$$

If $m < n$, then both sides of the equation is zero when $y = 0$; if $m = n$, then

$$P(x,0) = a_n x^n = a_m y^{n-m} \prod_{i=1}^{m}(x - \lambda_i y)\Big|_{y=0}$$

We deduce that for any $x, y \in \mathbb{C}$,

$$P(x,y) = a_m y^{n-m} \prod_{i=1}^{m}(x - \lambda_i y)$$

Since $\mathbb{C}$ is an infinite field, the equation also holds in $\mathbb{C}[x,y]$. Hence we have factorised $P(x,y)$ into linear polynomials over $\mathbb{C}$. □

Definition 1.21. Multiplicities, Tangent Lines

Suppose that $p(x,y) = 0$ defines an affine curve C in $\mathbb{C}^2$. The multiplicity of C at $(a,b) \in C$ is the smallest integer m such that

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j} P(a,b) \neq 0$$

for some $i, j \in \mathbb{N}$ with $i + j = m$.

The polynomial

$$\sum_{i+j=m} \frac{\partial^{i+j}}{\partial x^i \partial y^j} P(a,b) \frac{(x-a)^i (y-b)^j}{i!j!}$$

is homogeneous of degree m, and hence can be factorised into a product of m linear polynomials $\alpha(x-a) + \beta(y-b)$ by the previous lemma. Each linear polynomial defines a line in $\mathbb{C}^2$, called the tangent line of C at (a,b).

Remark. A point $(a,b) \in C$ is nonsingular if and only if it has multiplicity $m = 1$. In this case (a,b) has a unique tangent line given by

$$\frac{\partial P}{\partial x}(a,b)(x-a) + \frac{\partial P}{\partial y}(a,b)(y-b) = 0$$

The tangent lines of a projective curve can be defined by pulling back to the affine coordinates. In particular, if $(a,b,c) \in C$ is non-singular, the unique tangent line at (a,b,c) is given by

$$\frac{\partial P}{\partial x}(a,b,c)x + \frac{\partial P}{\partial y}(a,b,c)y + \frac{\partial P}{\partial z}(a,b,c)z = 0$$

If $P(x,y,z)$ is homogeneous of degree $d = 2$, then the curve C is called a **conic**. Note that such polynomial is defined by a **quadratic form** in $\mathbb{F}^3$, $P(v) = B(v,v)$.

Definition 1.22. Quadratic Form

$B : V \times V \to \mathbb{F}$ is called a quadratic form on the vector space V, if

- $B(v,w) = B(w,v)$;
- $B(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 B(v_1,w) + \lambda_2 B(v_2,w)$.

If we fix a basis $\{v_0, ..., v_n\}$ of V. Then B has the form

$$B(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n} M_{ij}x_i y_j = x^\intercal A y$$

for $x, y \in \mathbb{F}^{n+1}$, where $A \in M_{(n+1)\times(n+1)}(\mathbb{F})$ is a symmetric matrix and is called the **Gram matrix** of B.

Theorem 1.23. Diagonalisation of Quadratic Forms

Let V be a $(n+1)$-dimensional vector space. Let B be a quadratic form on V.

1. If $\mathbb{F} = \mathbb{C}$, then there exists a basis $\{v_0, ..., v_n\}$ of V such that

$$B(v,v) = \sum_{i=0}^{r} \lambda_i^2$$

2. If $\mathbb{F} = \mathbb{R}$, then there exists a basis $\{v_0, ..., v_n\}$ of V such that

$$B(v,v) = \sum_{i=0}^{r} \lambda_i^2 - \sum_{j=1}^{s} \lambda_{r+j}^2$$

where $v = \sum_{i=0}^{n} \lambda_i v_i$.

Proof. See *M1. Linear Algebra.* □

Remark. Suppose that C is a conic on $\mathbb{CP}^2$. Then C can be put into one of the following forms:

- $P(x,y,z) = x^2$. This is the double line $x = 0$.
- $P(x,y,z) = x^2 + y^2$. This is a pair of lines $x + \mathrm{i}y = 0$ and $x - \mathrm{i}y = 0$.
- $P(x,y,z) = x^2 + y^2 + z^2$. This is a non-singular conic.

Theorem 1.24. Rational Parametrisation of Conics

Let C be a *non-singular* conic in the projective plane $\mathbb{FP}^2$, and $A \in C$. Let $\mathbb{P}(U) \subseteq \mathbb{FP}^2$ be a projective line such that $A \notin \mathbb{P}(U)$. Then there exists a bijection $\alpha : \mathbb{P}(U) \to C$ such that $A, X, \alpha(X)$ are collinear for all $X \in \mathbb{P}(U)$.

Proof. Suppose the conic is defined by the nondegenerate quadratic form B. Let $a \in \mathbb{FP}^2$ be a representative vector for A, then $B(a,a) = 0$ since A lies on the conic. Let $x \in \mathbb{P}(U)$ be a representative vector for $X \in \mathbb{P}(U)$. Then a and x are linearly independent since X does not lie on the line $\mathbb{P}(U)$. Extend a, x to a basis a, x, y of $\mathbb{FP}^2$.

Now B restricted to the space spanned by a, x is not identically zero, because if it were, the matrix of B with respect to this basis would be of the form

$$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{pmatrix}$$

which is singular. So at least one of $B(x,x)$ and $B(a,x)$ is non-zero. Any point on the line AX is represented by a vector of the form $\lambda a + \mu x$ and this lies on the conic C if

$$0 = B(\lambda a + \mu x, \lambda a + \mu x) = 2\lambda\mu B(a,x) + \mu^2 B(x,x).$$

When $\mu = 0$ we get the point X. The other solution is $2\lambda B(a,x) + \mu B(x,x) = 0$ i.e. the point with representative vector

$$w = B(x,x)a - 2B(a,x)x$$

which is non-zero since the coefficients are not both zero. We define the map $\alpha : \mathbb{P}(U) \to C$ by $\alpha(X) = [w]$, which has the collinear property of the statement of the Theorem. If $Y \in C$ is distinct from A, then the line AY meets the line $\mathbb{P}(U)$ in a unique point, so α^{-1} is well-defined on this subset. By the definition of α, $\alpha(X) = A$ if and only if $B(a,x) = 0$. Since B is nonsingular $f(x) = B(a,x)$ is a non-zero linear map from V to F and so defines a line (the tangent to C at A), which hence meets $\mathbb{P}(U)$ in one point. Thus α has a well-defined inverse and is therefore a bijection. $\square$

Proposition 1.25. Five Points Determine a Conic

Let A, B, C, D, E be five points on a projective plane $\mathbb{FP}^2$ such that no three of them are collinear. Then there exists a unique conic passing through these points.

Proof. By assumption A, B, C, D are in general position. By General Position Theorem we may assume that $A = [1 : 0 : 0]$, $B = [0 : 1 : 0]$, $C = [0 : 0 : 1]$ and $D = [1 : 1 : 1]$. Suppose that $E = [\alpha_0 : \alpha_1 : \alpha_2]$. Let $\mathcal{C} : \sum_{i,j=0}^{2} \lambda_{i,j} x_i x_j = 0$ be a conic that contains the five points.

$A, B, C \in \mathcal{C}$ implies that $\lambda_{0,0} = \lambda_{1,1} = \lambda_{2,2} = 0$. So $\mathcal{C}$ has the form

$$\lambda_{0,1} x_0 x_1 + \lambda_{1,2} x_1 x_2 + \lambda_{2,1} x_2 x_0 = 0$$

$D, E \in \mathcal{C}$ implies that $(\lambda_{0,1}, \lambda_{1,2}, \lambda_{2,0}) \cdot (1, 1, 1) = 0$, $(\lambda_{0,1}, \lambda_{1,2}, \lambda_{2,0}) \cdot (\alpha_0, \alpha_1, \alpha_2) = 0$. Since $D \neq E$, $\langle (1, 1, 1) \rangle \neq \langle (\alpha_0, \alpha_1, \alpha_2) \rangle$. We deduce that $(\lambda_{0,1}, \lambda_{1,2}, \lambda_{2,0}) \in \langle (1, 1, 1),\ (\alpha_0, \alpha_1, \alpha_2) \rangle^{\perp}$, which is a 1-dimensional subspace. Hence the coefficients of the quadric is uniquely determined up to rescaling by a constant. The conic determined by the quadric is unique. $\square$

2 Intersection Theory

The main result in this section is the Bézout's Theorem, which claims that two plane curves of degree m and n respectively roughly intersect in mn points. We shall make the statement rigourous by defining the intersection multiplicity properly.

2.1 Resultants

Definition 2.1. Resultants

Let $p, q \in \mathbb{F}[x]$ given by

$$p(x) = \sum_{k=0}^{n} a_k x^k, \qquad q(x) = \sum_{k=0}^{m} b_k x^k, \qquad (a_n, b_m \neq 0)$$

Then the resultant of p and q is defined by the rank $(m+n)$ determinant:

$$\mathcal{R}_{p,q} := \det \begin{pmatrix} a_0 & a_1 & \cdots & a_n & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_n & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_m & 0 & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & b_m & \cdots & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & b_0 & b_1 & \cdots & b_m \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} m \text{ rows} \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} n \text{ rows} \end{matrix}$$

For $P(x, y, z)$ and $Q(x, y, z)$ in $\mathbb{C}[x, y, z]$, we identify $\mathbb{C}[x, y, z]$ as a subring of $\mathbb{F}[x]$, where $\mathbb{F} = \mathbb{C}(y, z)$. So the resultant of P and Q is a polynomial in $\mathbb{C}[y, z]$. It is denoted by $\mathcal{R}_{P,Q}(y, z)$.

Proposition 2.2. Properties of Resultants

Suppose that $p, q, r \in \mathbb{F}[x]$ and $P, Q, R \in \mathbb{C}[x, y, z]$. We assume that P, Q, R are homogeneous polynomials and $(1, 0, 0)$ is not a root of P, Q or R (this is to ensure that P, Q, R have the same degree when considered as polynomials in $\mathbb{C}[x, y, z]$ and in $\mathbb{C}[y, z]$).

1. p and q has a non-constant common factor if and only if $\mathcal{R}_{p,q} = 0$.
2. P and Q has a non-constant common factor if and only if $\mathcal{R}_{P,Q}(y, z) = 0$.
3. If P and Q are homogeneous of degree m and n respectively, then $\mathcal{R}_{P,Q}(y, z)$ is homogeneous of degree mn.
4. Let $\mathbb{F} = \mathbb{C}$. Suppose that $\lambda_1, .., \lambda_n$ are the roots of p and $\mu_1, ..., \mu_m$ are the roots of q. Then

$$\mathcal{R}_{p,q} = a_n^m b_m^n \prod_{i=1}^{n} \prod_{j=1}^{m} (\mu_j - \lambda_i)$$

5. $\mathcal{R}_{p,q}\mathcal{R}_{p,r} = \mathcal{R}_{p,qr}$.
6. $\mathcal{R}_{P,Q}(y, z)\mathcal{R}_{P,R}(y, z) = \mathcal{R}_{P,QR}(y, z)$.

