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Problem Sheet 4

C2.6: Introduction to Schemes

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Question 1

i) Suppose that $F, G \subseteq H$ are subsheaves. Show that $F = G \iff F_x = G_x$ for all $x \in X$

ii) Let $\varphi \colon F \to G$ be a morphism in $\mathsf{Ab}(X)$. Show that $\ker\varphi : U \longmapsto \ker(\varphi_U)$ is a sheaf (*whereas for* $\operatorname{coker}\varphi$ *and* $\operatorname{im}\varphi$ *we must sheafify*).

iii) Also show that $(\ker\varphi)_x = \ker(\varphi_x)$ and $(\operatorname{im}\varphi)_x = \operatorname{im}(\varphi_x)$.

Show that φ injective $\iff$ φ_x injective for all x, and φ surjective $\iff$ φ_x surjective for all x.

Deduce that $F \to G \to H$ in $\mathsf{Ab}(X)$ is exact $\iff$ $F_x \to G_x \to H_x$ exact for all x.

iv) Let $\varphi \colon F \to G$ be a morphism in $\mathsf{Ab}(X)$ with φ_x surjective for all x. Show that $\forall s \in G(U)$ and $x \in U$, there is open $V \subseteq U$ with $x \in V$ and $t \in F(V)$ with $\varphi(t) = s|_V$.

v) *"Surjectivity means local liftability."*

Show that $\varphi \colon F \to G$ is surjective $\iff$ for all $s \in G(U)$, there exists an open cover $U = \bigcup U_i$ and $t_i \in F(U_i)$ such that $F(t_i) = s|_{U_i}$.

vi) Let $X = \mathbb{C} \setminus \left\{\frac{1}{n} : n \in \mathbb{Z}_+\right\}$, $\mathcal{O}_X(U) := \{\text{holomorphic functions } U \to \mathbb{C}\}$, and $\mathcal{O}_X^*(U) := \{$nowhere zero holomorphic functions $U \to \mathbb{C}\}$

Show that $\exp : \mathcal{O}_X \to \mathcal{O}_X^*$ is surjective (*for* $f \in \mathcal{O}_X(U), \exp(f) \in \mathcal{O}_X^*(U)$ *is the complex exponential*) but $\exp_U : \mathcal{O}_X(U) \to \mathcal{O}_X^*(U)$ not surjective no matter how small the open $U \ni 0$ is.

vii) Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\varphi : F \to G$ be a homomorphism of $\mathcal{O}_X$-modules, where G is of **finite type**. Show that φ_x is surjective for some $x \in X \implies \varphi_U : F|_U \to G|_U$ surjective for some open $U \ni x$.

[*In notes 6.3 we used this: we had* $\mathcal{O}_{X,x}^{\oplus n} \cong F_x$, *and assuming* F *of finite type we claimed that* $\mathcal{O}_U^{\oplus n} \to F|_U$ *surjective on some open* $U \ni x$.]

viii) Let $\varphi : F \to G$ be a homomorphism of $\mathcal{O}_X$-modules, where F is of **finite type** and G is **coherent**. Show that φ_x is injective for some $x \in X \implies \varphi_U : F|_U \to G|_U$ injective for some open $U \ni x$.

[*Hint. First check that* $\ker\varphi$ *is of finite type (doesn't use injectivity of* φ_x*) by considering* $\ker(\mathcal{O}_X|_U^{\oplus n} \twoheadrightarrow F|_U \overset{\varphi_U}{\to} G|_U)$.]

Proof. i) "$\implies$" is trivial. For "$\impliedby$", suppose that $F_x = G_x$ for all $x \in X$. Let $U \subseteq X$ be open. Let $s \in F(U)$. Then $s_x \in F_x = G_x$ for all $x \in U$. We can pick representative $t' \in G(U_x)$ of s_x where $x \in U_x \subseteq U$. Then by the local-to-global condition there exists $t \in G(U)$ such that $t_x = s_x$ for all $x \in U$. Hence $t = s \in G(U)$. This implies that $F(U) \subseteq G(U)$. Symmetrically $G(U) \subseteq F(U)$. Hence $F(U) = G(U)$. Since U is arbitrary, $F = G$ as sheaves.

ii) First we check that $\ker\varphi : U \mapsto \ker(\varphi_U)$ is a presheaf. To check the functoriality, it suffices to check that it is compatible with restriction. Let $U \subseteq V$. Then

$$(\ker\varphi)(V)|_U = (\ker\varphi_V)|_U = \ker(\varphi_V|_U) = \ker\varphi_U = (\ker\varphi)(U)$$

Next we check the local-to-global condition. Let $\{U_i\}_{i\in I}$ be a family of open sets. For each i there is $s_i \in (\ker\varphi)(U_i) \subseteq F(U_i)$ such that

$$s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j} \in (\ker\varphi)(U_i \cap U_j) \subseteq F(U_i \cap U_j)$$

Let $U := \bigcup_{i\in I} U_i$. Since F is a sheaf, there exists a unique $s \in F(U)$ such that $s_i = s|_{U_i}$. Note that $\varphi_U(s)|_{U_i} = \varphi_{U_i}(s|_{U_i}) = 0 \in G(U_i)$. Since G is a sheaf, again by the local-to-global condition

$\varphi_U(s) = 0 \in G(U)$. Hence $s \in \ker\varphi_U = (\ker\varphi)(U)$. We conclude that $\ker\varphi$ is a sheaf.

iii) Note that

$$\begin{aligned} t \in \varinjlim_{U \ni x} (\ker\varphi)(U) &\iff \exists U \ni x \; \exists s \in (\ker\varphi)(U) \; s_x = t \\ &\iff \exists U \ni x \; \exists s \in F(U) \; (s_x = t \wedge \varphi_U(s) = 0 \in G(U)) \\ &\iff t \in \ker\varphi_x \end{aligned}$$

Therefore

$$\ker\varphi_x = \varinjlim_{U \ni x} (\ker\varphi)(U) = (\ker\varphi)_x$$

If $\operatorname{im}\varphi$ is the sheafification of $\underline{\operatorname{im}\varphi} : U \longmapsto \operatorname{im}\varphi_U$, then $(\operatorname{im}\varphi)_x = \operatorname{im}\varphi_x$ is just the definition of sheafification.

