Peize Liu
St. Peter's College
University of Oxford

Problem Sheet 5

Groups

B2: Symmetry & Relativity

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Question 1

Show that the Lorentz transformations in a single spatial direction form a group.

Proof. For simplicity we consider $1+1$ spacetime. A Lorentz boost is given by

$$L = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$$

or, using the rapidity,

$$L(\eta) = \begin{pmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{pmatrix}$$

The set of $1+1$ Lorentz transformations $\mathrm{SO}(1,1)^+$ is a subgroup of $\mathrm{GL}_2(\mathbb{R})$:

- Identity: For $\eta = 0$, $L(0) = \mathrm{diag}(1,1) = I$ is the identity in $\mathrm{SO}(1,1)^+$;
- Closed under composition: For $\eta_1, \eta_2 \in \mathbb{R}$,

$$\begin{aligned} L(\eta_1)L(\eta_2) &= \begin{pmatrix} \cosh\eta_1 & -\sinh\eta_1 \\ -\sinh\eta_1 & \cosh\eta_1 \end{pmatrix}\begin{pmatrix} \cosh\eta_2 & -\sinh\eta_2 \\ -\sinh\eta_2 & \cosh\eta_2 \end{pmatrix} \\ &= \begin{pmatrix} \cosh\eta_1\cosh\eta_2 + \sinh\eta_1\sinh\eta_2 & -(\cosh\eta_1\sinh\eta_2 + \sinh\eta_1\cosh\eta_2) \\ -(\cosh\eta_1\sinh\eta_2 + \sinh\eta_1\cosh\eta_2) & \cosh\eta_1\cosh\eta_2 + \sinh\eta_1\sinh\eta_2 \end{pmatrix} \\ &= \begin{pmatrix} \cosh(\eta_1+\eta_2) & -\sinh(\eta_1+\eta_2) \\ -\sinh(\eta_1+\eta_2) & \cosh(\eta_1+\eta_2) \end{pmatrix} = L(\eta_1+\eta_2) \end{aligned}$$

- Closed under inversion: For $\eta \in \mathbb{R}$:

$$L(\eta)^{-1} = \begin{pmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{pmatrix}^{-1} = \frac{1}{\cosh^2\eta - \sinh^2\eta}\begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} = \begin{pmatrix} \cosh(-\eta) & -\sinh(-\eta) \\ -\sinh(-\eta) & \cosh(-\eta) \end{pmatrix} = L(-\eta)$$

□

Question 2

Show that $e^{in\theta}$, with θ a constant, is a representation of the group of integers n under the addition operator. If $\theta = \pi/N$, how many elements does the representation have, and in what sense is it still a representation of the infinite group of integers?

Proof. The map $\rho : \mathbb{Z} \to \mathrm{GL}(\mathbb{C}) = \mathbb{C}^\times$ given by $\rho(n) = \mathrm{e}^{\mathrm{i}n\theta}$ is a group homomorphism, because for any $n, m \in \mathbb{Z}$,

$$\rho(n+m) = \mathrm{e}^{\mathrm{i}(m+n)\theta} = \mathrm{e}^{\mathrm{i}n\theta}\,\mathrm{e}^{\mathrm{i}m\theta} = \rho(n)\rho(m)$$

Hence it is an representation of the group $\mathbb{Z}$.

If $\theta = \pi/N$, where $N \in \mathbb{Z}$, then $\mathrm{e}^{\mathrm{i}2N\theta} = \mathrm{e}^{2\pi\mathrm{i}} = 1$ and $\rho(\mathbb{Z})$ has distinct elements $1, \mathrm{e}^{\mathrm{i}\theta}, \ldots, \mathrm{e}^{\mathrm{i}(2N-1)\theta}$. The image of the representation ρ has $2N$ elements.