Proof. 1. Let

$$p(x) = \sum_{k=0}^{n} a_k x^k, \qquad q(x) = \sum_{k=0}^{m} b_k x^k, \qquad (a_n, b_m \neq 0)$$

p and q have a non-constant common factor if and only if there exists polynomials φ and ψ with $\deg \varphi = n - 1$ and $\deg \psi = m - 1$ such that $p\psi = q\varphi$. Suppose that

$$\varphi(x) = \sum_{k=0}^{n-1} \alpha_k x^k, \qquad \psi(x) = \sum_{k=0}^{m} \beta_k x^k$$

We equate the coefficients of each x^j in the equation $p\psi = q\varphi$ and obtain:

$$\begin{aligned} a_0\beta_0 &= b_0\alpha_0 \\ a_0\beta_1 + a_1\beta_0 &= b_0\alpha_1 + b_1\alpha_0 \\ &\cdots \\ a_n\beta_{m-1} &= b_m\alpha_{n-1} \end{aligned}$$

The existence of a non-trivial solution $(\alpha_0, ..., \alpha_{n-1}, \beta_0, ..., \beta_{m-1})$ is equivalent to the vanishing of the determinant defining the resultant $\mathcal{R}_{p,q}$.

2. Without loss of generality we assume that $P(1,0,0) = Q(1,0,0) = 1$. Hence P and Q are monic polynomials in $R[x]$, where $R := \mathbb{C}[y,z]$. From *A3. Rings and Modules* we know that R is a unique factorisation domain. Its field of fraction is $\mathbb{F} = \mathbb{C}(y,z)$. We have

$$\begin{aligned} &P, Q \text{ has a non-constant common factor in } \mathbb{C}[x,y,z] \\ &\iff P, Q \text{ has a non-constant common factor in } R[x] \\ &\iff P, Q \text{ has a non-constant common factor in } \mathbb{F}[x] \quad \text{(Gauss' Lemma)} \\ &\iff \mathcal{R}_{P,Q}(y,z) = 0 \end{aligned}$$

3. By definition the resultant $\mathcal{R}_{P,Q}(y,z)$ of homogeneous polynomials $P(x,y,z)$ and $Q(x,y,z)$ of degrees n and m is the determinant of an $n+m$ by $n+m$ matrix whose ij th entry $r_{ij}(y,z)$ is a homogeneous polynomial in y and z of degree d_{ij} given by

$$d_{ij} = \begin{cases} n+i-j & \text{if } 1 \leqslant i \leqslant m \\ i-j & \text{if } m+1 \leqslant i \leqslant n+m. \end{cases}$$

Then $\mathcal{R}_{P,Q}(y,z)$ is a sum of terms of the form

$$\pm \prod_{i=1}^{n+m} r_{i\sigma(i)}(y,z)$$

where σ is a permutation of $\{1, \ldots, n+m\}$. Each such term is a homogeneous polynomial of degree

$$\begin{aligned} \sum_{i=1}^{n+m} d_{i\sigma(i)} &= \sum_{i=1}^{m}(n+i-\sigma(i)) + \sum_{i=m+1}^{m+n}(i-\sigma(i)) \\ &= nm + \sum_{i=1}^{m+n} i - \sum_{i=1}^{m+n} \sigma(i) \\ &= nm. \end{aligned}$$

Therefore $\mathcal{R}_{P,Q}(y,z)$ is a homogeneous polynomial of degree nm in y and z.

4. For each λ_i we associate an indeterminate x_i and each μ_j with y_j. Consider

$$\xi(x_1, ..., x_n, y_1, ..., y_m) := \prod_{i=1}^{n}\prod_{j=1}^{m}(y_j - x_i) \in \mathbb{C}[x_1, ..., x_n, y_1, ..., y_m]$$

We note that ξ is homogeneous of degree mn. On the other hand, let

$$p(x, x_1, ..., x_n) = a_n \prod_{i=0}^{n}(x - x_i), \qquad q(x, y_1, ..., y_m) = b_m \prod_{j=0}^{m}(x - y_j)$$

So $\mathcal{R}_{p,q} \in \mathbb{C}[x_1, ..., x_n, y_1, ..., y_m]$. From the proof of (3) we infer that $\mathcal{R}_{p,q}$ is homogeneous of degree mn. Moreover, by (1) we have that $\lambda_i = \mu_j$ implies $\mathcal{R}_{p,q} = 0$. Therefore $(x_i - y_j)$ divides $\mathcal{R}_{p,q}(x_1, ..., x_n, y_1, ..., y_m)$ for each i, j. We deduce that ξ divides $\mathcal{R}_{p,q}$. In particular

$$\mathcal{R}_{p,q}(x_1, ..., x_n, y_1, ..., y_m) = C \prod_{i=1}^{n}\prod_{j=1}^{m}(y_j - x_i)$$

We put $y_1 = \cdots = y_m = 0$ so that $q = b_m x^m$. In such case, the determinant defining $\mathcal{R}_{p,q}$ is upper triangular.

Therefore we have

$$\mathcal{R}_{p,q}(x_1, ..., x_n, 0, ..., 0) = a_0^m b_m^n = a_n^m b_m^n \prod_{i=1}^{n} (-x_i)^m$$

And hence $C = a_n^m b_m^n$. Finally,

$$\mathcal{R}_{p,q} = \mathcal{R}_{p,q}(\lambda_1, ..., \lambda_n, \mu_1, ..., \mu_n) = a_n^m b_m^n \prod_{i=1}^{n} \prod_{j=1}^{m} (\mu_j - \lambda_i)$$

5 & 6. These are direct corollaries of (4). □

Lemma 2.3. Existence of Intersections

Any two projective curves in $\mathbb{CP}^2$ intersects in at least one point.

Proof. Let C and D be projective curves defined by homogeneous polynomials $P(x, y, z) = 0$ and $Q(x, y, z) = 0$. By Proposition 2.2, either $\mathcal{R}_{P,Q}(y, z) = 0$ or $\mathcal{R}_{P,Q}(y, z)$ is a homogeneous polynomial. In the latter case, by Lemma 1.20 $\mathcal{R}_{P,Q}(y, z)$ factorises into a product of linear factors $(b_i z - c_i y)$. Therefore in both cases there exists $(b, c) \in \mathbb{C}^2 \setminus \{0\}$ such that $\mathcal{R}_{P,Q}(b, c) = 0$. Hence $p(x) := P(x, b, c)$ and $q(x) := Q(x, b, c)$ have a common root $a \in \mathbb{C}$. We deduce that $P(a, b, c) = Q(a, b, c) = 0$ and hence $[a : b : c] \in C \cap D$. □

Lemma 2.4. Weak Form of Bézout's Theorem

Suppose that C and D are projective curves in $\mathbb{CP}^2$ of degree n and m respectively. If they have no common components, then they intersect in at most mn points.

Proof. Let C and D be projective curves defined by homogeneous polynomials $P(x, y, z) = 0$ and $Q(x, y, z) = 0$. Suppose the a set of $(mn+1)$ distinct points lies in $C \cap D$. By applying a projective transformation we assume that $[1 : 0 : 0]$ does not lie in S, nor in a line joining any two points of S. In particular, $P(1, 0, 0), Q(1, 0, 0) \neq 0$. By the proof of the previous lemma, the resultant $\mathcal{R}_{P,Q}(y, z)$ is a product of linear factors $(b_i z - c_i y)$ with $(b_i, c_i) \in \mathbb{C}^2 \setminus \{0\}$.

For each $[a : b : c] \in S$, we have $P(a, b, c) = Q(a, b, c) = 0$ and hence the resultant $\mathcal{R}_{P,Q}(b, c) = 0$. Since $[1 : 0 : 0] \notin S$, $(b, c) \neq 0$. We have that $bz - cy$ divides $\mathcal{R}_{P,Q}(y, z)$. Furthermore, for $[a : b : c]$ and $[a' : b' : c']$ in S, we cannot have $[b : c] = [b' : c']$, because otherwise $[1 : 0 : 0]$ would lie on a line joining these points. Hence $\mathcal{R}_{P,Q}(y, z)$ has $(mn+1)$ distinct linear factors. It must be identically zero. We deduce that C and D has a common component. □

Corollary 2.5. Singular Points and Irreducibility

Suppose that C is a projective curve in $\mathbb{CP}^2$.

1. If C is non-singular, then C is irreducible;
2. If C is irreducible, then C has at most finitely many singular points.

Proof.

1. Suppose that C is reducible. Let C be defined by $PQ = 0$. By Lemma 2.3, there exists $[a : b : c] \in \mathbb{CP}^2$ such that $P(a, b, c) = Q(a, b, c) = 0$. By differentiating with respect to PQ we deduce that $[a : b : c]$ is a singular point of C.
2. Without loss of generality we assume that $[1 : 0 : 0] \notin C$. Then the defining polynomial $P(x, y, z)$ of C has non-zero coefficient of x^n. Then $\partial_x P$ is a non-zero homogeneous polynomial of degree $n-1$. Since C is irreducible, P and $\partial_x P$ are coprime. Then C and D have no common components, where D is the curve defined by $\partial_x P(x, y, z) = 0$. It follows from the weak form of Bézout's Theorem that C and D have at most $n(n-1)$ points of intersection. The singular points of C lie among these. □

Theorem 2.6. Pascal's Mystic Hexagon

The pairs of opposite sides of a hexagon inscribed in an irreducible conic on the projective plane meet in three collinear points.

Proof. Let $R(x, y, z) = 0$ defines the conic E. Let the successive 6 sides of the hexagon be defined by the linear polynomials $L_1, ..., L_6$. Let C and D be the curves defined by $P := L_1 L_2 L_3 = 0$ and $Q := L_4 L_5 L_6 = 0$ respectively. C and D has 9 points of intersection, 6 of which are the vertices of the hexagon, and the remaining 3 are the intersections of the opposite sides of the hexagon.

Let $[a : b : c] \in E \setminus (C \cap D)$. Then

$$Q(a, b, c)P(x, y, z) - P(a, b, c)Q(x, y, z) = 0$$

defines a cubic F which meets E in the six vertices plus the point $[a : b : c]$. By the weak form of the Béouzt' Theorem, E and F have a common component. Since the conic is irreducible, we must have

$$Q(a, b, c)P(x, y, z) - P(a, b, c)Q(x, y, z) = L(x, y, z)R(x, y, z)$$

Hence the linear polynomial L defines a line passing through the 3 points of intersection of the opposite sides of the hexagon. □

2.2 Bézout's Theorem

In order to state the strong form of Bézout's Theorem we must define the intersection multiplicity of two curves properly.

Definition 2.7. Intersection Multiplicity

Let C and D be projective curves in $\mathbb{CP}^2$. Let $p \in \mathbb{CP}^2$. We define the intersection multiplicity $\mathrm{I}_p(C, D)$ by:

- If $p \notin C \cap D$, then $\mathrm{I}_p(C, D) := 0$.
- If p lies on a common component of C and D, then $\mathrm{I}_p(C, D) := \infty$.
- If $p \in C \cap D$ but not in common components of C and D:
  - remove any common component of C and D to get the curves C', D';
  - apply a projective transformation such that $[1 : 0 : 0]$ does not lie in $C' \cup D'$, nor in a line joining distinct points of $C' \cap D'$, nor in any tangent line to C' and D' at a point of $C' \cap D'$. (We refer this choice of coordinates as $(*)$.)