In any Abelian category, a morphism is injective (*resp.* surjective) if and only if it is a monomorphism (*resp.* epimorphism). We have

$$\begin{aligned} \varphi \text{ is a monomorphism} &\iff \forall \psi \; (\varphi \circ \psi = 0 \implies \psi = 0) \\ &\iff \forall \psi \; \forall U \in \mathsf{Top}(X) \; (\varphi_U \circ \psi_U = 0 \implies \psi_U = 0) \\ &\iff \forall U \in \mathsf{Top}(X) \; \ker\varphi_U = 0 \\ &\iff \ker\varphi = 0 \\ &\iff \forall x \in X \; (\ker\varphi)_x = \ker\varphi_x = 0 \\ &\iff \forall x \in X \; \varphi_x \text{ is injective} \\ \varphi \text{ is an epimorphism} &\iff \forall \psi \; (\psi \circ \varphi = 0 \implies \psi = 0) \\ &\iff \forall \psi \; \forall x \in X \; (\psi_x \circ \varphi_x = 0 \implies \psi_x = 0) \\ &\iff \forall x \in X \; \varphi_x \text{ is surjective} \end{aligned}$$

Let $F \to G \to H$ be a sequence in $\mathsf{Ab}(X)$.

$$\begin{aligned} F \xrightarrow{\alpha} G \xrightarrow{\beta} H \text{ is exact at } G &\iff \ker\beta = \operatorname{im}\alpha \\ &\iff \forall x \in X \; (\ker\beta)_x = \ker\beta_x = \operatorname{im}\alpha_x = (\operatorname{im}\alpha)_x \\ &\iff \forall x \in X \; F_x \xrightarrow{\alpha_x} G_x \xrightarrow{\beta_x} H_x \text{ is exact at } G_x \end{aligned}$$

iv) Since $\varphi_x : F_x \to G_x$ is surjective, there exists $t_x \in F_x$ such that $\varphi_x(t_x) = s_x$. We choose a representative $t \in F(W)$ for some open $W \subseteq U$. Then $s_x = \varphi_x(t_x) = \varphi_W(t)_x$. Then $\varphi_W(t)|_V = s|_V$ for some open $V \subseteq W$. By replacing $t|_V$ with t we find a $t \in F(V)$ such that $\varphi_V(t) = s|_V$.

v) By (iv), for each $x \in U$ we can find open $U_x \subseteq U$ with $x \in U_x$ and $t^{(x)} \in F(U_x)$ such that $F(t^{(x)}) = s|_{U_x}$. It is clear that $\{U_x\}_{x \in U}$ is an open cover of U.

vi) We recall the following fact (corollary of Cauchy's Theorem and Riemann Mapping Theorem) from complex analysis:

> *Let $U \subseteq \mathbb{C}$ be a connected open set, and $f : U \to \mathbb{C}$ be a nowhere vanishing holomorphic function. Then U is simply-connected if and only if $f(z) = \exp(g(z))$ for some holomorphic function $g : U \to \mathbb{C}$.*

To show that $\exp\colon \mathcal{O}_X \to \mathcal{O}_X^*$ is surjective, it suffices to show that $\exp_z\colon \mathcal{O}_{X,z} \to \mathcal{O}_{X,z}^*$ is surjective for all $z \in X$. For $f_z \in \mathcal{O}_{X,z}^*$, we may pick a representative (f, U) for some open $U := V \cap X \ni z$ and nonwhere vanishing holomorphic $f : V \to \mathbb{C}$. We can take V to be an open disc in $\mathbb{C}$ so that

$f(z) = \exp(g(z))$ for some holomorphic $g : V \to \mathbb{C}$. Then (g, U) represents an element g_z in $\mathcal{O}_{X,z}$, and $\exp_z(g_z) = f_z$. So $\exp_z$ is surjective.

For any open set $V \subseteq \mathbb{C}$ with $0 \in V$, $\exp_U : \mathcal{O}_X(U) \to \mathcal{O}_X^*(U)$ is not surjective, where $U := V \cap X$. To see this, we note that there exists $B(0,r) \subseteq V$ for some small r and hence $1/n \in V$ for some $n \in \mathbb{N}$. Then $f(z) := \left(z - \frac{1}{n}\right)^{-1} \in \mathcal{O}_X^*(U)$. It does not admit a holomorphic logarithm because U is not simply connected.

vii) Since G is of finite type, there exists a neighbourhood $V \ni x$ such that $\pi_V : \mathcal{O}_V^{\oplus n} \to G|_V$ is surjective. At the stalk of x, we can construct $f_x : \mathcal{O}_{X,x}^{\oplus n} \to F_x$ as follows. We can pick generators $e_1, ..., e_n \in \mathcal{O}_{X,x}^{\oplus n}$. Let $f_x(e_i)$ be a element in the preimage $\varphi_x^{-1}(\pi_x(e_i))$ (we used that φ_x is surjective). Since $\mathcal{O}_{X,x}^{\oplus n}$ is a free module, this uniquely extends to a $\mathcal{O}_{X,x}$-module homomorphism f_x such that the following diagram commutes:

$$\begin{array}{ccc} & & \mathcal{O}_{X,x}^{\oplus n} \\ & \swarrow^{f_x} & \downarrow \pi_x \\ F_x & \xrightarrow[\varphi_x]{} & G_x \end{array}$$

Next we pick the representatives $(e_i, U_i), (f_x(e_i), V_i), (\pi_x(e_i), W_i)$ for $e_i, f_x(e_i), \pi_x(e_i)$. Let $U := V \cap \bigcap_{i=1}^n (U_i \cap V_i \cap W_i)$. Then the diagram lifts to a commutative diagram of $\mathcal{O}_X(U)$-modules:

$$\begin{array}{ccc} & & \mathcal{O}_X^{\oplus n}(U) \\ & \swarrow^{f_U} & \downarrow \pi_U \\ F(U) & \xrightarrow[\varphi_U]{} & G(U) \end{array}$$

It is clear that the diagram respects restrictions. So we have commutative diagram of $\mathcal{O}_X$-modules:

$$\begin{array}{ccc} & & \mathcal{O}_U^{\oplus n} \\ & \swarrow^{f_U} & \downarrow \pi_U \\ F|_U & \xrightarrow[\varphi_U]{} & G|_U \end{array}$$

Since π_U is surjective, so is $\varphi_U : F|_U \to G|_U$.

viii) *The categorical kernel* $\ker\varphi$ *is always a morphism. To clearify, the source object of* $\ker\varphi$ *shall be denoted by* $\operatorname{Ker}\varphi$*.*