I am not sure what the final part of the question is asking. If $\pi/\theta \in \mathbb{Q}$, then ρ has a non-trivial kernel and hence is not faithful. In fact it descends to a faithful representation of $\mathbb{Z}/\ker\rho$. If $\pi/\theta \notin \mathbb{Q}$, then $\rho : n \mapsto \mathrm{e}^{\mathrm{i}n\theta}$ is a faithful representation of $\mathbb{Z}$ in $\mathbb{C}$. □

Question 3

Write down a set of 3×3 matrices to represent the permutation group on three objects, such that the action of swapping the second and third objects is the matrix

$$(D_{132})^i_{\ j} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Show that this matrix representation is reducible by the following steps.

(i) Find a common eigenvector for all the D^i_j matrices.

(ii) Write down a suitable similarity transformation matrix S^i_j, such that $(D')^i_j = S^i_m D^m_n (S^{-1})^n_j$ with the common eigenvector becoming (1,0,0) in the new basis.

(iii) Show that the transformation matrices in the new basis take on block-diagonal form.

Proof. Consider the set $X = \{e_1, e_2, e_3\}$ and the free $\mathbb{C}$-vector space $\mathbb{C}^{\oplus X} \cong \mathbb{C}^3$. The permutation action of S_3 on X induces a left $\mathbb{C}[S_3]$-module structure on $\mathbb{C}^3$. The representation $\rho : S_3 \to \mathrm{GL}_3(\mathbb{C})$ afforded by the module is called the *permutation representation* of S_3. Now $\rho(S_3)$ is the set of the following matrices:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\quad\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\quad\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\quad\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\quad\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\quad\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

It is easy to check that $\mathbb{C}^3$ has sub-$\mathbb{C}[S_3]$-modules U, W:

$$U := \langle e_1 + e_2 + e_3 \rangle \qquad W := \left\{ \sum_{i=1}^{3} a_i e_i : \sum_{i=1}^{3} a_i = 0 \right\} = \langle e_1 - e_2,\ e_2 - e_3 \rangle$$

such that $\mathbb{C}^3 = U \oplus W$. Hence $\mathbb{C}^3$ is reducible.

(i) The common eigenvector is $e_1 + e_2 + e_3$, because for all $\sigma \in S_3$,

$$\rho(\sigma)(e_1 + e_2 + e_3) = \sigma \cdot (e_1 + e_2 + e_3) = e_{\sigma(1)} + e_{\sigma(2)} + e_{\sigma(3)} = e_1 + e_2 + e_3$$

as σ is a bijection on $\{1, 2, 3\}$.

(ii) It is clear that $\{e_1 + e_2 + e_3, e_1 - e_2, e_2 - e_3\}$ is a new basis of $\mathbb{C}^3$. The change-of-basis matrix is given by

$$S^{-1} = \begin{pmatrix}1&1&0\\1&-1&1\\1&0&-1\end{pmatrix}$$

which implies that

$$S = \begin{pmatrix}1/3&1/3&1/3\\2/3&-1/3&-1/3\\1/3&1/3&-2/3\end{pmatrix}$$

and that $S(1,1,1)^\intercal = (1,0,0)^\intercal$.

(iii) This is a theorem in linear algebra:

Suppose that $V = U \oplus W$, where U and W are T-stable subspaces of V. Take a basis B of U and C of W. Then $B \cup C$ is a basis of V with respect to which T takes the block diagonal form.

Now for any $\sigma \in S_3$, U and W are $\rho(\sigma)$-stable. So all $\rho(\sigma)$ take block diagonal form with respect to the basis $\{e_1 + e_2 + e_3, e_1 - e_2, e_2 - e_3\}$. □

Question 4

Show that the following matrix generates a rotation around the x axis

$$(J_1)^\mu_\nu = \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-i\\0&0&i&0\end{pmatrix}$$

using the exponential $R(\theta) = e^{-i\theta J_1}$ Show that the matrix

$$(K_1)^\mu_\nu = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

generates a boost in the x direction with $\Lambda(\eta) = e^{-i\eta K_1}$.