  Then define $\mathrm{I}_p(C, D)$ at $p = [a : b : c]$ to be the largest integer k such that $(bz - cy)^k$ divides the resultant $\mathcal{R}_{P',Q'}(y, z)$, where P' and Q' are the defining polynomials of C' and D' respectively.

Proposition 2.8. Properties of Intersection Multiplicity

Let C and D be projective curves in $\mathbb{CP}^2$.

1. $\mathrm{I}_p(C, D) = \mathrm{I}_p(D, C)$.
2. $p \notin C \cap D$ if and only if $\mathrm{I}_p(C, D) = 0$.
3. Two distinct lines meet at a point with multiplicity 1.
4. If C is the union of two components C_1 and C_2, then $\mathrm{I}_p(C, D) = \mathrm{I}_p(C_1, D) + \mathrm{I}_p(C_2, D)$.
5. If C and D are defined by P and Q respectively, and E is defined by $PR + Q$, then $\mathrm{I}_p(C, D) = \mathrm{I}_p(C, E)$.

Proof.

1. It follows from $|\mathcal{R}_{P,Q}(y, z)| = |\mathcal{R}_{Q,P}(y, z)|$.
2. If $p = [a : b : c] \in C \cap D$, then $P(x, b, c)$ and $Q(x, b, c)$ has a common root a. Then $\mathcal{R}_{P,Q}(b, c) = 0$ and hence $\mathcal{R}_{P,Q}(y, z)$ is divisible by $bz - cy$. We deduce that $\mathrm{I}_p(C, D) \geqslant 1$.
3. We can choose the coordinates such that the lines $ax + by = 0$ and $cx + dy = 0$ meet at $p = [0 : 0 : 1]$. Then the resultant is $(ad - bc)y$.
4. This follows from Proposition 2.2.6.

5. Without loss of generality we assume that P, Q have degree m, n respectively and $n > m$. Let

$$R(x,y,z) = \sum_{k=0}^{n-m} \rho_k(y,z)x^k$$

Suppose that $\mathcal{R}_{P,Q} = \det(a_{ij})$. Then $\mathcal{R}_{P,PR+Q} = \det(b_{ij})$, where

$$b_{ij} = \begin{cases} a_{ij} & i \leqslant m \\ a_{ij} + \displaystyle\sum_{k=i-m}^{i-n} \rho_{i-n-k} a_{kl} & i \geqslant m \end{cases}$$

We note that $\det(a_{ij}) = \det(b_{ij})$ by row operations. Hence $\mathrm{I}_p(C,D) = \mathrm{I}_p(C,E)$. □

Now the proof of Bézout's Theorem is easy.

Theorem 2.9. Bézout's Theorem

Let C and D be projective curves in $\mathbb{CP}^2$ of degree m and n respectively. If they have no common components, then

$$\sum_{p \in C \cap D} \mathrm{I}_p(C,D) = mn$$

Proof. Using the choice of coordinates $(*)$, we put P, Q to be homogeneous polynomials that define C and D respectively. Then the resultant is the product

$$\mathcal{R}_{P,Q}(y,z) = \prod_{i=1}^{k} (b_i z - c_i y)^{e_i}, \qquad \sum_{i=1}^{k} e_i = mn$$

Each such factor gives a point $p_i \in C \cap D$ with $\mathrm{I}_{p_i}(C,D) = e_i$. □

Proposition 2.10. Intersection Multiplicity 1

Let C and D be projective curves in $\mathbb{CP}^2$ and $p \in C \cap D$. Then $\mathrm{I}_p(C,D) = 1$ if and only if p is a non-singular point of C and D, and the tangent lines to C and D at p are distinct.

Proof. We may assume that C and D have no common component, and hence we may choose coordinates such that $p = [0 : 0 : 1]$ and the conditions of $(*)$ hold.

- First, we show that if $p \in C \cap D$ is a singular point of C, then $\mathrm{I}_p(C,D) > 1$.

  We wish to show that y^2 divides the resultant $\mathcal{R}_{P,Q}(y,z)$ of the polynomials $P(x,y,z)$ and $Q(x,y,z)$ defining C and D. If p is a singular point of C, we have

  $$\frac{\partial P}{\partial x}(0,0,1) = \frac{\partial P}{\partial y}(0,0,1) = P(0,0,1) = 0.$$

  Hence

  $$P(x,y,z) = a_0(y,z) + a_1(y,z)x + \ldots + a_n(y,z)x^n$$

  where y^2 divides $a_0(y,z)$ and y divides $a_1(y,z)$. Also $Q(0,0,1) = 0$ so

  $$Q(x,y,z) = b_0(y,z) + b_1(y,z)x + \ldots + b_m(y,z)x^m$$

  where y divides $b_0(y,z)$. Thus we can write

  $$b_0(y,z) = b_{01}yz^{m-1} + y^2 c_0(y,z)$$

  and

  $$b_1(y,z) = b_{10}z^{m-1} + yc_1(y,z)$$

for some homogeneous polynomials $c_0(y,z)$ and $c_1(y,z)$. If $b_{01} = 0$ then the first column of the determinant defining $\mathcal{R}_{P,Q}(y,z)$ is divisible by y^2 and hence y^2 divides $\mathcal{R}_{P,Q}(y,z)$ as required. If $b_{01} \neq 0$ then the first column is divisible by y; if we take out this factor y and subtract b_{10}/b_{01} times the first column from the second column then the second column becomes divisible by y. Hence again y^2 divides $\mathcal{R}_{P,Q}(y,z)$.

- If $\mathrm{I}_p(C,D) = 1$, p must be a nonsingular point of C and D. We need to show that the tangent lines coincide if and only if y^2 divides the resultant $\mathcal{R}_{P,Q}(y,z)$, or equivalently that the derivative of $\mathcal{R}_{P,Q}(y,1)$ vanishes at $y = 0$.

  Now by assumption $[1:0:0]$ does not lie on the tangent line to p for either curve so $\partial_x P(0,0,1) \neq 0$ and similarly for Q. The Implicit Function Theorem then tells us that in a suitable small neighbourhood, the solution x of $P(x,y,1) = 0$ is a holomorphic function of y. In other words, near $[0:0:1]$, the roots $\lambda_1(y), \mu_1(y)$ of P and Q, which coincide when $y = 0$, are holomorphic functions of y. Thus

  $$P(x,y,1) = (x - \lambda_1(y))\,\ell(x,y) \quad Q(x,y,1) = (x - \mu_1(y))\,m(x,y)$$

  for polynomials ℓ, m in x with coefficients which are holomorphic functions of y. Then the resultant $\mathcal{R}_{P,Q}(y,1) = (\lambda_1(y) - \mu_1(y))\,S(y)$ where $S(y)$ is holomorphic. Differentiating at $y = 0$,

  $$\left.\frac{\partial \mathcal{R}_{P,Q}(y,1)}{\partial y}\right|_{y=0} = (\lambda_1'(0) - \mu_1'(0))\,S(0) \qquad (\star)$$

  We shall show next that $S(0) \neq 0$.

  Since $\partial P/\partial x(0,0,1) \neq 0, x = 0$ is not a repeated root of $P(x,0,1)$ so for $i \neq 1$, $\lambda_i(0) \neq 0$ and similarly for Q. If $\lambda_i(0) = \mu_j(0)$ for $i,j > 1$ then $[0:0:1]$ and $[\lambda_i(0):0:1]$ are distinct points in $C \cap D$ and $[1:0:0]$ lies on the line joining them which contradicts our assumptions. Now $S(y)$ is a product of resultants and we see here that there is no other coincidence of roots than $\lambda_1(0) = \mu_1(0)$ at $y = 0$. Thus $S(0) \neq 0$.

  It follows that the derivative in equation $(\star)$ vanishes if and only if $\lambda_1'(0) - \mu_1'(0) = 0$. Now since $P(\lambda_1(y), y, 1) \equiv 0$, differentiating with respect to y gives

  $$\frac{\partial P}{\partial x}\lambda_1'(y) + \frac{\partial P}{\partial y} = 0$$

  and at $[0:0:1]$ by Euler's Relation $\partial_z P = nP = 0$, so the tangent line to C is $x - \lambda_1'(0)y = 0$ and to D is $x - \mu_1'(0)y = 0$. Hence the tangents coincide if and only if $\lambda_1'(0) - \mu_1'(0) = 0$, which proves the theorem. □

Corollary 2.11

Let C and D be projective curves in $\mathbb{CP}^2$ of degree m and n respectively. Suppose that every $p \in C \cap D$ is a non-singular point of C and D, and the tangent lines to C and D at p are distinct. Then $C \cap D$ contains exactly mn points.

2.3 Cubic Curves

We shall use the intersection multiplicity to classify cubic curves on $\mathbb{CP}^2$. We shall show that all nonsingular cubic has the form

$$y^2 z = x(x-z)(x-\lambda z)$$

for some $\lambda \in \mathbb{C} \setminus \{0,1\}$.

Definition 2.12. Inflection Points

Suppose that $p \in C$ is a non-singular point of the projective curve C. p is called an inflection point of C if there exists a line L through p such that $\mathrm{I}_p(C,L) \geqslant 3$. Note that L is necessarily tangent to C.

Let $P(x,y,z)$ be a homogeneous polynomial of degree d. The **Hessian** of P is defined by

$$\mathcal{H}_P(x,y,z) := \det(\partial_i \partial_j P)$$

So $\mathcal{H}_P$ is a homogeneous polynomial of degree $3(d-2)$.

Proposition 2.13. Charaterisations of Inflection Points

A non-singular point $p = [a_0 : a_1 : a_2] \in C$ is an inflection point of C if and only if $\mathcal{H}_P(a_), a_1, a_2) = 0$, where P is the defining polynomial of C.

Proof. For convenience we set

$$P_i := \partial_i P(a_0, a_1, a_2), \qquad P_{ij} := \partial_i \partial_j P(a_0, a_1, a_2)$$

Consider the line L as the set

$$\{[a_0 + t\alpha_0 : a_1 + t\alpha_1 : a_2 + t\alpha_2] : t \in \mathbb{C}\}$$

Note that $\mathrm{I}_p(C, L) \geqslant 3$ if and only if t^3 divides $P(a_0 + t\alpha_0, a_1 + t\alpha_1, a_2 + t\alpha_2)$. We use Taylor Theorem to expand it:

$$P(a_0 + t\alpha_0, a_1 + t\alpha_1, a_2 + t\alpha_2) = P(a_0, a_1, a_2) + t\sum_{i=0}^{2} P_i\alpha_i + \frac{1}{2}t^2 \sum_{i=0}^{2}\sum_{j=0}^{2} P_{ij}\alpha_i\alpha_j + t^3 R$$

We note that t^3 divides this if and only if

$$\sum_{i=0}^{2} P_i\alpha_i = \sum_{i=0}^{2}\sum_{j=0}^{2} P_{ij}\alpha_i\alpha_j = 0 \tag{1}$$

The polynomials P and $\partial_i P$ are homogeneous of degree n and $n-1$ respectively. By using Euler's relation, we have

$$\sum_{i=0}^{2}\sum_{j=0}^{2} P_{ij}a_i a_j = (n-1)\sum_{i=0}^{2} P_i a_i = n(n-1)P(a_0, a_1, a_2) = 0$$

and

$$\sum_{i=0}^{2}\sum_{j=0}^{2} P_{ij}a_i\alpha_j = (n-1)\sum_{i=0}^{2} P_i\alpha_i = 0$$

First we assume that (1) holds. We note that the quadratic form defined by (P_{ij}) on $\mathbb{C}^3$ vanishes completely on $\langle a, \alpha\rangle$. Hence it is degenerate on $\mathbb{C}^3$ and $\mathcal{H}_P(a) = \det(P_{ij}) = 0$.