Let $y \in X$. Since F is of finite type, there exists open $W \subseteq X$ and a surjection $\pi_W : \mathcal{O}_W^{\oplus n} \to F|_W$. Since G is coherent, we know that $\operatorname{Ker}(\varphi_W \circ \pi_W)$ is of finite type. Then, with possible restriction to a smaller open set, we have a surjection $\sigma_W : \mathcal{O}_W^{\oplus m} \to \operatorname{Ker}(\varphi_W \circ \pi_W)$. By universal property of kernel, there exists unique a morphism $j_W : \operatorname{Ker}(\varphi_W \circ \pi_W) \to \operatorname{Ker}\varphi_W$. The composite $j_W \circ \sigma_W : \mathcal{O}_W^{\oplus m} \to \operatorname{Ker}\varphi_W$ is surjective.

$$\begin{array}{ccccccc} & & \operatorname{Ker}(\varphi_W \circ \pi_W) & \hookrightarrow & \mathcal{O}_W^{\oplus n} & & \\ & \nearrow^{\sigma_W} & \downarrow j_W & & \downarrow \pi_W & & \\ \mathcal{O}_W^{\oplus m} & \xrightarrow[j_W \circ \sigma_W]{} & \operatorname{Ker}\varphi_W & \hookrightarrow & F|_W & \xrightarrow{\varphi_W} & G|_W \end{array}$$

This shows that $\operatorname{Ker}\varphi$ is of finite type.Now we are given that φ_x is injective for some $x \in X$. Then $(\operatorname{Ker}\varphi)_x = 0$. Since $\operatorname{Ker}\varphi$ is of fintie type, there exist an open $V \subseteq X$ and a surjection $\alpha_V : \mathcal{O}_V^{\oplus p} \to (\operatorname{Ker}\varphi)|_V$. The morphism on the stalk $\alpha_x : \mathcal{O}_{X,x}^{\oplus p} \to (\operatorname{Ker}\varphi)_x$ sends the generators $a_1, ..., a_p$ to $0 \in (\operatorname{Ker}\varphi)_x$. We pick representatives (a_i, U_i) for each a_i and let $U := V \cap \bigcap_{i=1}^p U_i$. Then α_x lifts to a surjective zero map $\alpha_U : \mathcal{O}_U^{\oplus p} \to (\operatorname{Ker}\varphi)_U$. Hence $(\operatorname{Ker}\varphi)_U = 0$, and $\varphi_U : F|_U \to G|_U$ is injective. $\square$

Question 2

Motivation: Why is Nakayama's Lemma useful in geometry? "Transferring information from pointwise to infinitesimal to local":

Recall ***Nakayama's Lemma:***

> *Let R be a ring, $\mathfrak{p} \in \operatorname{Spec} R$, and M be a finitely generated R-module. If $n_1, ..., n_d \in M$ is a basis for the $\kappa(\mathfrak{p})$-vector space $M_\mathfrak{p} \otimes_{R_\mathfrak{p}} \kappa(\mathfrak{p})$, then $n_1, ..., n_d$ generate the R_f-module M_f for some $f \in R \setminus \mathfrak{p}$ (indeed it is a minimal generating set).*

(*When R is a local ring with max ideal $\mathfrak{m}$, this becomes $M/\mathfrak{m}M = \langle n_1, ..., n_d \rangle \implies M = \langle n_1, ..., n_d \rangle$*)

i) Let $(X, \mathcal{O}_X)$ be a scheme and $F \in \mathsf{QCoh}(X)$ of finite type. Then we call $F(x) = F_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ the **fibre**.

Given $s_1, \ldots, s_n \in F(U)$ on open $U \ni x$, if $(s_1)_x, \ldots, (s_n)_x$ generate the fibre then possibly after shrinking U, show that $s_1, ..., s_n$ also generate $F|_U$. Deduce that:

- if $F(x) = 0$ then $F|_U = 0$ some open $U \ni x$.
- $x \mapsto \dim F(x)$ is upper-semi-continuous, i.e. $\{\dim < d\} \subseteq X$ is open (*since integer-valued can also take $\leqslant d$.*)

ii) *Algebra fact. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence in R-Mod. Suppose that M_2 is flat. Then M_3 flat $\iff$ $M_1/IM_1 \to M_2/IM_2$ injective for all ideal $I \subseteq R$.*

Let $F \in \mathcal{O}_X$-Mod be locally finitely presented (*e.g.* $F \in \mathsf{Coh}(X)$). Prove that $F \in \mathsf{Vect}(X) \iff F$ is a flat $\mathcal{O}_X$-module.

[*Hint. Rewrite the algebra fact in case R local ring, you will use the case that I is the maximal ideal. The key is to reach an exact sequence of type $0 \to N_x/\mathfrak{m}_x N_x \to K(x)^{\oplus n} \to F_x/\mathfrak{m}_x F_x \to 0$ and use (i).*]

Proof. i) WLOG reduce to the affine case: $X = \operatorname{Spec} R$. $F = \widetilde{M}$ for some R-module M. $x = \mathfrak{p}$ is a prime ideal in R. $F_x = M_\mathfrak{p}$. $\mathcal{O}_{X,x} = R_\mathfrak{p}$. $\kappa(x) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. The fibre $F(x) = M_\mathfrak{p} \otimes_{R_\mathfrak{p}} \kappa(\mathfrak{p})$. By Nakayama Lemma above, $(s_1)_x, ..., (s_n)_x$ generates $F(x)$ implies that $s_1, ..., s_n$ generate M_f for some $f \in R \setminus \mathfrak{p}$. Note that $\widetilde{M}|_{D_f} = \widetilde{M_f}$, so we take $U = D_f$.

- If $F(x) = 0$, then $(0)_x$ generates $F(x)$. By the previous part 0 generates $F|_U$. So $F|_U = 0$.
- If $\dim F(x) = d$ at x, then we can find $(s_1)_x, ..., (s_d)_x$ that generate $F(x)$. So $s_1, ..., s_d$ generates $F|_U$ for some $U \ni x$. Then $\dim F(y) \leqslant d$ for $y \in U$, since $(s_1)_y, ..., (s_d)_y$ generate $F(y)$. This shows that $\{\dim < d\}$ is open.

ii) If F is a vector bundle, then locally $F|_U$ is free and hence is flat. Since flatness is a local property, F is flat.