Write the matrix form of the generator K_2 for infinitesimal boosts along the y axis. Multiply an infinitesimal boost along x by another along y. What does the form of the matrix indicate about whether non-aligned Lorentz transformations can form a group?

Proof. Recall that the exponential of a matrix is defined by

$$\mathrm{e}^A := \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

where the RHS converges absolutely for any $A \in M_n(\mathbb{C})$. For diagonalisable matrix $A = PDP^{-1}$, where $D = \operatorname{diag}(\lambda_1, \lambda_2)$, we have $A^n = PD^nP^{-1}$, and hence

$$\mathrm{e}^A = \sum_{n=0}^{\infty} \frac{1}{n!} PD^nP^{-1} = P\left(\sum_{n=0}^{\infty} \frac{D^n}{n!}\right)P^{-1} = P\operatorname{diag}\left(\sum_{n=0}^{\infty} \frac{\lambda_1^n}{n!}, \sum_{n=0}^{\infty} \frac{\lambda_2^n}{n!}\right)P^{-1} = P\operatorname{diag}(\mathrm{e}^{\lambda_1}, \mathrm{e}^{\lambda_2})P^{-1}$$

First we diagonalise $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$: The characteristic polynomial is $x^2 + 1$, so the eigenvalues are $\pm\mathrm{i}$. The corresponding eigenvectors are $(\mathrm{i}, 1)$ and $(-\mathrm{i}, 1)$. So

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}\begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix}^{-1}$$

Hence

$$R(\theta) = \mathrm{e}^{-\mathrm{i}\theta J_1} = \exp\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\theta \\ 0 & 0 & \theta & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{i} & -\mathrm{i} \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} \mathrm{e}^0 & 0 & 0 & 0 \\ 0 & \mathrm{e}^0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & 0 & 0 & \mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{i} & -\mathrm{i} \\ 0 & 0 & 1 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2}(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}) & -\frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}) \\ 0 & 0 & \frac{1}{2\mathrm{i}}(\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}) & \frac{1}{2}(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Hence the Lie subalgebra $J_1 \in \mathfrak{so}(1,3)$ generates the subgroup of $\mathrm{SO}(1,3)$ corresponding to rotation about the x-axis.

Next we diagonalise $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$: The characteristic polynomial is $x^2 - 1$, so the eigenvalues are ± 1. The corresponding eigenvectors are $(1, -1)$ and $(1, 1)$.

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$$

Hence

$$\Lambda(\eta) = \mathrm{e}^{-\mathrm{i}\eta K_1} = \exp\begin{pmatrix} 0 & -\eta & 0 & 0 \\ -\eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{\eta} & 0 & 0 & 0 \\ 0 & \mathrm{e}^{-\eta} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{1}{2}(\mathrm{e}^{\eta} + \mathrm{e}^{-\eta}) & \frac{1}{2}(-\mathrm{e}^{\eta} + \mathrm{e}^{-\eta}) & 0 & 0 \\ \frac{1}{2}(-\mathrm{e}^{\eta} + \mathrm{e}^{-\eta}) & \frac{1}{2}(\mathrm{e}^{\eta} + \mathrm{e}^{-\eta}) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence K_1 generates the subgroup corresponding to boost in the x-axis.

The boost in the y-axis gives the matrix in Lie algbra:

$$K_2 = \begin{pmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{so}(1,3)$$

$$K_2 K_1 = \begin{pmatrix} 0 & 0 & -\mathrm{i} & 0 \\ 0 & 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Note that $(K_2 K_1)^2 = 0$. The one-parameter subgroup generated by $K_2 K_1$ is given by

$$\mathrm{e}^{-\mathrm{i}\alpha K_2 K_1} = I - \mathrm{i}\alpha(K_2 K_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \mathrm{i}\alpha & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This clearly does not represent a Lorentz boost. The Lorentz boosts do not form a subgroup of $\mathrm{SO}(1,3)^+$. □

Question 5

The canonical $j = 1$ representation of the generators of 3D rotations can be derived from following rules:

$$\begin{aligned} J_3|m\rangle &= m|m\rangle \\ J_\pm|m\rangle &= [j(j+1) - m(m\pm 1)]^{1/2}|m\pm 1\rangle \\ J_\pm &= J_1 \pm i J_2 \end{aligned}$$

Write down matrices representing the J_i generators using the basis $\{|1\rangle, |0\rangle, |-1\rangle\}$.