Conversely, we assume that $\det(P_{ij}) = 0$. Let α defines the tangent to C at p. Extend a, α to a basis $\{a, \alpha, \beta\}$ of $\mathbb{C}^3$. The quadratic form with respect to the basis has matrix of the form

$$M = \begin{pmatrix} 0 & 0 & * \\ 0 & * & * \\ * & * & * \end{pmatrix}$$

Then we have $M_{20}M_{11}M_{20} = 0$. But

$$M_{02} = M_{20} = \sum_{i=0}^{2}\sum_{j=0}^{2} P_{ij}a_i\beta_j = (n-1)\sum_{i=0}^{2} P_i\beta_i \neq 0$$

for otherwise β lies on the tangent $a + t\alpha$. Hence we must have

$$M_{11} = \sum_{i=0}^{2}\sum_{j=0}^{2} P_{ij}\alpha_i\alpha_j = 0$$

Hence the equation (1) holds. □

Lemma 2.14

Suppose that $P(x, y, z)$ is a homogeneous polynomial of degree $d > 1$. Then

$$z^2\mathcal{H}_P(x, y, z) = (d-1)^2 \begin{pmatrix} \partial_x^2 P & \partial_x\partial_y P & \partial_x P \\ \partial_y\partial_x P & \partial_y^2 P & \partial_y P \\ \partial_x P & \partial_y P & \frac{d}{d-1}P \end{pmatrix}$$

Proof. Starting from the Hessian

$$\mathcal{H}_P(x,y,z) = \det \begin{pmatrix} \partial_x^2 P & \partial_x\partial_y P & \partial_x\partial_z P \\ \partial_y\partial_x P & \partial_y^2 P & \partial_y\partial_z P \\ \partial_z\partial_x P & \partial_z\partial_y P & \partial_z^2 P \end{pmatrix}$$

We perform some elementary row operations:

$$\begin{aligned} z\mathcal{H}_P(x,y,z) &= \det \begin{pmatrix} \partial_x^2 P & \partial_x\partial_y P & \partial_x\partial_z P \\ \partial_y\partial_x P & \partial_y^2 P & \partial_y\partial_z P \\ z\partial_z\partial_x P & z\partial_z\partial_y P & z\partial_z^2 P \end{pmatrix} \\ &= \det \begin{pmatrix} \partial_x^2 P & \partial_x\partial_y P & \partial_x\partial_z P \\ \partial_y\partial_x P & \partial_y^2 P & \partial_y\partial_z P \\ x\partial_x^2 P + y\partial_y\partial_x P + z\partial_z\partial_x P & x\partial_x\partial_y P + y\partial_y^2 P + z\partial_z\partial_y P & x\partial_x\partial_z P + y\partial_y\partial_z P + z\partial_z^2 P \end{pmatrix} \\ &= (d-1)\det \begin{pmatrix} \partial_x^2 P & \partial_x\partial_y P & \partial_x\partial_z P \\ \partial_y\partial_x P & \partial_y^2 P & \partial_y\partial_z P \\ \partial_x P & \partial_y P & \partial_z P \end{pmatrix} \qquad \text{(using Euler's Relation of } \partial_x P, \partial_y P, \partial_z P) \end{aligned}$$

Then we perform some elementary column operations:

$$\begin{aligned} z^2\mathcal{H}_P(x,y,z) &= (d-1)\det \begin{pmatrix} \partial_x^2 P & \partial_x\partial_y P & z\partial_x\partial_z P \\ \partial_y\partial_x P & \partial_y^2 P & z\partial_y\partial_z P \\ \partial_x P & \partial_y P & z\partial_z P \end{pmatrix} \\ &= (d-1)\det \begin{pmatrix} \partial_x^2 P & \partial_x\partial_y P & x\partial_x^2 P + y\partial_x\partial_y P + z\partial_x\partial_z P \\ \partial_y\partial_x P & \partial_y^2 P & x\partial_y\partial_x P + y\partial_y^2 P + z\partial_y\partial_z P \\ \partial_x P & \partial_y P & x\partial_x P + y\partial_y P + z\partial_z P \end{pmatrix} \\ &= (d-1)\det \begin{pmatrix} \partial_x^2 P & \partial_x\partial_y P & (d-1)\partial_x P \\ \partial_y\partial_x P & \partial_y^2 P & (d-1)\partial_y P \\ \partial_x P & \partial_y P & dP \end{pmatrix} \qquad \text{(using Euler's Relation of } \partial_x P, \partial_y P, P) \\ &= (d-1)^2\det \begin{pmatrix} \partial_x^2 P & \partial_x\partial_y P & \partial_x P \\ \partial_y\partial_x P & \partial_y^2 P & \partial_y P \\ \partial_x P & \partial_y P & dP/(d-1) \end{pmatrix} \end{aligned}$$

□

Remark. In calculus we define the inflection point of the function $f(x)$ to be the point x_0 such that $f''(x_0) = 0$.

Let C be defined by $P(x,y,z)$. By applying a projective transformation we assume that $P(0,0,1) = 0$ and $\partial_y P(0,0,1) \neq 0$. Applying the Implicit Function Theorem to $P(x,y,1)$ we know that there exists a holomorphic function $g : U \to V$ such that $g(0) = 0$ and $P(x,y,1) = 0$ if and only if $y = g(x)$. In particular we differentiate the equation with respect to x twice to get

$$g''(x) = (\partial_y P)^{-3}\det \begin{pmatrix} \partial_x^2 P & \partial_x\partial_y P & \partial_x P \\ \partial_y\partial_x P & \partial_y^2 P & \partial_y P \\ \partial_x P & \partial_y P & 0 \end{pmatrix} = \frac{\mathcal{H}_P(x,y,1)}{(d-1)^2(\partial_y P)^3}$$

We note that $[a:b:1]$ is an inflection point of C if and only if $\mathcal{H}_P(a,b,1) = 0$, if and only if $g''(a) = 0$. So the definition of inflection points we give is consistent with that in calculus.

Lemma 2.15

Suppose that C is an irreducible curve in $\mathbb{CP}^2$ of degree d. Then $d = 1$ if and only if every point of C is an inflection point.

Proof. The forward direction is trivial. For the backward direction, suppose that every point of C is an inflection point. As in the previous lemma, the local homogeneous function $y = g(x)$ satisfies $g(0) = 0$ and $g''(x) = 0$ for all $x \in U$. Hence $g(x) = \lambda x$. Thus the polynomial $p(x) := P(x,\lambda,1) = 0$, and $P(x,\lambda x,1)$ is divisible by $(y - \lambda x)$. Since P is irreducible, we must have $d = 1$. □

Proposition 2.16. Number of Inflection Points

Suppose that C is a non-singular curve in $\mathbb{CP}^2$ of degree d.

1. If $d \geqslant 2$, then C has at most $3d(d-2)$ inflection points;
2. If $d \geqslant 3$, then C has at least one inflection point.

Proof. Let C be defined by the polynomial P. For $d \geqslant 3$, $\mathcal{H}_P$ defines a projective curve H of degree $3(d-2)$. The inflection points of C are exactly the points of intersection of C and H.

1. If $d = 2$, then $\mathcal{H}_P$ is a constant polynomial. Hence C has no inflection points.

   Since C is non-singular, it is irreducible. For $d \geqslant 3$, $\mathcal{H}_P$ defines a projective curve H. If C and H have a common component, then P divides $\mathcal{H}_P$. Therefore every point on C is an inflection point. It follows from the previous lemma that $d = 1$. Therefore C and H have no common component, then by the weak form of Bézout's Theorem, C and H have at most $3d(d-2)$ points of intersection.

2. By the above discussion, this follows from Lemma 2.3 immediately. □

Theorem 2.17. Normal Form of Cubic Curves

Let C be a non-singular curve in $\mathbb{CP}^2$. Then C is equivalent under a projective transformation to the curve defined by

$$y^2 z = x(x-z)(x-\lambda z)$$

for some $\lambda \in \mathbb{C} \setminus \{0, 1\}$.

Proof. By the previous proposition, C has an inflection point. We can choose coordinates such that $[0:1:0]$ is an inflection point of C and $z = 0$ is the tangent to C at $[0:1:0]$. Then C is defined by $P(x, y, z) = 0$, where

$$P(0,1,0) = \partial_x P(0,1,0) = \partial_y P(0,1,0) = \mathcal{H}_P(0,1,0) = 0, \qquad \partial_z P(0,1,0) \neq 0$$

Using Lemma 2.14 with the role of y and z reversed, we have

$$y^2 \mathcal{H}_P(x,y,z) = 4 \det \begin{pmatrix} \partial_x^2 P & \partial_x P & \partial_x \partial_z P \\ \partial_x P & \frac{3}{2} P & \partial_z P \\ \partial_z \partial_x P & \partial_z P & \partial_z^2 P \end{pmatrix}$$

Hence

$$0 = \mathcal{H}_P(0,1,0) = -4(\partial_z P)^2 \partial_x^2 P \implies \partial_x^2 P(0,1,0) = 0$$

Therefore P must have the form

$$P(x,y,z) = yz(\alpha x + \beta y + \gamma z) + \varphi(x,z)$$

where $\varphi(x,z)$ is homogeneous in x and z of degree 3. Since $\beta = \partial_z P(0,1,0) \neq 0$, we apply the projective transformation

$$[x:y:z] \mapsto \left[x : y + \frac{\alpha x + \gamma z}{2\beta} : z\right]$$

so that C is defined by

$$\beta y^2 z + \psi(x,z) = 0$$

Since C is non-singular, it is irreducible, and hence z^3 does not divide $\psi(x,z)$. We have

$$\psi(x,z) = A(x-az)(x-bz)(x-cz)$$

for some scalars A, a, b, c. Finally we apply the projective transformation

$$[x:y:z] \mapsto \left[\frac{x-az}{b-a} : y : z\right]$$

and with a suitable rescaling of y, we obtain

$$y^2 z = x(x-z)(x-\lambda z)$$

$\lambda \notin \{0, 1\}$ because C is non-singular. $\square$

Corollary 2.18. Number of Inflection Points of a Cubic Curve

A non-singular cubic curve in $\mathbb{CP}^2$ has exactly 9 inflection points.

Proof. Let C be a non-singular cubic curve defined by P, and H be a cubice curve defined by $\mathcal{H}_P$. By Corollary 2.11 of Bézout's Theorem, it suffices to show that each inflection point is a non-singular point of H, and the tangent lines to C and H at p are distinct.