Conversely, suppose that F is flat. Since F is lovally finitely presented, locally we have

$$\mathcal{O}_U^{\oplus m} \to \mathcal{O}_U^{\oplus n} \to F|_U \to 0$$

Consider the exact sequence

$$0 \to \operatorname{Ker} \varphi \to \mathcal{O}_U^{\oplus n} \to F|_U \to 0$$

Strategy: Shrink U to make $N := \operatorname{Ker} \varphi = 0$. Suffices to show $N(x) = 0$.

Since F is a flat $\mathcal{O}_X$-module, the stalk F_x is a flat $\mathcal{O}_{X,x}$-module. First we localise the above exact sequence:

$$0 \to N_x \to \mathcal{O}_{X,x}^{\oplus n} \to F_x \to 0$$

We use the algebra fact. Since $\mathcal{O}_{X,x}^{\oplus n}$ and F_x are flat, $N_x/\mathfrak{m}_x N_x \to \mathcal{O}_{X,x}^{\oplus n}/\mathfrak{m}_x\mathcal{O}_{X,x}^{\oplus n} \cong \kappa(x)^{\oplus n}$ is injective. Note that $N_x/\mathfrak{m}_x N_x \cong N_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathfrak{m}_x$. We tensor $\mathcal{O}_{X,x}/\mathfrak{m}_x$ to the short exact above. We have that

$$o \to N_x/\mathfrak{m}_x N_x \to \kappa(x)^{\oplus n} \to F_x/\mathfrak{m}_x F_x \to 0$$

is right exact. It is in fact exact because the first nonzero map is injective.

If $n = \dim F_x/\mathfrak{m}_x F_x = \dim F(x)$, then the second nonzero map is an isomorphism and we are done. If $n > \dim F(x)$, let $(s_1)_x, ..., (s_m)_x$ generate $F(x)$ ($m < n$). By (i) we have $\mathcal{O}_U^{\oplus m} \twoheadrightarrow F|_U \to 0$. Then we take N to be the kernel of the above map and follow the same argument.

Note: We need the fact: consider $\ker f \to R^{\oplus m} \to M \to 0$ and $\ker g \to R^{\oplus m} \to M \to 0$. If $\ker f$ is finitely generated, then so is $\ker g$. □

Question 3

Motivation: $\mathsf{Vect}(\operatorname{Spec} R) \longleftrightarrow \widetilde{M}$ *for finitely generated projective* R*-module* M

Let $X = \operatorname{Spec} R$. M is an R-module. Consider the following conditions.

(0) $\widetilde{M} \in \mathsf{Vect}(X)$. i.e. locally free of finite rank. i.e. there is a cover $X = \bigcup D_{f_i}$ with M_{f_i} finitely generated free R_{f_i}-module.

(1) M is finitely presented and flat.

(2) M is finitely presented and $M_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module for all maximal ideals $\mathfrak{m}$ (*can also use all prime ideals*).

(3) M is the direct summand of a finite rank free module.

(4) M is finitely generated and projective.

i) Prove that (0) $\iff$ (1).

[*Hint. For* $\Longleftarrow$ *use 2.(ii). For* $\Longrightarrow$ *compare the proof in Sec 3.1 of notes, use tricks from Sec 3.0, and use fact that in the short exact sequence* $0 \to K \to M_1 \to M_2 \to 0$, M_1 *is finitely generated and* M_2 *is finitely presented implies that* K *is finitely generated.*]

ii) Prove that (0) & (1) $\iff$ (2), and (4) $\iff$ (3) $\implies$ (1).

iii) Finally prove that (0) $\implies$ (4).

[*Hint. use Sec 3.0 of notes and the fact about localisation: for* M *finitely presented,* $S^{-1}\operatorname{Hom}_R(M, N) = \operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$.]

Proof. i) "(1) $\implies$ (0)": Suppose that M is finitely presented and flat. We need to prove that $\widetilde{M}$ is locally finitely presented and flat as a $\mathcal{O}_X$-module. Since flatness is a local property, we have

$$\begin{aligned} M \text{ is a flat } R\text{-module} &\iff \forall \mathfrak{p} \in \operatorname{Spec} R: \ M_\mathfrak{p} \text{ is a flat } R_\mathfrak{p}\text{-module} \qquad (\textit{A\&M Prop. 3.10}) \\ &\iff \forall \mathfrak{p} \in X: \ \widetilde{M}_\mathfrak{p} \text{ is a flat } \mathcal{O}_{X,\mathfrak{p}}\text{-module} \\ &\iff \widetilde{M} \text{ is a flat } \mathcal{O}_X\text{-module} \end{aligned}$$

Since M is finitely presented, we have an exact sequence

$$R^{\oplus m} \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0$$

$R^{\oplus m}, R^{\oplus n}, M$ are global sections of $\mathcal{O}_X$-modules $\mathcal{O}_X^{\oplus m}, \mathcal{O}_X^{\oplus n}, \widetilde{M}$. So $\widetilde{M}$ is locally finitely presented. Now by Question 2.(ii), we have $\widetilde{M} \in \mathsf{Vect}(X)$.

"(0) $\implies$ (1)": Suppose that $\widetilde{M} \in \mathsf{Vect}(X)$. There is a finite cover of X by basic open sets D_{f_i} with $M_{f_i} \cong (R_{f_i})^{\oplus n_i}$ for some n_i.

We can take the generators $e_1, ..., e_{n_i}$ of M_{f_i}. Let $e_j = \varphi_i(h_j)/f_j^{m_j}$, where $h_j \in M$ and $\varphi_i : M \to M_{f_i}$ is the localisation map. Take $A_i := \{h_1, ..., h_{n_i}\}$, and $A = \bigcup_i A_i$. Therefore we have a R-module homomorphism $\alpha \colon R^{\oplus A} \to M$. Since $X = \bigcup_i D_{f_i}$, for any $\mathfrak{p} \in X$, the induced morphism on the stalk $\alpha_{\mathfrak{p}} : R_{\mathfrak{p}}^{\oplus A} \to M_{\mathfrak{p}}$ is surjective. Hence α is surjective. M is finitely generated. We have a short exact sequence

$$0 \longrightarrow K \longrightarrow R^{\oplus A} \longrightarrow M \longrightarrow 0$$

Now we take the localisation to f_i again. Since localisation is exact, we have the short exact sequence

$$0 \longrightarrow K_{f_i} \longrightarrow R_{f_i}^{\oplus A} \longrightarrow M_{f_i} \longrightarrow 0$$

By the hint, since M_{f_i} is finitely presented and $R_{f_i}^{\oplus A}$ is finitely generated, then K_{f_i} is finitely generated. Similar to the proof above, we have that K is finitely generated. Hence M is finitely presented.