Verify that the generators satisfy the same Lie algebra as that of the $SO(3)$ group, i.e.,

$$[J_j, J_k] = i\sum_m \epsilon_{jkm} J_m$$

Using the appropriate spherical harmonics as basis of the $j = 1$ representation space, show that J_3 generates rotations around $\hat{z}$. Similarly, show that the spherical harmonics representing a direction in the yz plane are rotated around $\hat{\mathbf{x}}$ by J_1.

Proof. When $j = 1$,

$$\begin{aligned} J_1|m\rangle &= \frac{1}{2}(J_+ + J_-)|m\rangle = \frac{1}{2}\sqrt{2 - m(m+1)}\,|m+1\rangle + \frac{1}{2}\sqrt{2 - m(m-1)}\,|m-1\rangle \\ J_2|m\rangle &= \frac{1}{2\mathrm{i}}(J_+ - J_-)|m\rangle = \frac{1}{2\mathrm{i}}\sqrt{2 - m(m+1)}\,|m+1\rangle - \frac{1}{2\mathrm{i}}\sqrt{2 - m(m-1)}\,|m-1\rangle \\ J_3|m\rangle &= m|m\rangle \end{aligned}$$

The spin-1 representation of the generates are given by

$$J_1 = \frac{\sqrt{2}}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad J_2 = \frac{\sqrt{2}}{2\mathrm{i}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Verify the commutation relations:

$$[J_1, J_2] = \frac{1}{2\mathrm{i}}\left(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right)$$

$$= \frac{1}{2\mathrm{i}}\left(\begin{pmatrix}-1&0&1\\0&0&0\\-1&0&1\end{pmatrix}-\begin{pmatrix}1&0&1\\0&0&0\\-1&0&-1\end{pmatrix}\right)=\frac{1}{2\mathrm{i}}\begin{pmatrix}-2&0&0\\0&0&0\\0&0&2\end{pmatrix}=\mathrm{i}J_3$$

$$[J_2,J_3]=\frac{\sqrt{2}}{2\mathrm{i}}\left(\begin{pmatrix}0&1&0\\-1&0&1\\0&-1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}-\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\begin{pmatrix}0&1&0\\-1&0&1\\0&-1&0\end{pmatrix}\right)$$

$$=\frac{\sqrt{2}}{2\mathrm{i}}\left(\begin{pmatrix}0&0&0\\-1&0&-1\\0&0&0\end{pmatrix}-\begin{pmatrix}0&1&0\\0&0&0\\0&1&0\end{pmatrix}\right)=\frac{\sqrt{2}}{2\mathrm{i}}\begin{pmatrix}0&-1&0\\-1&0&-1\\0&-1&0\end{pmatrix}=\mathrm{i}J_1$$

$$[J_3,J_1]=\frac{\sqrt{2}}{2}\left(\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}-\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\right)$$

$$=\frac{\sqrt{2}}{2}\left(\begin{pmatrix}0&1&0\\0&0&0\\0&-1&0\end{pmatrix}-\begin{pmatrix}0&0&0\\1&0&-1\\0&0&0\end{pmatrix}\right)=\frac{\sqrt{2}}{2}\begin{pmatrix}0&1&0\\-1&0&1\\0&-1&0\end{pmatrix}=\mathrm{i}J_2$$

In summary, $[J_j,J_k]=\mathrm{i}\epsilon_{jkm}J_m$. $\{J_1,J_2,J_3\}$ generates the Lie algebra $\mathfrak{so}(3)$.