Let p be an inflection point. By a projective transformation we may assume that $p = [0 : 1 : 0]$ and

$$P(x, y, z) = y^2 z - x(x - z)(x - \lambda z)$$

The remaining work is a simple calculation:

$$\partial_x P(0, 1, 0) = \partial_y P(0, 1, 0) = 0, \qquad \partial_z P(0, 1, 0) = 1$$

$$\partial_x \mathcal{H}_P(0, 1, 0) = 24, \qquad \partial_y \mathcal{H}_P(0, 1, 0) = 0, \qquad \partial_z(0, 1, 0) = -8(\lambda + 1)$$

Hence the conditions are satisfied. $\square$

3 Topological Properties

We study the topological properties of the algebraic curves, which are examples of Riemann surfaces. Some results are from *B3.2. Geometry of Surfaces* with details omitted.

3.1 Riemann Surfaces

Riemann surfaces are one-dimensional complex manifolds:

Definition 3.1. Riemann Surfaces

Suppose that X is a Hausdorff, second-countable topological space. X is called a Riemann surface, if there exists a family $\mathscr{A} := \{(U_i, \varphi_i) : i \in I\}$ such that

1. $\{U_i : i \in I\}$ is an open cover of X;
2. $\varphi : U_i \to \mathbb{C}$ is a homeomorphism onto its image;
3. for $U_i \cap U_j \neq \varnothing$, the transition map

$$\varphi_j^{-1} \circ \varphi_i\Big|_{\varphi_i(U_i \cap U_j)} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

   is a biholomorphism.

$\mathscr{A}$ is called an *altas* of X. (U_i, φ_i) is called a *coordinate chart*, and φ_i is called a *holomorphic coordinate* on U_i.

Example 3.2. Non-Singular Algebraic Curves

The non-singular algebraic curves on $\mathbb{CP}^2$ are Riemann surfaces.

Proof. Let C be a nonsingular projective algebraic curve defined by $P(x, y, z) = 0$. Every point lies in an affine open set of $\mathbb{CP}^2$ which is homeomorphic to $\mathbb{C}^2$. On $z \neq 0$ its equation is $P(x, y, 1) = 0$ and if C is nonsingular one of $\partial P/\partial x, \partial_y P$ is non-zero. Suppose it is the latter, then at $(x, y) = (a, b)$ on the curve, the implicit function theorem tells us that there are neighbourhoods V and W of a and b in $\mathbb{C}$ and a holomorphic function $g : V \to W$ such that for $x \in V$ and $y \in W$, $P(x, y, 1) = 0$ if and only if $y = g(x)$. Hence for $(x, y) \in C \cap (V \times W)$ the function x has an inverse $x \mapsto (x, g(x))$, and this is a local coordinate for C.

If $\partial_x P$ is non-vanishing we can do the same interchanging the roles of x and y, and get $x = h(y)$. Where both $\partial_x P$ and $\partial_y P$ are non-vanishing, $y = g(h(y))$ and we have an invertible holomorphic function relating the two local coordinates.

On the affine set $y \neq 0$, the equation of the curve is $P(\widetilde{x}, 1, \widetilde{z})$ where, when $z \neq 0$, $\widetilde{x} = x/y, \widetilde{z} = 1/y$ and it is easy to see that the holomorphic coordinates on the intersection of these two open sets is holomorphic and invertible. □

Definition 3.3. Holomorphic Maps

Let X and Y be Riemann surfaces. A continuous map $f : X \to Y$ is called a holomorphic map, if for each $x \in X$, we can find some coordinate chart (U, φ_U) of x in X, and (W, ψ_W) of $f(x)$ in Y, such that the composition map

$$\psi_W \circ f \circ \varphi_U^{-1} : \varphi_U(U) \to \mathbb{C}$$

is holomorphic.

Definition 3.4. Meromorphic Functions

Suppose that X is a Riemann surface. a meromorphic function is a holomorphic map $f : X \to \mathbb{CP}^1 \cong \mathbb{C}_\infty$ such that f is not identically equal to ∞ on any connected component of X.

Remark. The requirement that f is not identically equal to ∞ on any connected component of X is necessary for the meromorphic functions to form a field.

Proposition 3.5. Local Form of Holomorphic Maps

Suppose that $f : X \to Y$ is a holomorphic map between Riemann surfaces X and Y. For $x \in X$, we can choose holomorphic coordinate charts (U, φ_U) of x in X, and (W, ψ_W) of $f(x)$ in Y, such that $\varphi_U(U)$ and $\psi_W(W)$ contain an open neighbourhood of $0 \in \mathbb{C}$, and

$$\psi_W \circ f \circ \varphi_U^{-1} : z \mapsto z^n$$

We say that n is the **ramification index** of f at x, and write $\nu_f(x) = n$.

Remark. With some abuse of language, we often say that f is locally the map $f(z) = z^n$.

Proof. Let $F := \psi_W \circ f \circ \varphi_U^{-1}$. By translation we can always assume that φ_U and ψ_W contains an open neighbourhood of $0 \in \mathbb{C}$, and $F(0) = 0$. If 0 is a zero of F of order n, then F has the Taylor series around 0:

$$F(z) = \sum_{k=n}^{\infty} a_k z^k, \qquad a_n \neq 0$$

Then $G(z) := F(z)^{1/n} = a_n^{1/n} z + \mathrm{o}(z)$ and $G'(0) = a_n^{1/n} \neq 0$. By Implicit Function Theorem, there exists a injective holomorphic function $g : \Omega \to \mathbb{C}$ such that $g(0) = 0$ and $F(g(z))^{1/n} = z$. Now we replace the holomorphic coordinate φ_U by $g^{-1} \circ \varphi_U$. Then

$$\widetilde{F}(z) = \psi_W \circ f \circ \varphi_U^{-1} \circ g(z) = F(g(z)) = z^n$$

□

Remark. Geometrically, the valency of f at x tells us how many solutions there are to the equation $f(x) = y$.

Definition 3.6. Ramification Points, Branch Points

Suppose that $f : X \to Y$ is a holomorphic map between Riemann surfaces X and Y. For $x \in X$, if $\nu_f(x) > 1$, then $x \in X$ is called a ramification point; $f(x) \in Y$ is called a branch point.

Note that for $\nu_f(x) > 1$, locally $f(z) = z^n$ and $f'(z) = nz^{n-1} \neq 0$ in a deleted neighbourhood of 0. Hence the set $\{x \in X : \nu_f(x) > 1\}$ is discrete. If X is compact, the set is finite. We have the following definition:

Definition 3.7. Degree of Holomorphic Map

Suppose that $f : X \to Y$ is a non-constant holomorphic map between *compact connected* Riemann surfaces X and Y. Fix $y \in Y$. We define the degree of f to be

$$\deg f := \sum_{x \in f^{-1}(\{y\})} \nu_f(x)$$

Lemma 3.8

$\deg f$ does not depend on the choice of $y \in Y$.

Proof. Using connectedness and the fact that $y \mapsto \sum_{x \in f^{-1}(\{y\})} \nu_f(x)$ is locally constant. See *B3.2. Goemetry of Surfaces* for detail. □

Corollary 3.9

If $y \in Y$ is not a branch point, then $\deg f = \operatorname{card} f^{-1}(\{y\})$.

3.2 Degree-Genus Formula

We need the following theorem from *B3.2. Geometry of Surfaces*:

Theorem 3.10. Riemann-Hurwitz Formula

Suppose that $f : X \to Y$ is a non-constant holomorphic map between compact connected Riemann surfaces X and Y. Then we have

$$\chi(X) = \deg f \cdot \chi(Y) - b(f)$$

where $\chi(X)$ and $\chi(Y)$ are the **Euler characteristics** of X and Y, and

$$b(f) := \sum_{y \in Y} \sum_{x \in f^{-1}(\{y\})} (\nu_f(x) - 1) = \sum_{y \in Y} \left(\deg f - \operatorname{card} f^{-1}(\{y\})\right)$$

is called the **branching index** of f.

Proof. Omitted. See *B3.2. Geometry of Surfaces.* □

Proposition 3.11

Suppose that C is a non-singular projective curve in $\mathbb{CP}^2$. Then C is compact, connected, and orientable.

Proof. Compactness is trivial. Since C is a Riemann surface, it is orientable (the transition maps are holomorphic and hence are angle-preserving). It remains to show that C is connected. Below is some idea of the proof.

First consider the special curve $x^d + y^d - z^d = 0$. The intersection multiplicity of $y - z = 0$ with the curve is clearly d, so $[0 : 1 : 1]$ is a ramification point of the map $f : C \to \mathbb{CP}^1 \cong \mathbb{C}_\infty$, $[x : y : z] \mapsto y/z$ with ramification index d. This means that $f^{-1}(U)$ is connected for a small neighbourhood U of $1 \in \mathbb{C}$. If there is another connected component C_0, then $1 \neq f(C_0)$ but then f maps C_0 to $\mathbb{C} \cong \mathbb{CP}^1 \setminus \{1\}$ and so f is a constant c. But then the line $y - cz = 0$ divides P but we have assumed that C is nonsingular. So this curve is definitely connected.

Next we shall show that the space of non-singular curves of degree d is path-connected. The condition for non-singularity is the vanishing of a polynomial in the coefficients. If we take curves define by P and Q, then $tP(x, y, z) + (1 - t)Q(x, y, z)$ for $t \in \mathbb{C}$ will be nonsingular, unless a polynomial in t vanishes at a finite number of points or is identically zero. In the latter case, we can replace this path between P and Q by a series of such complex "intervals" for which the singular curves are given by the vanishing of a polynomial in t. Either way, we can avoid a finite number of points in $\mathbb{C}$ by a real path joining P to Q, and so have a path of curves all of which are non-singular.

It remains to show that two non-singular curves are homeomorphic if they are connected by a path. Assuming this, if we start with $P(x, y, z) = x^n + y^n - z^n$, then since $P = 0$ is connected, so is the curve defined by Q.

Suppose that $P(x, y, z, t)$ is a homogeneous polynomial in x, y, z whose coefficients depend smoothly on the real parameter t, so that for each $t \in \mathbb{R}$ the curve $P(x, y, z, t) = 0$ is non-singular. It is easy to check that the subset $P(x, y, z, t) = 0$ of $\mathbb{CP}^2 \times \mathbb{R}$ has the structure of a 3-dimensional (real) smooth manifold M. We can introduce a Riemannian metric on M by embedding $\mathbb{CP}^2 \times \mathbb{R}$ into $\mathbb{R}^N$ for some $N \in \mathbb{N}$.

Now consider the surface $N_c := \{(x, y, z, t) : t = c\} \subseteq M$. We can find a normal vector field $X_c : p \mapsto X(p) \in (\mathrm{T}_p N_c)^\perp$ on N_c. As $\dim(\mathrm{T}_p N_c)^\perp = 1$, we can normalise the vector such that $\langle X_c(p), X_c(p) \rangle = 1$ for all $p \in N_c$. We vary c and get a vector field X on M which is normal to N_c at every point of M. The flow of X gives a local diffeomorphism from $t(0) = c$ to $t(s) = c + s$ for small s. By connectedness this extends for t to all $c \in \mathbb{R}$. □

By the classification of compact connected topological surfaces, we note that a non-singular curve C is determined up to homeomorphism by its Euler characteristic. We have $\chi(C) = 2 - 2g$ for some $g \in \mathbb{N}$. The integer g is called the **genus** of C.