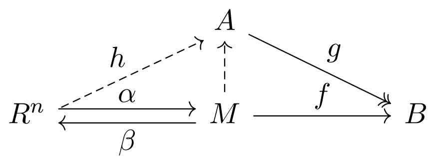
In particular, $\widetilde{M}$ is locally finitely presented. We use Question 2.(ii) again to deduce that $\widetilde{M}$ is a flat $\mathcal{O}_X$-module. Hence M is a flat R-module.

ii) "(0)(1) $\implies$ (2)": Since $\widetilde{M}$ is locally free of finite rank, it is free of finite rank at the stalk level. This means that $\widetilde{M}_{\mathfrak{p}} = M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in X$.

"(2) $\implies$ (0)": First we prove that $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module for all $\mathfrak{p} \in X$. For any $\mathfrak{p} \in X$, let $\mathfrak{m}$ be a maximal ideal containing $\mathfrak{p}$. Suppose that $M_{\mathfrak{m}} \cong R_{\mathfrak{m}}^{\oplus I}$. Then $M_{\mathfrak{p}} \cong (M_{\mathfrak{m}})_{\mathfrak{p}} \cong (R_{\mathfrak{m}}^{\oplus I})_{\mathfrak{p}} \cong R_{\mathfrak{p}}^{\oplus I}$. Hence $M_{\mathfrak{p}}$ is free.

Next, consider $\mathfrak{p} \in X$ and isomorphism $\alpha : R_{\mathfrak{p}}^{\oplus I} \to M_{\mathfrak{p}}$. Since M is fintiely presented, so is $M_{\mathfrak{p}}$. In particular we may take I to be a finite set. Since $\widetilde{M}$ is of finite type, by the (first) lemma in Section 6.2 of the notes, there exists an open $U \ni \mathfrak{p}$ and a morphism of $\mathcal{O}_U$-modules $\varphi : \mathcal{O}_U^{\oplus I} \to \widetilde{M}|_U$ such that $\varphi_{\mathfrak{p}} = \alpha$. Using the same method in the prood of "(0) $\implies$ (1)", we can prove that $\operatorname{Ker}\varphi(V)$ is finitely generated for any open $V \subseteq U$. Hence $\operatorname{Ker}\varphi$ is of finite type. By the second lemma in Section 6.2 of the notes, we deduce that φ is an isomorphism. Hence $\mathcal{O}_U^{\oplus I} \cong \widetilde{M}|_U$. In other words, $\widetilde{M} \in \mathsf{Vect}(X)$.

"(3) $\implies$ (4)": Suppose that M is direct summand of a finite rank free module. That is, $R^{\oplus n} = M \oplus N$. Let $g : A \to B$ be any epimorphism and $f : M \to B$ be any R-module homomorphism. Then we have the commutative diagram:



$\alpha : M \oplus N \to M$ and $\beta : M \to M \oplus N$ are the projection and inclusion maps. Since R^n is free, it is projective, and hence there exist a lift h of $f \circ \alpha$. Then $h \circ \beta$ is a lift of f. Hence M is projective.

"(4) $\implies$ (3)": Suppose that M is finitely generated and projective. Then we have a short exact sequence

$$0 \longrightarrow K \longrightarrow R^{\oplus n} \longrightarrow M \longrightarrow 0$$

Since M is projective, the sequence splits. We have $R^{\oplus n} \cong M \oplus K$. Hence M is a direct summand of a finite rank free module.

"(3)(4) $\implies$ (1)": Suppose that M is finitely generated and projective. Then $\operatorname{Tor}_n^R(A, M) = 0$ for any R-module A and $n \geqslant 1$. In particular, let $0 \to A \to B \to C$ be a short exact sequence. We look at the induced long exact sequence of Tor. Since $\operatorname{Tor}_1^R(C, M) = 0$, we have an exact sequence

$$0 = \operatorname{Tor}_1^R(C, M) \longrightarrow A \otimes_R M \longrightarrow B \otimes_R M \longrightarrow C \otimes_R M \longrightarrow 0$$

Hence $- \otimes_R M$ is exact. M is flat. Since M is finitely generated, the short exact sequence $0 \to K \to R^{\oplus n} \to M \to 0$ gives a finite presentation for M.

iii) "(0) $\implies$ (4)": In the proof of "(0) $\implies$ (1)" we have shown that M is finitely generated. To prove that M is exact, we need to prove that $\operatorname{Hom}_R(M, -)$ is exact. Let $0 \to A \to B \to C \to 0$ be a short exact sequence. Consider the sequence

$$0 \longrightarrow \operatorname{Hom}_R(M, A) \longrightarrow \operatorname{Hom}_R(M, B) \longrightarrow \operatorname{Hom}_R(M, C) \longrightarrow 0$$

Since M is finitely presented, by the hint, we look at the localisation at f_i:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Hom}_R(M, A)_{f_i} & \longrightarrow & \operatorname{Hom}_R(M, B)_{f_i} & \longrightarrow & \operatorname{Hom}_R(M, C)_{f_i} & \longrightarrow & 0 \\ & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\ 0 & \longrightarrow & \operatorname{Hom}_{R_{f_i}}(M_{f_i}, A_{f_i}) & \longrightarrow & \operatorname{Hom}_{R_{f_i}}(M_{f_i}, B_{f_i}) & \longrightarrow & \operatorname{Hom}_{R_{f_i}}(M_{f_i}, C_{f_i}) & \longrightarrow & 0 \end{array}$$

We know that M_{f_i} is free as an R_{f_i}-module. So $\operatorname{Hom}_{R_{f_i}}(M_{f_i}, -)$ is exact. The above sequence is exact. Now as $\{D_{f_i}\}$ covers X, by the local algebra theorem, we deduce that the sequence before localisation is exact. Hence M is projective. □

Question 4

i) Let $X = \operatorname{Spec} R$. M is an R-module.

- Show that $\mathcal{L} = \widetilde{M}$ is a line bundle $\iff \forall \mathfrak{p} \in X\ \exists f \in R \setminus \mathfrak{p}\colon M_f \cong R_f$.
- Show that $\mathcal{L} = \widetilde{M}$ is a line bundle $\iff$ M is a finitely generated projective R-module with $\dim_{\kappa(\mathfrak{p})} M \otimes \kappa(\mathfrak{p}) = 1$ for all $\mathfrak{p} \in X$.

Deduce that every line bundle on $\mathbb{A}^1_k = \operatorname{Spec} k[t]$ is trivial.