The spherical harmonics:

$$Y_{1,1}(\theta,\varphi)=-\sqrt{\frac{3}{8\pi}}\sin\theta\,\mathrm{e}^{\mathrm{i}\varphi},\qquad Y_{1,0}(\theta,\varphi)=\sqrt{\frac{6}{8\pi}}\cos\theta,\qquad Y_{1,-1}(\theta,\varphi)=\sqrt{\frac{3}{8\pi}}\sin\theta\,\mathrm{e}^{-\mathrm{i}\varphi}$$

The one-parameter subgroup of SO(3) generated by J_3 is

$$R(\theta)=\mathrm{e}^{-\mathrm{i}\alpha J_3}=\begin{pmatrix}\mathrm{e}^{-\mathrm{i}\alpha}&0&0\\0&0&0\\0&0&\mathrm{e}^{\mathrm{i}\alpha}\end{pmatrix}$$

Hence we have

$$R(\alpha)Y_{1,1}(\theta,\varphi)=\mathrm{e}^{-\mathrm{i}\alpha}\,Y_{1,1}(\theta,\varphi)=Y_{1,1}(\varphi-\alpha);\qquad R(\alpha)Y_{1,0}(\theta,\varphi)=Y_{1,0}(\theta,\varphi)=Y_{1,0}(\theta,\varphi-\alpha),\qquad R(\alpha)Y_{1,-1}(\theta,\varphi)=Y_{1,-1}(\theta,\varphi-\alpha)$$

So the action of $\mathrm{e}^{-\mathrm{i}\alpha J_3}$ is equivalent to a coordinate transformation $\theta\mapsto\theta$, $\varphi\mapsto\varphi-\alpha$, which is a rotation around z-axis by an angle α.

For J_1, we may have to choose a different parametrisation of S^2. I am not sure about what this question want me to show explicitly.

(*What is the general relationship between the angular momentum operator in quantum mechanics and the Lie algebra* $\mathfrak{so}(3)$*?*)

□

Question 6

Consider the Dirac equation of a fermion field in the presence of an electromagnetic field A^μ

$$\left(i\gamma^\mu\partial_\mu+q\gamma^\mu A_\mu-m\right)\psi=0$$

where the γ^μ are 4×4 matrices with the anti-commutation relationship

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=-2g^{\mu\nu}$$

(The important thing to keep in mind here is merely that there are 4 distinct matrices; you shouldn't need the anti-commutation relationship itself.) Apply the local gauge transformation to the fermion field

$$\psi\to\psi'=e^{iq\alpha}\psi$$

where α is also a function of spacetime. Show that local gauge invariance can be restored by applying a simultaneous gauge transformation to the electromagnetic field,

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \chi$$

What must the relationship be between χ and α?

Proof. With $\psi \mapsto \mathrm{e}^{\mathrm{i}q\alpha}\psi$ and $A_\mu \mapsto A_\mu + \partial_\mu\chi$, the gauge invariance requires that

$$\left(\mathrm{i}\gamma^\mu\partial_\mu + q\gamma^\mu(A_\mu + \partial_\mu\chi) - m\right)\mathrm{e}^{\mathrm{i}q\alpha}\psi = 0$$

Since $\partial_m u\left(\mathrm{e}^{\mathrm{i}q\alpha}\psi\right) = \partial_\mu\psi + \mathrm{i}q\,\mathrm{e}^{\mathrm{i}q\alpha}\psi\partial_\mu\alpha$, we have

$$-q\gamma^\mu\mathrm{e}^{\mathrm{i}q\alpha}\psi\partial_\mu\alpha + q\gamma^\mu\mathrm{e}^{\mathrm{i}q\alpha}\psi\partial_\mu\chi = 0$$

✓

Hence α and χ must satisfy

$$\gamma^\mu\partial_\mu(\alpha - \chi) = 0$$

□

✓

$\Rightarrow \alpha = \chi + c$ for constant c $\Rightarrow$ global $U(1)$