Lemma 3.12. Ramification Index and Intersection Multiplicity

Suppose that C is a non-singular projective curve such that $[0 : 0 : 1] \notin C$. $f : C \to \mathbb{CP}^1$ given by $[x : y : z] \mapsto [x : y]$ is a well-defined meromorphic function. The ramification points of f are those $p \in C$ at which the tangent line T_p at p passes through $[0 : 0 : 1]$. The ramification index $\nu_f(p)$ is the intersection multiplicity $\mathrm{I}_p(C, T_p)$.

Proof. Let $p = [a : b : c] \in C$ be a ramification point. It could be shown that if $c = 0$ and $\partial_z P(a,b,c) \neq 0$, then f is locally the identity map and cannot have a ramification point at p. Suppose that $c \neq 0$. Let C be defined by the homogeneous polynomial P. Without loss of generality assume that $\partial_y P(a,b,c) \neq 0$ and use x as a local coordinate to represent the curve as $y = g(x)$. The map f is then locally $g(x)/x$ if $a \neq 0$ and $x/g(x)$ if $a = 0$. Assume the first case, then $F' = 0$ if and only if $xg'(x) - g(x) = 0$. But $P(x, g(x), 1) = 0$ so

$$\frac{\partial P}{\partial x} + g'(x)\frac{\partial P}{\partial y} = 0$$

At the ramification point $b = g(a) = ag'(a)$, so

$$a\frac{\partial P}{\partial x}(a,b,c) + b\frac{\partial P}{\partial y}(a,b,c) = 0$$

but from Euler's Relation this means that $\partial_z P(a,b,c) = 0$. Hence the tangent line passes through $[0 : 0 : 1]$.

The tangent line T_p is locally $y - b = g'(a)(x - a)$. If $\nu_f(p) = n$, then we have

$$g(x) = g'(a)x + (x-a)^n h(x)$$

where h is holomorphic with $h(a) \neq 0$. Since $P(x, g(x), 1) = 0$, by putting $x = a + t$, we have

$$P(a + t, b + g'(a)t + t^n k(t), 1) = 0$$

Note that $\mathrm{I}_p(C, T_p) = m \in \mathbb{N}$ if and only if $P(a + t, b + g'(a)t, 1)$ is divisible by t^m but not t^{m+1}. We deduce that $\mathrm{I}_p(C, T_p) = n$ if and only if $\nu_f(p) = n$. ☐

Theorem 3.13. Degree-Genus Formula

Suppose that C is a non-singular projective curve of degree d. Then the genus of C is given by

$$g = \frac{1}{2}(d-1)(d-2)$$

Remark. One proof of the theorem is to explicitly construct a non-singular curve D of degree d with genus $g = \frac{1}{2}(d-1)(d-2)$. And then use the idea of the proof in Proposition 3.11, that is, connect C and D by a path in the space of non-singular curves of degree d and argue that they are diffeomorphic. We take another proof here, which uses the Riemann-Hurwitz Formula above.

Proof. If $d = 1$, then we already know that C is homeomorphic to the Riemann surface, and hence $g = 0$. Now suppose that $d \geqslant 2$.

We can choose coordinates such that $[0 : 0 : 1] \notin C$. Consider the meromorphic function $f : C \to \mathbb{CP}^1$ in the previous lemma. By the lemma, for $p \in C$, $\nu_f(p) > 2$ if and only if $\mathrm{I}_p(C, T_p) > 2$, if and only if p is an inflection point of C. We know that C has finitely many inflection points. So we can apply a projective transformation such that $[0 : 0 : 1]$ does not lie on any tangent line to C at the inflection points. Therefore $\nu_f(p) = 2$ at every ramification point of f.

Next we shall show that f has exactly $d(d-1)$ ramification points. Let C be defined by P and D be defined by $\partial_z P$. From the proof of the above lemma, we note that p is a ramification point of f if and only if $p \in C \cap D$. We need to verify that the conditions in Corollary 2.11 of Bézout's Theorem are satisfied.

Let $p = [a_0 : a_1 : a_2] \in C \cap D$. Now $\left(\partial_z\partial_x P, \partial_z\partial_y P, \partial_z^2 P\right)$ is not identically zero at p because this would make the Hessian of C vanish and we know that p is not an inflection point. This shows that D is nonsingular here. Suppose that the tangents of C and D coincide then $\left(\partial_z\partial_x P, \partial_z\partial_y P, \partial_z^2 P\right)$ is a multiple of $\left(\partial_x P, \partial_y P, \partial_z P\right)$. As in the proof of Proposition 2.13, we use the quadratic form B defined by the matrix of partial derivatives $\partial_i\partial_j P$. Then $B(a, a) = 0 = B(a, \alpha)$ where the tangent line joins p and $q := [\alpha_0 : \alpha_1 : \alpha_2]$ Put $v = (0, 0, 1)$. By the Euler's Relation

$$a_0\partial_z\partial_x P + a_1\partial_z\partial_y P + a_2\partial_z^2 P = (n-1)\partial_z P = 0$$

since $\partial_z P(a_0, a_1, a_2) = 0$. This gives $B(a, v) = 0$. Moreover since $\partial_z^2(a_0, a_1, a_2) = \lambda\partial_z P(a_0, a_1, a_2) = 0$, we have $B(v, v) = 0$. Since p is not an inflection point, $\det B \neq 0$ so from

$$0 = B(a, a) = B(a, \alpha) = B(a, v)$$

We deduce that $v = \mu a + \nu\alpha$. But then

$$0 = B(v, v) = \nu^2 B(\alpha, \alpha)$$

and, as in the proof of Proposition 2.13, this gives $\det B = 0$ unless $\nu = 0$. But then $p = [0 : 0 : 1]$ which we have specifically excluded. We conclude that the tangents are distinct and the conditions for Corollary 2.11 of Bézout's Theorem hold. Since C has degree d and D has degree $d - 1$, we deduce that $\text{card}(C \cap D) = d(d - 1)$.

Finally, we note that f has $d(d-1)$ ramification points, each of which has ramification index 2. By Riemann-Hurwitz Formula,

$$2 - 2g = \chi(X) = d\chi(\mathbb{CP}^1) - b(f) = 2d - d(d - 1)$$

We deduce that $g = \frac{1}{2}(d - 1)(d - 2)$. □

As predicted by the degree-genus formula, a cubic curve with degree 3 is homeomorphic to a torus with genus 1. We give an explicit construction to this fact.

Example 3.14. Weierstrass $\wp$-Function and Cubic Curve

We wish to construct an explicit homeomorphism from a torus X to a cubic curve C on $\mathbb{CP}^2$.

Proof. Fix $\omega_1, \omega_2 \in \mathbb{R}$ which are linearly independent over $\mathbb{C}$. Consider the lattice $\Lambda := \{m\omega_1 + n\omega_2 : m, n \in \mathbb{Z}\} \subseteq \mathbb{C}$. We define the **Weierstrass $\wp$-function** to be

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right)$$

From Complex Analysis we can show that $\wp$ is meromorphic and is doubly-periodic with respect to Λ. So we can take $\wp$ as a meromorphic function $\wp : \mathbb{C}/\Lambda \to \mathbb{CP}^1$. It has 4 ramification points: 0, $\omega_1.2$, $\omega_2/2$, and $(\omega_1 + \omega_2)/2$. We put

$$e_1 := \wp\Big(\frac{\omega_1}{2}\Big), \qquad e_2 := \wp\Big(\frac{\omega_2}{2}\Big), \qquad e_3 := \wp\left(\frac{\omega_1 + \omega_2}{2}\right)$$

We can prove by Complex Analysis that $\wp'(z)$ is a cubic polynomial such that

$$\wp'(z)^2 = 4(z - e_1)(z - e_2)(z - e_3)$$

which corresponds to an affine cubic curve in $\mathbb{C}^2$:

$$y^2 = 4(x - e_1)(x - e_2)(x - e_3)$$

Now $\wp : \mathbb{C}/\Lambda \to \mathbb{CP}^1$ is surjective by Riemann-Hurwitz Formula. Moreover, since $\wp(-z) = \wp(z), \wp'(-z) = -\wp'(z)$ so for each value of x there is a z for both values of y.

Therefore $z \mapsto [\wp(z), \wp'(z), 1]$ defines a homeomorphism from $\mathbb{C}/\Lambda$ to C, where $\mathbb{C}/\Lambda$ is topologically a torus by construction. □

4 Riemann-Roch Theorem

4.1 Divisors and Differentials

Definition 4.1. Divisors

Suppose that C is an algebraic curve in $\mathbb{CP}^2$. A divisor D on C is a formal sum

$$D = \sum_{p \in C} n_p p$$

where $n_p \in \mathbb{Z}$ for each $p \in C$ and $n_p = 0$ for all but finitely many points.

All divisors on C form a free Abelian group on the set C.

The degree of D is defined by

$$\deg D := \sum_{p \in C} n_p$$

Suppose that $f : C \to \mathbb{CP}^1$ is a non-zero meromorphic function on the algebraic curve C. Suppose that f has zeros at $p_1, ..., p_k$ with multiplicities $m_1, ..., m_k$ and poles at $q_1, ..., q_\ell$ with multiplicities $n_1, ..., n_\ell$. Then we define the **divisor of** f to be

$$\operatorname{div}(f) := \sum_{i=1}^{k} m_i q_i - \sum_{i=1}^{\ell} n_i q_i$$

We note that $\operatorname{div}(f)$ determines f up to a scalar multiple. Because if $\operatorname{div}(f) = \operatorname{div}(\widetilde{f})$, then $f/\widetilde{f}$ is a holomorphic function on C. Since C is compact, by Liouville's Theorem $f/\widetilde{f}$ is constant.

We note that if $f : C \to \mathbb{CP}^1$ is meromorphic, then $\deg \operatorname{div}(f) = 0$, because the number of zeros (counting multiplicity) is $f^{-1}(\{0\})$ and the number of poles (counting multiplicity) is $f^{-1}(\{\infty\})$. The two numbers are equal to the degree of f.

Definition 4.2. Effective Divisors, Principal Divisors, Linear Equivalence

A divisor $D = \sum_p n_p p$ is called an effective divisor, if $n_p \geqslant 0$ for all p.

We write $D \geqslant D'$ if $D - D'$ is an effective divisor.

A divisor D is called a principal divisor, if $D = \operatorname{div}(f)$ for some meromorphic function $f : C \to \mathbb{CP}^1$.

Divisors D, D' are said to be linearly equivalent, if $D - D'$ is a principal divisor. We write $D \sim D'$.

Remark. Since $\deg \operatorname{div}(f) = 0$, we have $\deg D = \deg D'$ if $D \sim D'$.