[*Hint. Structure theorem for finitely generated modules over PID.*]

ii) Let $F \in \mathsf{Vect}(X)$. Describe the transition function of the **dual** $F^\vee := \operatorname{Hom}_{\mathcal{O}_X}(F, \mathcal{O}_X)$.

Deduce that for a line bundle $\mathcal{L}$, the transition function of $\mathcal{L}^\vee$ is the inverse of that of $\mathcal{L}$ and

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee = \mathcal{L} \otimes_{\mathcal{O}_X} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \cong \mathcal{O}_X$$

via the natural evaluation map.

iii) Let M, N be R-modules. Suppose that $\varphi : M \otimes_R N \xrightarrow{\cong} R$. Pick $m_i \in M, n_i \in N$ with $\varphi\left(\sum_{i=1}^d m_i \otimes n_i\right) = 1$. Check that $M \to M, m \mapsto \sum \varphi(m \otimes n_i)\, m_i$ is an isomorphism which factorises as $M \to R^d \to M$ and deduce that M is a summand of R^d.

iv) Show that $\mathcal{L} \in \mathcal{O}_X\text{-}\mathsf{Mod}$ is a line bundle $\iff$ there exists $F \in \mathsf{QCoh}(X)$ such that $F \otimes_{\mathcal{O}_X} \mathcal{L} \cong \mathcal{O}_X$ (*definition of $\mathcal{L}$ being **invertible sheaf***). (*In fact, enough to require $F \in \mathcal{O}_X$-*Mod*, but tricky*)

[*Hint. combine (iii) with 3.(3).*]

Proof. i) • By definition (and using Section 7.6),

$$\begin{aligned} \widetilde{M} \text{ is a line bundle} &\iff \forall \mathfrak{p} \in X\ \exists U \in \mathsf{Top}(X)\ (\mathfrak{p} \in U \wedge \widetilde{M}|_U \cong \mathcal{O}_U) \\ &\iff \forall \mathfrak{p} \in X\ \exists f \in R \setminus \mathfrak{p}\ (\widetilde{M}|_{D_f} \cong \mathcal{O}_{D_f}) \\ &\iff \forall \mathfrak{p} \in X\ \exists f \in R \setminus \mathfrak{p}\ (M_f \cong R_f) \end{aligned}$$

- Note that for $\mathfrak{p} \in X$,

$$M \otimes_R \kappa(\mathfrak{p}) = M \otimes_R (R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}) \cong M \otimes_R R_\mathfrak{p} \otimes_{R_\mathfrak{p}} (R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}) \cong M_\mathfrak{p} \otimes_{R_\mathfrak{p}} (R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}) \cong M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$$

  Therefore,

$$\dim_{\kappa(\mathfrak{p})}(M \otimes_R \kappa(\mathfrak{p})) = 1 \iff M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \cong \kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \text{ as } \kappa(\mathfrak{p})\text{-modules}$$

  If $\widetilde{M}$ is a line bundle, then using $M_f \cong R_f$ and taking direct limits, we have $M_\mathfrak{p} \cong R_\mathfrak{p}$. Hence $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \cong \kappa(\mathfrak{p})$. Conversely, suppose that M is finitely generated and projective, then by Question 3, $\widetilde{M}$ is a vector bundle over X. And we have $M_\mathfrak{p} \cong (R_\mathfrak{p})^{\oplus n}$ for some $n \in \mathbb{N}$. Then $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \cong \kappa(\mathfrak{p})^{\oplus n}$. Then $\dim_{\kappa(\mathfrak{p})}(M \otimes_R \kappa(\mathfrak{p})) = 1$ implies that $n = 1$. Hence $M_\mathfrak{p} \cong R_\mathfrak{p}$ for all $\mathfrak{p} \in X$. This shows that $\widetilde{M}$ is a line bundle.

Suppose that $\mathcal{L}$ is a line bundle on $\mathbb{A}^1_k$. Then $\mathcal{L} \in \mathsf{QCoh}(\mathbb{A}^1_k)$ and by Section 7.6 of the notes, we have $\mathcal{L} = \widetilde{M}$ for some $k[t]$-module M. By the above result, M is finitely generated and projective. Since $k[t]$ is a PID, and M is torsion-free (since it is projective), by the structure theorem $M \cong (k[t])^{\oplus n}$ for some $n \in \mathbb{N}$. Let $\mathfrak{p} = \langle t \rangle$. Then $\kappa(\langle t \rangle) = k$. We have $\dim_k(M \otimes_{k[t]} k) = \dim_k k^n = n$. Since $\mathcal{L}$ is a line bundle, we deduce that $n = 1$. So $\mathcal{L} = \mathcal{O}_{\mathbb{A}^1_k}$ is a trivial line bundle.

ii) Without loss of generality we assume that X is connected. Let $\{U_1, ..., U_n\}$ be an open cover of X. For each U_i we can pick a local trivialisation of $\mathcal{L}$ as $\sigma_i : \mathcal{L}|_{U_i} \to \mathcal{O}_{U_i}^{\oplus n}$. If $U_{ij} := U_i \cap U_j \neq \varnothing$, the transition function $\alpha_{ij} : \mathcal{O}_{U_{ij}}^{\oplus n} \to \mathcal{O}_{U_{ij}}^{\oplus n}$ is the morphism such that the following diagram commutes:

$$\begin{array}{ccc} & \mathcal{L}|_{U_{ij}} & \\ \swarrow^{\sigma_i} & & \searrow^{\sigma_j} \\ \mathcal{O}_{U_{ij}}^{\oplus n} & \xrightarrow{\alpha_{ij}} & \mathcal{O}_{U_{ij}}^{\oplus n} \end{array}$$

Let β_{ij} be the corresponding transition function of $\mathcal{L}^\vee$. We claim that $\beta_{ij} = (\alpha_{ij}^{-1})^\top = \alpha_{ji}^\top$ as matrices over $\mathcal{O}_X(U_{ij})$. (*To be proven...*)

Suppose that $\mathcal{L}$ is a line bundle. Let $\mathrm{ev}\colon \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \to \mathcal{O}_X$ be the natural evaluation map. Let $\{U_1, ..., U_n\}$ be an open cover of X such that $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$. It is trivial that the restriction ev_{U_i} is an isomorphism. Then ev is given by gluing ev_{U_i}.

iii) Following the question, $\varphi : M \otimes_R N \to R$ is an isomorphism and $\varphi(\sum_i m_i \otimes n_i) = 1$. Let $\alpha : M \to M$ be the R-module homomorphism given by $m \mapsto \sum_i \varphi(m \otimes n_i)m_i$. In fact, α is the composition of the isomorphisms:

$$\begin{array}{ccccccccccc} M & \longrightarrow & M \otimes_R R & \xrightarrow{\mathrm{id} \otimes \varphi^{-1}} & M \otimes_R (M \otimes_R N) & \longrightarrow & (M \otimes_R N) \otimes_R M & \xrightarrow{\varphi \otimes \mathrm{id}} & R \otimes_R M & \longrightarrow & M \\ m & \longmapsto & m \otimes 1 & \longmapsto & \sum_i m \otimes (m_i \otimes n_i) & \mapsto & \sum_i (m \otimes n_i) \otimes m_i & \longmapsto & \sum_i \varphi(m \otimes n_i) \otimes m_i & \mapsto & \sum_i \varphi(m \otimes n_i)m_i \end{array}$$

Note that $\alpha : M \to M$ factorises as

$$\begin{array}{ccccc} M & \longrightarrow & R^d & \longrightarrow & M \\ m & \longmapsto & (\varphi(m \otimes n_1), ..., \varphi(m \otimes n_d)) & \longmapsto & \sum_{i=1}^d \varphi(m \otimes n_i)m_i \\ & & (r_1, ..., r_n) & \longmapsto & \sum_{i=1}^d r_i m_i \end{array}$$

The map $R^d \to M$ extends to a short exact sequence $0 \to K \to R^d \to M \to 0$. Since $R^d \to M$ admits a section, the sequence splits. Hence M is a direct summand of R^d.

iv) Suppose that $\mathcal{L}$ is a line bundle. By (ii) we have that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \cong \mathcal{O}_X$, where $\mathcal{L}^\vee \in \mathsf{QCoh}(X)$ because it is also a line bundle.

Conversely, suppose that F is a $\mathcal{O}_X$-module such that $F \otimes_{\mathcal{O}_X} \mathcal{L} \cong \mathcal{O}_X$. We have $F = \widetilde{M}$ for some R-module M. For $\mathfrak{p} \in X$, we have a $\mathcal{O}_{X,\mathfrak{p}}$-module isomorphism $\varphi_\mathfrak{p} : \mathcal{L}_\mathfrak{p} \otimes_{\mathcal{O}_{X,\mathfrak{p}}} F_\mathfrak{p} to \mathcal{O}_{X,\mathfrak{p}}$. Let $m_i \in \mathcal{L}_\mathfrak{p}$ and $n_i \in F_\mathfrak{p}$ such that $\varphi_\mathfrak{p}(\sum_{i=1}^d m_i \otimes n_i) = 1 \in \mathcal{O}_{X,\mathfrak{p}}$. We choose representatives $(m_i, U_i), (n_i, V_i)$ for m_i, n_i. Let $U = \bigcap_{i=1}^d (U_i \cap V_i)$. Then $\varphi_\mathfrak{p}$ lifts to a $\mathcal{O}_U$-module isomorphism $\varphi_U : \mathcal{L}|_U \otimes_{\mathcal{O}_U} F|_U \cong \mathcal{O}_U$. By the same method in (iii), we can show that $\mathcal{L}|_U$ is a direct summand of $\mathcal{O}_U^{\oplus d}$. Hence $\mathcal{L}$ is a vector bundle. In particular, $\mathcal{L}$ is quasi-coherent and $\mathcal{L} = \widetilde{M}$ for some R-module M. Similarly, F is also quasi-coherent and $F = \widetilde{N}$ for some R-module N. In particular we have $M \otimes_R N \cong R$ as R-modules.

Finally, to show that $\mathcal{L}$ is a line bundle, by (i) it suffices to show that $\dim_{\kappa(\mathfrak{p})}(M \otimes_R \kappa(\mathfrak{p})) = 1$. Note that

$$\begin{aligned} 1 &= \dim_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}) = \dim_{\kappa(\mathfrak{p})}(R \otimes_R \kappa(\mathfrak{p})) = \dim_{\kappa(\mathfrak{p})}(M \otimes_R N \otimes_R \kappa(\mathfrak{p})) \\ &= \dim_{\kappa(\mathfrak{p})}((M \otimes_R \kappa(\mathfrak{p})) \otimes_{\kappa(\mathfrak{p})} (N \otimes_R \kappa(\mathfrak{p}))) \\ &= \dim_{\kappa(\mathfrak{p})}(M \otimes_R \kappa(\mathfrak{p})) \cdot \dim_{\kappa(\mathfrak{p})}(N \otimes_R \kappa(\mathfrak{p})) \end{aligned}$$

Hence $\dim_{\kappa(\mathfrak{p})}(M \otimes_R \kappa(\mathfrak{p})) = 1$. This finishes the proof. ☐

Question 5

Fact. every line bundle on $\mathbb{A}_k^n$ is trivial.

i) Calculate $\operatorname{Pic}(\mathbb{P}^n) = \{$isomorphism classes of line bundles on $\mathbb{P}^n\}$ with group operation $- \otimes_{\mathcal{O}_{\mathbb{P}^n}} -$. Indeed show it is $\cong \mathbb{Z}$, generated by $\mathcal{O}(1)$.

ii) Compute $\Gamma(\mathbb{P}^n, \mathcal{O}(d))$ for $d \in \mathbb{Z}$. $(\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d})$

iii) Let p be the point $(x) \in \operatorname{Spec} k[x] = A_0$ in $\mathbb{P}_k^1 = A_0 \cup A_1$. Show that $\mathcal{O}(-1) \cong$ ideal sheaf of $\{p\}$.

Let Z be the closed subscheme $\operatorname{Spec}(k[x]/x^d) \subseteq A_0 \subseteq \mathbb{P}_k^1$. Show that $\mathcal{O}(-d) \cong$ ideal sheaf of Z.

What is the ideal sheaf of d closed points $\{p_1, ..., p_d\} \subseteq \mathbb{P}^1$?

iv) Show that if two graded R-mods M, N over graded ring R satisfy $M_n \cong N_n$ (in graded sense) for $n \geqslant d$, then $\widetilde{M} = \widetilde{N}$. (*See Sec 10 notes*)

Proof. i) Let $U_0, ..., U_n$ be the affine open sets of $\mathbb{P}_k^n$. From the fact we know that $\varphi_i : \mathcal{L}|_{U_i} \to \mathcal{O}_{U_i}$ is an isomorphism. The transition function $\alpha_{ij} : \mathcal{O}_{U_{ij}} \to \mathcal{O}_{U_{ij}}$ is a unit in the ring R_{ij}.