Definition 4.3. Vector Space $\mathcal{L}(D)$

Let D be a divisor on the algebraic curve C. We define $\mathcal{L}(D)$ to be the set of meromorphic functions f such that $\operatorname{div}(f) + D \geqslant 0$ together with the zero function. Then $\mathcal{L}(D)$ is a finite-dimensional vector space. We denote $\ell(D) := \dim \mathcal{L}(D)$.

Proposition 4.4. Properties of $\mathcal{L}(D)$

1. $\mathcal{L}(D)$ is a finite-dimensional vector space.
2. If $\deg D < 0$, then $\mathcal{L}(D) = \{0\}$.
3. If $D \sim D'$, then $\ell(D) = \ell(D')$.
4. The projective space $\mathbb{P}(\mathcal{L}(D))$ is in bijective correspondence with the effective divisors equivalent to D.

Proof. 1. We write D as

$$D = \sum_{i=1}^{k} m_i p_i - \sum_{i=1}^{\ell} n_i q_i$$

where $p_1, ..., p_k, q_1, ..., q_\ell \in C$ and $m_1, ..., m_k, n_1, ..., n_\ell \in \mathbb{Z}_+$. Then $\mathcal{L}(D)$ is the set of meromorphic functions f which have poles of order $\leqslant m_i$ at p_i, and zeros of order $\geqslant n_i$ at q_i (together with the zero function). This clearly forms a vector space.

To show that $\mathcal{L}(D)$ is finite-dimensional, we write $f \in \mathcal{L}(D)$ at each pole p_i in local coordinate z:

$$f(z) = \sum_{j=1}^{m_i} a_{i,j} z^{-j} + h_i(z)$$

where h_i is a holomorphic function in an open neighbourhood of $z = 0$. This defines a map $\varphi_i : \mathcal{L}(D) \to \mathbb{C}^{m_i}$, $f \mapsto (a_{i,1}, ..., a_{i,m_i})$ for each pole p_i. We have $\varphi : \mathcal{L}(D) \to \mathbb{C}^{m_1} \times \cdots \times \mathbb{C}^{m_k}$ given by $\varphi = (\varphi_1, ..., \varphi_k)$. Note that

$$f \in \ker \varphi = \bigcap_{i=1}^{k} \ker \varphi_i \implies f \text{ is holomorphic on } C \implies f = \text{const}$$

Hence $\dim \ker \varphi \leqslant 1$. By Rank-Nullity Theorem, we deduce that $\dim \mathcal{L}(D) < \infty$.

2. If $\operatorname{div}(f) + D \geqslant 0$, then $0 \leqslant \deg \operatorname{div}(f) + \deg D = \deg D$.
3. If $D = D' + \operatorname{div}(g)$ for some meromorphic function $g : C \to \mathbb{CP}^1$, then $f \mapsto fg$ defines an isomorphism from $\mathcal{L}(D)$ to $\mathcal{L}(D')$.
4. From the discussion below Definition 4.1, $\mathbb{P}(\mathcal{L}(D))$ is in bijective correspondence with the set of divisors $\{\operatorname{div}(f) : f \in \mathcal{L}(D) \setminus \{0\}\}$.

   For each $f \in \mathcal{L}(D)\setminus\{0\}$, by definition $\operatorname{div}(f)+D$ is an effective divisor and is linearly equivalent to D. Conversely, if $D \sim D'$ and D' is effective, then there exists a meromorphic function f such that $\operatorname{div}(f) + D = D' \geqslant 0$. Hence $f \in \mathcal{L}(D)$. □

Definition 4.5. Meromorphic Differentials

Suppose that X is a Riemann surface. Let f, g be meromorphic functions on X. The meromorphic differential $f\mathrm{d}g$ is an equivalence class such that

$$f\mathrm{d}g = \widetilde{f}\mathrm{d}\widetilde{g} \iff \left(f \circ \varphi^{-1}\right)\left(g \circ \varphi^{-1}\right)' = (\widetilde{f} \circ \varphi^{-1})\left(\widetilde{g} \circ \varphi^{-1}\right)'$$

for any coordinate chart (U, φ) on X.

Remark. Alternatively, we can define a meromorphic differential η on C to be a collection of meromorphic functions $\{\eta_i : \varphi_i(U_i) \to \mathbb{CP}^1 \mid i \in I\}$ on the open subsets of $\mathbb{C}$, such that for $x \in U_i \cap U_j$,

$$\eta_i \circ \varphi_i(x) = \eta_j \circ \varphi_j(x) \cdot (\varphi_j \circ \varphi_i^{-1})'(\varphi_i(x))$$

In such sense, we have $f\mathrm{d}g = \eta$, where

$$\eta_i = \left(f \circ \varphi^{-1}\right)\left(g \circ \varphi^{-1}\right)$$

From the definition, we see immediately that the integral of meromorphic differential along a path γ is well-defined:

$$\int_\gamma \eta := \int_\gamma \eta_i$$

For two meromorphic differentials η and ζ, where ζ is not identically zero, the ratio η/ζ gives a well-defined meromorphic function $f : X \to \mathbb{CP}^1$ such that

$$f(x) = \frac{\eta \circ \varphi_U(x)}{\zeta \circ \varphi_U(x)}$$

for any coordinate chart (U, φ_U) with $x \in U$. We write $\eta = f\zeta$ in this case.

Definition 4.6. Canonical Divisors

Suppose that $\omega = f\mathrm{d}g$ is a non-zero meromorphic differential on the algebraic curve C. Then we define the zeros and poles of ω to be those of $\left(f \circ \varphi^{-1}\right)\left(g \circ \varphi^{-1}\right)'$, which are well-defined. So we can define the divisor $\operatorname{div}(\omega)$ just as meromorphic functions.

A divisor κ is called a canonical divisor, if $\kappa \sim \operatorname{div}(\omega)$.

We note that, if η and ζ are meromorphic differentials, let $f := \eta/\zeta$ and then

$$\operatorname{div}(\eta) = \operatorname{div}(f\zeta) = \operatorname{div}(f) + \operatorname{div}(\zeta) \sim \operatorname{div}(\zeta)$$

So all canonical divisors are linearly equivalent and have the same degree.

Proposition 4.7. Degree of Canonical Divisor

Suppose that C is a non-singular projective curve on $\mathbb{CP}^2$ of genus g. Then the degree of a canonical divisor κ is

$$\deg \kappa = 2g - 2$$

Proof. Choose coordinates such that $[0:0:1] \notin C$. Let C be defined by the homogeneous polynomial P. Then $\partial_z P$ vanishes at finitely many points. By applying a projective transformation we assume that if $P(a,b,c) = \partial_z P(a,b,c) = 0$ then $b \neq 0$.

Consider the meromorphic function $f : C \to \mathbb{CP}^1$, $[x:y:z] \mapsto [x:y]$. We shall count the zeros and poles of the meromorphic differential $\mathrm{d}f$. At a point $[a:b:c] \in C$:

- If $\partial_z P(a,b,c) \neq 0$ and $b \neq 0$, we take $w = x/y$ as a holomorphic coordinate on $\mathbb{CP}^1$. Then $\mathrm{d}f = \mathrm{d}w$ has no zeros or poles.
- If $b = 0$, then $a \neq 0$ and $\partial_z P(a,b,c) \neq 0$. Take $w = y/x$ as a holomorphic coordinate on $\mathbb{CP}^1$. $\mathrm{d}f = -w^{-2}\mathrm{d}w$ has a pole of order 2. Note that $y = 0$ is nonwhere tangent to C. By Corollary 2.11 of Bézout's Theorem, there are exactly d such points on C.
- If $\partial_z P(a,b,c) = 0$, then $[a:b:c]$ is a ramification point of C. By the proof of degree-genus formula, there are $d(d-1)$ such points, each with ramification index 2. Therefore $\mathrm{d}f$ has exactly $d(d-1)$ simple zeros.

We deduce that $\mathrm{d}f$ has $d(d-1)$ simple zeros and d double poles. By degree-genus formula,

$$\deg \kappa = \deg \operatorname{div}(\mathrm{d}f) = d(d-1) - 2d = 2g - 2$$ □

4.2 Proof of Riemann-Roch Theorem

In this subsection, we fix C to be a non-singular projective curve in $\mathbb{CP}^2$ of genus g. We fix κ to be a canonical divisor on C.

Definition 4.8. Divisor Class H

Suppose that L is a line in $\mathbb{CP}^2$. We consider the divisor defined by

$$H = \sum_{p \in C \cap L} \mathrm{I}_p(C, L)p$$

By Bézout' Theorem we know that $\deg H$ is the degree of the curve C.

Note that if $L : ax + by + cz = 0$ and $L' : a'x + b'y + c'z = 0$ are two lines, then

$$f = \frac{ax + by + cz}{a'x + b'y + c'z}$$

is a meromorphic function. So the divisor H of the line L is linearly equivalent to the divisor H' of L'.

First we investigate the structure of $\mathcal{L}(mH)$.

Proposition 4.9. Dimensional of $\mathcal{L}(mH)$

Let the divisor H given by the above definition. We have

$$\ell(mH) \geqslant \deg(mH) - g + 1$$

Proof. Suppose that C has degree d. Let C be defined by $P(x, y, z) = 0$ and L by $R(x, y, z) = 0$. If $Q(x, y, z)$ is any homogeneous polynomial of degree m then

$$\frac{Q(x, y, z)}{R(x, y, z)^m}$$

defines a meromorphic function f on C such that $\mathrm{div}(f) + mH \geqslant 0$, that is, an element of $\mathcal{L}(mH)$. Moreover, two such polynomials define the same function on C if and only if their difference is divisible by $P(x, y, z)$. Let $\mathbb{C}_k[x, y, z]$ denotes the space of homogeneous polynomials of degree k in x, y, z. Then

$$\begin{aligned}\ell(mH) &\geqslant \dim\left(\mathbb{C}_m[x, y, z]/P(x, y, z)\mathbb{C}_{m-d}[x, y, z]\right)\\ &= \dim \mathbb{C}_m[x, y, z] - \dim \mathbb{C}_{m-d}[x, y, z]\\ &= \frac{1}{2}(m+1)(m+2) - \frac{1}{2}(m-d+1)(m-d+2) = md + \frac{1}{2}d(3-d)\end{aligned}$$

By degree-genus formula, $d(3-d)/2 = 1 - g$. And we know that $md = \deg(mH)$. We deduce that

$$\ell(mH) \geqslant \deg(mH) - g + 1$$

□

Lemma 4.10

A meromorphic differential on a compact Riemann surface cannot have a single simple pole.

Sketch of Proof.