We know that $\operatorname{Pic} \mathbb{P}_k^n \cong \check{H}^1(\mathbb{P}_k^1, \mathcal{O}_{\mathbb{P}_k^1})$. (α_{ij}) are identified if they differ by a factor from k.

Line bundle $\mathcal{O}(m)$ defined by $(\alpha_{ij}) = \left(\dfrac{x_i}{x_j}\right)^m$. $\mathcal{O}(m) = \mathcal{O}(1)^{\otimes m}$, where tensoring comes from multiplying the transition functions.

$\operatorname{Pic} \mathbb{P}_k^n \cong \mathbb{Z}$.

ii) For $d < 0$, $\Gamma(\mathbb{P}_k^n, \mathcal{O}(d)) = 0$; for $d > 0$, $\Gamma(\mathbb{P}_k^n, \mathcal{O}(d)) \cong k[x_0, ..., x_n]_d$ is the d-th grading.

iii) ☐

Question 6

i) Let C be an Abelian category. Show that if every object $M \in \mathsf{C}$ has a injective morphism $M \to I$ into an injective object, then every object M admits an injective resolution. (*We say* C *has "enough injectives'.*)

Fact. Ab *has enough injectives.*

ii) Let $F \in \mathsf{Ab}(X)$. Pick $I_x \in \mathsf{Ab}$ such that $F_x \to I_x$ is an injective morphism and I_x is an injective object in Ab. Show that $I := \prod_{x \in X} (\varphi_x)_* I_x \in \mathsf{Ab}(X)$ (*inclusion map* $\varphi_x : \{x\} \hookrightarrow X$ *of a point*) is an injective object admitting an injective morphism $F \to I$. (*Hence* $\mathsf{Ab}(X)$ *has enough injectives.*)

Proof. i) We construct an injective resolution for M inductively. Firstly, since C has enough injectives, we have a monomorphism $\epsilon : M \to I^0$ where I^0 is some injective object. Now suppose that we have constructed

$$M \xhookrightarrow{\epsilon} I^0 \xrightarrow{d^0} \cdots \longrightarrow I^{n-1} \xrightarrow{d^{n-1}} I^n$$

Let $\operatorname{coker} d^{n-1} : I^n \to C^n$ be the cokernel of d^{n-1}. Since C has enough injectives, we have a monomorphism $\phi^n : C^n \to I^{n+1}$. Let $d^n := \phi^n \circ \operatorname{coker} d^{n-1} : I^n \to I^{n+1}$. Then $\ker d^n = \ker \operatorname{coker} d^{n-1} = \operatorname{im} d^{n-1}$. Hence the sequence is exact at I^n. By induction, we have constructed a sequence which is exact at each injective object I^n:

$$M \xhookrightarrow{\epsilon} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \xrightarrow{d^2} I^3 \longrightarrow \cdots$$

This is an injective resolution for M.

ii) We use the following criterion for an object being injective:

I ia an injective object if and only if $\operatorname{Hom}(-, I)$ *is an exact functor.*

We define a morphism $\varphi : F \to I$ by the following. For an open set $U \subseteq X$, let $\varphi_U : F(U) \to I(U) \cong \prod_{x \in U} (\varphi_x)_* I_x$, $s \mapsto \prod_{x \in U} \iota_x(s_x)$, where $\iota_x : F_x \to I_x$ is the given monomorphism. Note that φ is an monomorphism if and only if φ_U is injective for all open $U \subseteq X$. Let $s, s' \in F(U)$ such that $\varphi_U(s) = \varphi_U(s')$. By definition we have $\prod_{x \in U} \iota_x(s_x) = \prod_{x \in U} \iota_x(s'_x)$. Then $\iota_x(s_x - s'_x) = 0$ for all $x \in U$. Since ι_x is injective, $s_x = s'_x$ for all $x \in U$. Hence (Question 2 of Sheet 1) $s = s' \in F(U)$. So φ_U is injective. We deduce that $\varphi : F \to I$ is an monomorphism.

Then we shall show that $\operatorname{Hom}(-, I)$ is exact. Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathsf{Ab}(X)$. Take $x \in X$. Then $0 \to A_x \to B_x \to C_x \to 0$ is exact by Question 1. Since I_x is injective, $\operatorname{Hom}_{\mathsf{Ab}}(-, I_x)$ is exact. So we have a short exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathsf{Ab}}(C_x, I_x) \longrightarrow \operatorname{Hom}_{\mathsf{Ab}}(B_x, I_x) \longrightarrow \operatorname{Hom}_{\mathsf{Ab}}(A_x, I_x) \longrightarrow 0$$

The functor $\prod_{x \in X}$ is exact by Question 4 of Sheet 3 of *C2.2 Homological Algebra*. So we have a short exact sequence

$$0 \longrightarrow \prod_{x \in X} \operatorname{Hom}_{\mathsf{Ab}}(C_x, I_x) \longrightarrow \prod_{x \in X} \operatorname{Hom}_{\mathsf{Ab}}(B_x, I_x) \longrightarrow \prod_{x \in X} \operatorname{Hom}_{\mathsf{Ab}}(A_x, I_x) \longrightarrow 0$$

Finally, note that

$$\operatorname{Hom}_{\mathsf{Ab}(X)}(A, I) = \operatorname{Hom}_{\mathsf{Ab}(X)}\left(A, \prod_{x \in X} (\varphi_x)_* I_x\right)$$

$$\cong \prod_{x \in X} \mathrm{Hom}_{\mathsf{Ab}(X)}(A, (\varphi_x)_* I_x)$$
$$\cong \prod_{x \in X} \mathrm{Hom}_{\mathsf{Ab}}(\varphi_x^{-1} A, I_x)$$
$$\cong \prod_{x \in X} \mathrm{Hom}_{\mathsf{Ab}}(A_x, I_x)$$

So we have an short exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathsf{Ab}(X)}(C, I) \longrightarrow \mathrm{Hom}_{\mathsf{Ab}(X)}(B, I) \longrightarrow \mathrm{Hom}_{\mathsf{Ab}(X)}(A, I) \longrightarrow 0$$

We deduce that $\mathrm{Hom}_{\mathsf{Ab}(X)}(-, I)$ is exact. So I is an injective object. In conclusion, $\mathsf{Ab}(X)$ has enough injectives. □