Suppose for a contradiction that p is the simple pole of the differential ω. It has non-zero residue and so taking a coordinate neighbourhood of p, and surrounding it with a small contour Γ, we have

$$\int_\Gamma \omega \neq 0$$

Now triangulate C such that each triangle lies in a coordinate neighbourhod and p lies in the interior of one triangle, Δ_0. By Cauchy's theorem the integral of ω around each triangle Δ_i $(i \neq 0)$ is zero and the integrations along adjacent edges of different triangles cancel (like the proof of the Gauss-Bonnet Theorem). But then the integral around Δ_0 vanishes which is a contradiction. □

Lemma 4.11

Let D be a divisor on C. For any point $p \in C$,

$$0 \leqslant \ell(D+p) - \ell(\kappa - D - p) - \ell(D) + \ell(\kappa - D) \leqslant 1$$

Proof. Firstly $f \in \mathcal{L}(D)$ if and only if $\mathrm{div}(f) + D \geqslant 0$ which clearly implies that $\mathrm{div}(f) + D + p \geqslant 0$, so that $\mathcal{L}(D) \subseteq \mathcal{L}(D+p)$ and

$$\ell(D+p) \geqslant \ell(D).$$

Suppose that

$$D = \sum_{i=1}^{k} m_i p_i - \sum_{i=1}^{\ell} n_i q_i$$

Take $f \in \mathcal{L}(D+p)$. If p is not one of the p_i or q_j then f has at most a simple pole at p. The condition for f to lie in $\mathcal{L}(D)$ is thus a single linear condition, the vanishingof the coefficient of $(z-a)^{-1}$. If $p = p_i$, then in the Laurent expansion around p we have:

$$f(z) = \frac{a_{m_i+1}}{(z-z_i)^{m_i+1}} + \mathrm{o}\Big((z-z_i)^{-(m_i+1)}\Big)$$

and here for f to lie in $\mathcal{L}(D)$ is the vanishing of a_{m_i+1}. If $p = q_i$ then f has a zero of order at least $(n_i - 1)$ at q_i and to lie in $\mathcal{L}(D)$, must have a zero of order n_i. This is again one linear condition. In all cases we see that

$$\ell(D+p) \leqslant \ell(D) + 1$$

Applying this to D and $\kappa - D - p$, we see that the lemma holds so long as we can eliminate the case

$$\ell(D+p) - \ell(D) = 1 \text{ and } \ell(\kappa - D) - \ell(\kappa - D - p) = 1$$

Suppose for a contradiction that this holds. Then there is a meromorphic function f with $\operatorname{div}(f) + D + p \geqslant 0$ but $\neg \operatorname{div}(f) + D \geqslant 0$, so $-p$ is the only negative term in $\operatorname{div}(f) + D$. Similarly there is g such that $\operatorname{div}(g) + \kappa - D \geqslant 0$ but $\operatorname{div}(g) + \kappa - D - p \ngeqslant 0$ which means that p does not appear in the divisor $\operatorname{div}(g) + \kappa - D$. Thus in

$$0 \leqslant \operatorname{div}(f) + D + p + \operatorname{div}(g) + \kappa - D = \operatorname{div}(fg) + \kappa + p$$

the positive element p is not cancelled.

But κ is the divisor of a meromorphic differential ω, which means that $fg\omega$ is a meromorphic differential with a single simple pole at p. This is impossible by the previous lemma. ☐

Theorem 4.12. Riemann-Roch Theorem

Let D be a divisor on C. Then

$$\ell(D) - \ell(\kappa - D) = \deg D - g + 1$$

Proof. We shall show that

$$\ell(D) - \ell(\kappa - D) \geqslant \deg D - g + 1$$

Suppose that

$$D = \sum_{i=1}^{k} m_i p_i - \sum_{i=1}^{\ell} n_i q_i$$

We choose lines $a_i x + b_i y + c_i z = 0$ that pass through the points p_i. Then the divisor (as in Definition 4.8) of

$$\prod_{i=1}^{k} (a_i x + b_i y + c_i z)^{m_i}$$

is of the form

$$\sum_{i=1}^{k} m_i p_i + \sum_{j=1}^{N} r_j = D + \sum_{i=1}^{s} x_i$$

where $x_1, ..., x_s \in C$. Furthmore,

$$D + \sum_{i=1}^{s} x_i \sim m_0 H$$

where $m_0 := \sum_{i=1}^{k} m_i$. By adding more points to $D + \sum_{i=1}^{s} x_i$, we can choose sufficiently large $m > m_0$ such that

$$\deg(\kappa - mH) = 2g - 2 - m \deg H < 0 \quad \text{and} \quad D + \sum_{i=1}^{r} x_i \sim mH$$

By Proposition 4.9, we have

$$\ell\left(D + \sum_{i=1}^{r} x_i\right) - \ell\left(\kappa - D - \sum_{i=1}^{r} x_i\right) = \ell(mH) - \ell(\kappa - mH) = \ell(mH)$$

$$\geqslant \deg(mH) - g + 1 = \deg\left(D + \sum_{i=1}^{r} x_i\right) - g + 1 = \deg D + r - g + 1$$

From the previous lemma,

$$\ell\left(D + \sum_{i=1}^{r} x_i\right) - \ell\left(\kappa - D - \sum_{i=1}^{r} x_i\right) \leqslant \ell\left(D + \sum_{i=1}^{r-1} x_i\right) - \ell\left(\kappa - D - \sum_{i=1}^{r-1} x_i\right) + 1$$

Repeating the process r times, we have

$$\ell\left(D + \sum_{i=1}^{r} x_i\right) - \ell\left(\kappa - D - \sum_{i=1}^{r} x_i\right) \leqslant \ell(D) - \ell(\kappa - D) + r$$

Combining the above inequalities,

$$\ell(D) - \ell(\kappa - D) + r \geqslant \deg D + r - g + 1$$

which proves our claim.

Now it remains to show the reversed inequality. We replace D by $\kappa - D$ in the above equation:

$$\ell(\kappa - D) - \ell(D) \geqslant \deg(\kappa - D) - g + 1 = 2g - 2 - \deg D - g + 1 = -\deg D + g - 1$$

We conclude that

$$\ell(D) - \ell(\kappa - D) = \deg D - g + 1$$

□

4.3 Applications

Definition 4.13. Holomorphic Differentials

A meromorphic differential on a Riemann surface is called a holomorphic differential, if it has no poles.

Corollary 4.14. Holomorphic Differentials and Genus

The vector space of holomorphic differentials on a non-singular projective curve in $\mathbb{CP}^2$ has dimension g, the genus of the curve.

Proof. The vector space is $\mathcal{L}(\kappa)$, where κ is a canonical divisor. We take $D = 0$ and use Riemann-Roch Theorem:

$$\ell(0) - \ell(\kappa) = 1 - g$$

We note that $\mathcal{L}(0)$ is the space of holomorphic functions on C, which are constants and hence 1-dimensional. It follows that $\ell(\kappa) = g$. □

Remark. We can actually write down these differentials. First consider the affine part of the curve given by $P(x, y, 1) = 0$. Then x is a local coordinate where $\partial_y P \neq 0$ so consider the differential

$$\omega = \frac{\mathrm{d}x}{\partial_y P(x, y, 1)}$$

At first sight this seems to have poles where the denominator vanishes but this is just where the role of x as a local coordinate breaks down. Since the curve is nonsingular, at such points $\partial_x P \neq 0$ and from the chain rule, on the curve $\partial_x P \mathrm{d}x + \partial_y P \mathrm{d}y = 0$, so that ω can also be written, using y as a coordinate, as

$$\omega = -\frac{\mathrm{d}y}{\partial_x P(x, y, 1)}$$

This form has no poles and no zeros in the affine part of the curve. Now look at C near $z = 0$. We have

$$\frac{\mathrm{d}(x/z)}{\partial_y P(x/z, y/z, 1)} = \frac{\mathrm{d}(x/z)}{\partial_y P(1, y/x, z/x)(x/z)^{n-1}} = \frac{-\mathrm{d}(z/x)(x/z)^2}{\partial_y P(1, y/x, z/x)(x/z)^{n-1}}$$

and so

$$\omega = \frac{-z^{n-3}\mathrm{d}z}{\partial_y P(1,y,z)}$$

and has a zero of order $n-3$ where $z=0$. This tells us that $\kappa \sim (n-3)H$, and so we can obtain a holomorphic differential by writing

$$\frac{Q(x,y,1)\mathrm{d}x}{\partial_y P(x,y,1)}$$

for a homogeneous polynomial $Q(x,y,z)$ of degree $n-3$. The dimension of the space of polynomials of this degree is $(n-2)(n-1)/2$ which is g from the degree-genus formula. Riemann-Roch Theorem therefore tells us that *every holomorphic differential is obtained from a polynomial this way.*

Theorem 4.15. Chow's (周炜良) Theorem

The meromorphic functions on a non-singular projective curve in $\mathbb{CP}^2$ are rational.

Proof. Let C be a non-singular projective curve of degree d (and genus g). We consider the vector space $\mathcal{L}(mH)$ with $m > d-3$. Note that

$$\deg(\kappa - mH) = 2g-2-md = d(d-m-3) < 0$$

By Riemann-Roch Theorem, we have

$$\ell(mH) = \deg(mH) - g + 1$$

But by the proof of Proposition 4.9, we have seen that the subspace of $\mathcal{L}(mH)$ in which every meromorphic function is of the form

$$\frac{Q(x,y,z)}{R(x,y,z)^m}$$

has dimension $\deg(mH) - g + 1$. Therefore every meromorphic function in $\mathcal{L}(mH)$ is given by that form.

Now let f be a meromorphic function on C. Take lines L_i that pass through the poles p_i of f. We have, for some $m \in \mathbb{N}$,

$$f + H_1 + \cdots + H_m \geqslant 0$$

The same argument shows that f is a rational function. $\square$

Theorem 4.16. Abel's Theorem on the Group Law of Cubics

Let C be a non-singular cubic curve on $\mathbb{CP}^2$. Let e be an inflection point on C. There is an unique additive group structure on C such that e is the identity and $p_1+p_2+p_3=0$ if and only if p_1,p_2,p_3 are the three points of intersection (counting multiplicity) of C with a line.

Proof. The addition of divisors is commutative and associative. This is also true of their linear equivalence classes since if $p = p' + (f), q = q' + (g)$ then $p+q = p'+q'+(fg)$. We noted earlier that p is linearly equivalent to q only if $g=0$, so for any curve C of genus $g>0$ the equivalence class of p determines p uniquely. We could also use Riemann-Roch Theorem: if $D=p$ then since $\deg D = 1 > 0 = \deg\kappa$ we have $\ell(\kappa - D) = 0$ and

$$\ell(D) = 1+1-1 = 1$$

For the cubic, with $g=1$, we take an inflection point e and map $p \mapsto [p-e]$ into the group of equivalence classes of degree zero divisors. From the above, this is injective. Moroever $[e-e]=[0]$ is clearly an identity.

Then $p+q$ maps to $[p+q-2e]$ and we want to show that this is of the form $[s-e]$. The line $ax+by+cz=0$ joining p and q (or the tangent at p if $p=q$) meets the degree 3 curve in a third point r by Bézout's theorem. Let $a'x+b'y+c'z=0$ be the tangent at e, then the divisor of its intersection with C is $3e$ since e is an inflection point. If $f = (ax+by+cz)/(a'x+b'y+c'z)$, the divisor of f is

$$(f) = p+q+r-3e$$

which shows that $[p+q-2e] = [e-r]$. Now take $q=e$ in this expression, then $[p-e]=[e-p']$ for some p' which we call the inverse of p. In general then

$$[p + q - 2e] = [e - r] = [r' - e]$$

as required, proving that C is closed under the addition law. $\square$

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